Maximum Mass and Radial Modes of Hybrid Star in Presence of Strong Magnetic Field

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It is believed that neutron stars (NS) consist of hadronic and exotic states like strange quark or color superconducting matter. Stars having a quark core surrounded by a mixed phase followed by hadronic matter, may be considered as hybrid stars (HS). The mixed phase is well proportionate of both the hadron and quark phases. A huge magnetic field is predicted in core as well as on surface of the neutron star. Here we study the effect of this strong magnetic field on the equation of states(EOS) of HS matter. We further study the hadron-quark phase transition in the interiors of NS giving rise to HS in presence of strong magnetic field. We finally study the effect of strong magnetic field on maximum mass and eigen-frequencies of radial pulsation of such type of HS. For the EOS of hadronic matter, we have considered RMF(Relativistic Mean Field) theory and we incorporated the effect of strong magnetic fields leading to Landau quantization of the charged particles. For the EOS of quark phase we use the simple MIT bag model. We have assumed Gaussian parametrization to incorporate the density dependence of both bag pressure and magnetic field. We have constructed the intermediate mixed phase by using Glendenning conjecture. We found that magnetic field softens the EOS of both the phases, as a result the maximum mass is reduced for HS and period of oscillation is increased significantly for primary mode.

Key Words : Neutron Stars; Magnetic Fields; Equation of States; Phase Transition; Oscillation: Radial Modes.

Introduction

If the central density of neutron stars exceed the nuclear saturation density \( n_0 \sim 0.15 \text{ fm}^{-3} \), then the compact stars might contain deconfined and chirally restored quark matter. Recently, (Demorest et al., 2010) the mass measurement of millisecond pulsar PSR J1614-2230 and pulsar J1903+0327 has set a new mass limit for compact stars to be \( M = 1.97 \pm 0.04 \, M_\odot \) and \( M = 1.667 \pm 0.021 \, M_\odot \) respectively (Freire

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et al., 2010). This measurement for the first time has set a very strong limit on parameters of the EOS of matter under extreme conditions (Weber, 1999; Glendenning, 2000).

Generally there are two classes of compact stars with quark matter. The first is the so-called (strange) quark stars (SS) of absolutely stable strange quark matter. The second are the so-called hybrid stars (HS), along with the hadronic matter they have quarks matter in their interior. In between the quark and the hadronic phase, a quark-hadron mixed phase exists. The size of the core depends on the critical density for the quark-hadron phase transition and the EOS describing the matter phases.

New observations suggests that in some pulsars, the surface magnetic field can be as high as $10^{14}$-10$^{15}$G. It has also been attributed that the observed giant flares, SGR 0526-66, SGR 1900+14 and SGR 1806-20 (Palmer et al., 2005), are the manifestation of such strong surface magnetic field in those stars. Such stars are separately assigned as magnetars. If we assume flux conservation from a progenitor star, we can expect the central magnetic field of such stars as high as $10^{17}$-10$^{18}$G. Such strong fields are bound to affect the NS properties. It can modify either the metric describing the star (Bocquet et al., 1995; Cardall et al., 2001) or it can modify the EOS of matter of the star. The effect of strong magnetic field, both for nuclear matter (Broderick et al., 2000; Chakrabarty et al., 1997; Chen et al., 2005; Wei et al., 2006; Cheng et al., 2002) and quark matter (Chakrabarty and Sahu, 1996; Felipe et al., 2008; Ghosh and Chakrabarty, 2001a,b) has been studied earlier in detail.

There are several investigations of vibrating neutron stars and the simple dimensional analysis suggest that the period of fundamental mode would be of the order of milliseconds (Cameron, 1965). The value of the period of oscillation strictly depends on the equation of state along with some constraints on the parameters in both hadron and quark phase. Here we investigate the effect of magnetic field on equation of states of both the matter phases (quark phase and hadron phase). Then we construct the mixed phase equation of state in presence of strong magnetic field and use this to calculate the maximum mass as well as the period of oscillation of HS.

The paper is organized as follows. In Section 2 we discuss the relativistic nuclear EOS and the effect of Landau quantization due to magnetic field on the charged particles. In Section 3 we employ the simple MIT bag model for the quark matter EOS and the effect of magnetic field on the quarks (also due to Landau quantization). In Section 4 we construct the mixed phase region by Glendenning construction. In Section 5 we describe the equation for infinitesimal radial pulsations of a non rotating star given by Chandrasekhar (1964) in general relativistic formalism. We show our results in Section 6 for the density dependent bag parameter in quark matter and varying magnetic field for the mixed HS. Finally we summarize our results and draw some conclusion in Section 7.
Magnetic Field in Hadronic Phase

Hadrons are the degrees of freedom for the EOS at normal nuclear density. To describe the hadronic phase, we use a non-linear version of the relativistic mean field (RMF) model with hyperons (TM1 parametrization) which is widely used to construct EOS for NS. In this model the baryons interact with mean meson fields (Boguta and Bodmer, 1977; Glendenning and Moszkowski, 1991; Ghosh et al., 1995; Sugahara and Toki, 1994; Schaffner and Mishustinm, 1996; Mallick 2013; Mohanta et al., 2014).

For the beta equilibrated matter the conditions is
\[ \mu_i = b_i \mu_B + q_i \mu_e, \]  
where \( b_i \) and \( q_i \) are the baryon number and charge (in terms of electron charge) of species \( i \), respectively. \( \mu_B \) is the baryon chemical potential and \( \mu_e \) is the electron chemical potential.

For charge neutrality, the condition is
\[ \rho_c = \sum_i q_i n_i, \]  
where \( n_i \) is the number density of species \( i \).

For the magnetic field we choose the gauge to be, \( A^\mu \equiv (0, -yB, 0, 0) \), \( B \) being the magnitude of magnetic field. For this particular gauge choice we can write \( \vec{B} = B \hat{z} \). Due to the magnetic field, the motion of the charged particles are Landau quantized in the perpendicular direction to the magnetic field. The momentum of the \( x-y \) plane is quantized and hence the energy of the \( n \)th Landau level is (Landau and Lifshitz, 1965) given by
\[ E_i = \sqrt{p_i^2 + m_i^2 + |q_i|B(2n + s + 1)}, \]  
where \( n=0, 1, 2, ... \), are the principal quantum numbers for allowed Landau levels, \( s = \pm 1 \) refers to spin up(+) and down(-) and \( p_i \) is the component of particle (species \( i \)) momentum along the field direction. Setting \( 2n + s + 1 = 2\tilde{\nu} \), where \( \tilde{\nu} = 0, 1, 2, ... \), we can rewrite the single particle energy eigenvalue in the following form
\[ E_i = \sqrt{p_i^2 + m_i^2 + 2\tilde{\nu}|q_i|B} = \sqrt{p_i^2 + \tilde{m}_i^2}, \]  
where the \( \tilde{\nu} = 0 \) state is singly degenerate. It should be remembered that for baryons the mass is \( m_b^* \).

The total energy density of the system can be written as (Mallick, 2013; Mohanta et al., 2014).
\[ \varepsilon = \frac{1}{2} m_\omega^2 \omega_0^2 + \frac{1}{2} m_\rho^2 \rho_0^2 + \frac{1}{2} m_\sigma^2 \sigma^2 + \frac{1}{2} m_{\sigma^*}^2 \sigma^*^2 + \frac{1}{2} m_{\phi^0}^2 \phi_0^2 + \frac{3}{4} d \omega_0^4 + U(\sigma) \]  
\[ + \sum_b \bar{\varepsilon}_b + \sum_t \varepsilon_t + \frac{B^2}{8\pi^2}, \]  
(5)
where the last term is the contribution from the magnetic field. The general expression for the pressure is given by

\[ p = \sum_i \mu_i n_i - \varepsilon. \]  

(6)

**Magnetic Field in Quark Phase**

Considering the simple MIT bag model for the quark matter in presence of magnetic field, we assume that the quarks are non-interacting. The current masses of u and d quarks are extremely small, e.g., 5 and 10 MeV respectively, whereas, for s-quark the current quark mass is taken to be 150 MeV.

The thermodynamic potential in presence of strong magnetic field \(B(> B^{(c)}, \text{critical value})\) is given by (Chakrabarty and Sahu, 1996; Sahu et al., 2002; Sahu and Patra, 2001)

\[ \Omega_i = -\frac{q_i|q_i|B}{4\pi^2} \int dE \sum_\nu \frac{d\rho_i}{dE_i} \ln[1 + \exp(\mu_i - E_i)/T]. \]  

(7)

For the zero temperature, the Fermi distribution is approximated by a step function. By interchanging the order of the summation over \(\nu\) and integration over \(E\) one gets,

\[ \Omega_i = -\frac{2q_i|q_i|B}{4\pi^2} \sum_\nu \int_{\mu}^\infty \frac{dE_i}{\sqrt{m_i^2 + 2\nu|q_i|B}} \frac{dE_i}{\sqrt{E_i^2 - m_i^2 - 2\nu|q_i|B}}. \]  

(8)

The total energy density and pressure of the strange quark matter is given by (Mallick, 2013; Mohanta et al., 2014)

\[ \varepsilon = \sum_i \Omega_i + B_G + \sum_i n_i \mu_i \]

\[ p = -\sum_i \Omega_i - B_G, \]  

(9)

where \(B_G\) is the bag constant.

**Phase Transition and Mixed Phase**

With the above given hadronic and quark EOS, we now perform the Glendenning construction (Glendenning, 1992) for the mixed phase, which determines the range of baryon density where both phases coexist. Here one allows both the hadron and quark phases to be separately charged, preserving the total charge neutrality as a whole in the mixed phase. Thus the matter can be treated as a two-component system, and can be parametrized by two chemical potentials, usually the pair \((\mu_e, \mu_n)\), i.e., electron and baryon chemical...
potential. To maintain mechanical equilibrium, the pressure of the two phases are equal. Satisfying the chemical and beta equilibrium the chemical potential of different species are connected to each other. The Gibbs condition for mechanical and chemical equilibrium at zero temperature between both phases is given by

$$p_{HP}({\mu}_e, {\mu}_n) = p_{QP}({\mu}_e, {\mu}_n) = p_{MP}. \quad (10)$$

This equation gives the equilibrium chemical potentials of the mixed phase corresponding to the intersection of the two phases. At lower densities below the mixed phase, the system is in the charge neutral hadronic phase, and for higher densities above the mixed phase the system is in the charge neutral quark phase. As the two surfaces intersect, one can calculate the charge densities $\rho_{c}^{HP}$ and $\rho_{c}^{QP}$ separately in the mixed phase. If $\chi$ is the volume fraction occupied by quark matter in the mixed phase, we have

$$\chi \rho_{c}^{QP} + (1 - \chi) \rho_{c}^{HP} = 0. \quad (11)$$

Therefore the energy density $\epsilon_{MP}$ and the baryon density $n_{MP}$ of the mixed phase can be obtained as

$$\epsilon_{MP} = \chi \epsilon_{QP} + (1 - \chi) \epsilon_{HP}, \quad (12)$$

$$n_{MP} = \chi n_{QP} + (1 - \chi) n_{HP}. \quad (13)$$

### Radial Pulsation of Non-Rotating Star

The equation for infinitesimal radial pulsation of a non-rotating star was given by Chandrasekhar (Chandrasekhar, 1964) and in general relativistic formalism, has the following form (Sahu et al., 2002)

$$X \frac{d^2 \xi}{dr^2} + Y \frac{d \xi}{dr} + Z \xi = \tilde{\sigma}^2 \xi. \quad (14)$$

Here $\xi(r)$ is the Lagrangian fluid displacement and $c\tilde{\sigma}$ is the characteristic eigenfrequency ($c$ is the speed of light). The quantities $X, Y, Z$ depend on the equilibrium profiles of the pressure $p$ and density $\rho$ of the star and are represented by:

$$X = \frac{-e^{-\lambda} e^{\nu}}{p + \rho c^2} \Gamma p, \quad (15)$$

$$Y = \frac{-e^{\lambda} e^{\nu}}{p + \rho c^2} \left\{ \Gamma p \left( \frac{1}{2} \frac{d \nu}{dr} + \frac{1}{2} \frac{d \lambda}{dr} + \frac{2}{r} \right) + p \frac{d \Gamma}{dr} + \Gamma \frac{dp}{dr} \right\}, \quad (16)$$

$$Z = \frac{-e^{\lambda} e^{\nu}}{p + \rho c^2} \left\{ \frac{4}{r} \frac{dp}{dr} - \frac{(dp/dr)^2}{p + \rho c^2} - A \right\} + \frac{8 \pi G}{c^4} e^{\nu} p. \quad (17)$$
\( \Gamma \) is the adiabatic index defined as
\[
\Gamma = \left( 1 + \frac{\rho c^2}{p} \right) \frac{dp}{d(\rho c^2)}
\]  
(18)

and
\[
A = \frac{d\lambda}{dr} \frac{\Gamma p}{r} + \frac{2 p d\Gamma}{r^2} - \frac{2\Gamma p}{r^2} - \frac{1}{4} \frac{4 dp}{dr} \left( \frac{d\lambda}{dr} \frac{\Gamma p}{r} + \frac{2 p d\Gamma}{r^2} + 2\Gamma \frac{dp}{dr} - \frac{8\Gamma p}{r} \right) 
\]
\[- \frac{1}{2} \frac{\Gamma p}{r} \left( \frac{d\nu}{dr} \right)^2 - \frac{1}{2} \frac{\Gamma p}{r} \frac{d^2\nu}{dr^2} \]  
(19)

To solve the pulsation equation (14), the boundary conditions are
\[
\xi (r = 0) = 0,
\]
(20)
\[
\delta p (r = R) = -\xi \frac{dp}{dr} - \Gamma p \frac{e^{\nu/2}}{r^2} \frac{\partial}{\partial r} \left( r^2 e^{-\nu/2} \xi \right) \bigg|_{r=R} = 0.
\]
(21)

It is important to note that \( \xi \) is finite when \( p \) vanishes at \( r = R \). The pulsation equation (14) is a Strum-Liouville eigenvalue equation for \( \tilde{\sigma}^2 \), subject to the boundary conditions Eq. (20) and (21). As a consequence the eigenvalues \( \tilde{\sigma}^2 \) are all real and form an infinite discrete sequence \( \tilde{\sigma}^2_0 < \tilde{\sigma}^2_1 < ... < \tilde{\sigma}^2_n < ......., \) with the corresponding eigenfunction \( \xi_0(r), \xi_1(r), ..., \xi_n(r) \), where \( \xi_n(r) \) has \( n \) nodes. It immediately follows that if fundamental radial mode of a star is stable \( (\tilde{\sigma}_0 > 0) \), then all the radial modes are stable. We note that Eqs.(15-19) depend on the pressure and density profiles, as well as on the metric functions \( \lambda(r), \nu(r) \) of the non-rotating star configuration. Those profiles are obtained by solving the Oppenheimer-Volkof equation of hydrostatic equilibrium (Misner et al., 1970).
\[
\frac{dp}{dr} = -G \left( \rho + \frac{p}{c^2} \right) \frac{m + 4\pi r^3 p/c^2}{r^2 (1 - 2Gm/rc^2)},
\]
(22)
\[
\frac{dm}{dr} = 4\pi r^2 \rho,
\]
(23)
\[
\frac{d\nu}{dr} = \frac{2G (m + 4\pi r^3 p/c^2)}{r^2 c^2 (1 - 2Gm/rc^2)},
\]
(24)
\[
\lambda = -\ln \left( 1 - 2Gm/rc^2 \right).
\]
(25)

Eqs (9)-(12) can be numerically integrated for a given equation of state \( p(\rho) \) and given central density to obtain the radius \( R \) and mass \( M = m(R) \) of the star. Therefore the basic input to solve the structure and
pulsation equations is the equations of state, \( p = p(\rho) \). It has been seen (Burgio et al., 2001) that structure parameters of neutron stars are mainly dominated by the equation of state at high densities, specifically around the core. Since the oscillation features are governed by structure profiles of neutron stars, it is expected to possess marked sensitivity on the high density equation of state as well. We employed equations of states of mixed phase region in presence as well as absence of magnetic field to calculate the period of oscillation \( P (= 2\pi/c\tilde{\sigma}) \) of both HS.

**Results**

The magnetic field in HS changes the EOS of the matter. The single particle energy is Landau quantized, and thereby it changes all the other thermodynamic variable of the EOS, namely the number density, pressure and the energy density.

We parametrized the bag constant in such a way that it attains a value \( B_\infty \), asymptotically at very high densities. The experimental range of \( B_\infty \) is given in Burgio et al. (Burgio et al., 2002), and from there we choose the value \( B_\infty = 130 \text{MeV} \). With such assumptions we then construct a Gaussian parametrization as given by (Burgio et al., 2002)

\[
B_G(n_b) = B_\infty + (B_g - B_\infty) \exp \left[ -\beta \left( \frac{n_b}{n_0} \right)^2 \right].
\]  

(26)

The value \( B_\infty \), is the lowest one which it attains at asymptotic high density in quark matter, and is fixed at \( 130 \text{MeV} \). The quoted value of bag pressure, is at the hadron and mixed phase intersection point denoted by \( B_g \) in the equation. The value of \( B_G \) decreases with increase in density and attain \( B_\infty = 130 \text{MeV} \) asymptotically, the rate of decrease of the bag pressure being governed by parameter \( \beta \).

We assume that the parametrization of the magnetic field strength depends on the baryon number density. Therefore we assume a simple density dependence, as given by (Chakrabarty et al., 1997)

\[
B (n_b/n_0) = B_s + B_0 \left\{ 1 - e^{-\alpha \frac{n_b}{n_0}} \right\}^\gamma,
\]

(27)

where \( \alpha \) and \( \gamma \) are two parameters determining the magnetic field profile with given \( B_s \) and \( B_0 \), \( n_b \) being the baryon number density. The value of \( B \) mainly depends on \( B_0 \), and is quite independent of \( B_s \). Therefore, we vary the field at the center, whereas surface field strength is taken to be \( B_s = 10^{14} \text{G} \). We keep \( \gamma \) fixed at 2, and vary \( \alpha \) to have the field variation. In the above parametrization, the magnetic field strength depends on the baryon number density. However, at each density the field is uniform and constant.

In Fig. 1, we have plotted pressure against energy density having density dependent bag pressure \( 170 \text{MeV} \) for \( \alpha = 0.01 \). It is clear that magnetic field softens the EOS as well as broadens the mixed phase.
region. In Fig. 2, we have plotted Gravitational mass (in solar mass unit) against energy density for zero and non zero value of magnetic field. In Fig. 3, we have plotted period of oscillation (in seconds) against gravitational mass for zero and non zero value of magnetic field. We notice that there is a kink for both magnetic and non magnetic case which corresponds to phase transition from hadron phase to quark phase. Kink is prominent for primary mode. Presence of magnetic field also increases the period of oscillation in higher mass region.

Figure 1: Pressure (\(MeV/fm^3\)) against energy density (\(MeV/fm^3\)) for zero and non zero value of magnetic field. B=0 (dotted line) and B=10\(^{17}\) G (solid line)

Figure 2: Gravitational mass in solar mass unit against energy density (\(MeV/fm^3\)) for zero and non zero value of magnetic field. B=0 (dotted line) and B=10\(^{17}\) G (solid line)
Figure 3: Period of oscillation in seconds against gravitational mass in solar mass for zero and non zero value of magnetic field. $B=0$ (dotted lines) and $B=10^{17}$ G (solid lines).

Conclusions

We have presented a calculation of the period of oscillations of neutron stars by using the radial pulsation equations of non rotating neutron stars, as given by Chandrasekhar (Chandrasekhar 1964) in the general relativistic formalism. To solve the radial pulsation equations, one needs a structure profile of non rotating neutron stars, obtained by employing realistic equations of state. The equations of state for hadron matter used here were derived from the relativistic formalisms with quark phases at higher densities. Then, the equations of state were constructed by using the Glendenning’s condition for mechanical and chemical equilibrium as a function of baryon and electron density at the mixed phase, comprising hadron as well as quark phases. The main conclusion of our work is that the presence of magnetic field broadens the mixed phase region, the period of oscillation shows a kink around the point where mixed phase starts, in primary as well as in the higher modes, which is the distinct signature of quark matter onset in neutron star. The presence of magnetic field increases period of oscillation of fundamental as well as in higher mode at maximum mass but the effect is significant in fundamental mode.

References

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