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THE GAUSS-BONNET THEOREM

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In this note we give a proof of the Gauss-Bonnet theorem for Riemannian manifolds (of any dimension) using Morse theory.

Key words : Chern Weil theory; Euler characteristic.

1. Introduction

In this note we give a proof of the celebrated Gauss-Bonnet theorem which was the first result to relate curvature and topology. The first formulations of the theorem were for the curvature tensor of the (unique torsion free) Levi-Civita connection associated to a Riemannian metric. However after Chern-Weil theory one knows that the theorem applies to the curvature of *any* $SO(m)$-connection on the tangent bundle of the manifold of dimension $m(= 2 \cdot n)$. For our proof of the (more general version of the) Gauss-Bonnet Theorem, we make use of Morse theory. The basic idea is that outside the set of critical points of a Morse function on a Riemannian manifold M, the tangent bundle admits a reduction of the structure group to $SO(2 \cdot n - 1)$ (dimension of the manifold being $2 \cdot n$) and we use a $SO(2 \cdot n)$-connection on the tangent bundle of M (the Riemannian metric gives a reduction of structure group of the tangent bundle to $SO(2 \cdot n)$) which outside suitably small neighbourhoods of the critical points is a $SO(2n-1)$ connection. The form that has to be integrated in the Gauss-Bonnet Theorem vanishes outside these neighbourhoods since the determinant of a skew symmetric matrix in odd dimensions vanishes. Thus the theorem is reduced to computing ceratin local integrals. Equating integrals arising out of such connections corresponding to two Morse functions one of which has all the critical points of the other with the same indices, but has exactly two additional (cancelling) critical points of index q and $(q+1)$, one sees that these local integrals corresponding to critical points of different pairities are negatives of each other. One knows from Morse theory that if M is a smooth compact manifold and f is a Morse function on M, then M has the homotopy type of a CW-complex with exactly as many q-cells as there are critical points of f of index q. It is immediate from this that the Gauss-Bonnet integral equals $\lambda_n \cdot \chi(M)$ Where $\chi(M)$ is the Euler-Poincare characteristic of M, for a suitable constant λ_n depending only on the dimension $m = 2 \cdot n$. The constant λ_1 can be computed to be $2 \cdot \pi$ for the unit sphere in dimension 2 using the Riemannian connection for the (constant curvature) metric on it induced from $\mathbb{R}^3$ and then one sees easily that $\lambda_n = (2 \cdot \pi)^n$.

2. Statement of the Theorem

Let M be a smooth oriented manifold of dimension m and g a Riemannian metric on M. We denote by $T(M)$ the tangent bundle of M. The Riemannian metric gives a reduction of strucure group of the tangent bundle (or equivalently, of the principal $GL(m, \mathbb{R})$-bundle associated to $T(M)$) to the special orthogonal group $SO(m)$. We denote by P this $SO(m)$ bundle. The underlying set of P has the following description: a point $\xi \in P$ is an isometric isomorphism $\xi : \mathbb{R}^m \simeq T_b$ of $\mathbb{R}^m$ with the standard inner product, on T_b (the tangent space at $b \in M$) (for some $b \in M$) equipped with the inner product defined by g and preserving orientation: $\mathbb{R}^m$ is given the orienation determined by its standard (ordered) basis $\{e_i\}_{1 \leq i \leq n}$. The projection π of P on M assigns to the element ξ in P the point b in M. A smooth structure on P is obtained as follows. Let $\{(U_\alpha, f_\alpha) | \alpha \in A\}$ be an atlas for the smooth structure on M; we denote the coordinates in (U_α, f_α), $\{_\alpha x_i\}_{1 \leq i \leq m}$. For $1 \leq i \leq m$ let $_\alpha X_i$ be the vector field on U_α defined by setting, for $b \in U_\alpha$, $_\alpha X_i(b) = \partial / \partial_\alpha x_i|_b$. Let $\{_\alpha Z_i \ 1 \leq i \leq m\}$ be the vector fields in U_α obtained from the $\{_\alpha X_i | 1 \leq i \leq m\}$ by the Schmidt orthogonalization process (for the inner product defined by g on T_b). We can then define a map $\Phi_\alpha : \tilde{U}_\alpha = \pi^{-1}(U_\alpha) \to U_\alpha \times SO(m)$. For $\xi \in \pi^{-1}(U_\alpha)$, $\Phi_\alpha(\xi) = (\pi(\xi)), T(\xi))$ where $T(\xi)$ is the (special orthogonal) matrix defined by setting $\xi^{-1}(X_i) = \sum_{1 \leq j \leq m} T(\xi)_{ij} \cdot e_j$ ($\{e_i | 1 \leq i \leq m\}$ is the standard basis of $\mathbb{R}^m$). The topology on P is the one for which the $\tilde{U}_\alpha$ are open for all $\alpha \in A$ and the maps Φ_α are homeomorphisms. Then - as is easily seen - there is a unique smooth structure on P such that $\{\Phi_\alpha | \alpha \in A\}$ are smooth diffeomorphisms.

In the sequel we denote by G the special orthogonal group $SO(m)$ and its Lie algebra (of left translation invariant vector fields) by $\mathfrak{g}$: it has a natural identification with the Lie algebra of skew-symmetric matrices. Any $X \in \mathfrak{g}$ defines a vector field $\underline{X}$ on P - the 1-parameter group of $\underline{X}$ is given as follows: for $t \in \mathbb{R}$ and $\xi \in P$, $\xi \to \xi \cdot exp\, t \cdot X$ where exp is the exponential map of $\mathfrak{g}$ in G. Recall that a conection on the principal G-bundle P is a smooth 1-form ω on P with values in $\mathfrak{g}$ satisfying the following conditions: (i) For a tangent vector $v \in T_\xi$ and $g \in G$, $\omega(dg(v)) = Ad\; g^{-1}(\omega(v))$ and (ii) $\omega(\underline{X}_\xi) = X$ for all $X \in \mathfrak{g}$ and $\xi \in P$. The curvature form of the connection ω is the (smooth) $\mathfrak{g}$-valued 2-form $K_\omega = d\,\omega + \omega \wedge \omega$ (here for tangent vectors v, w in $T_\xi, \xi \in P$, $(\omega \wedge \omega)(v, w) = [\omega(v), \omega(w)] - [\omega(w), \omega(v)] = 2 \cdot [\omega(v), \omega(w)]$). One checks easily that $K_\omega(v, w) = 0$ whenever $d\pi(v)$ or $d\pi(w)$ is zero. K_ω satisfies the Bianchi identity: $dK_\omega + \omega \wedge K_\omega = 0$.

Let $S(\mathfrak{g})$ be the symmetric algebra on $\mathfrak{g}$ and for an integer $p \geq 0$, $S^p(\mathfrak{g}) \subset S(\mathfrak{g})$, the pth symmetric power of $\mathfrak{g}$. The adjoint action of G on $\mathfrak{g}$ extends to an action on $S(\mathfrak{g})$ preserving the algebra structure and the gradation (in which $S^p(\mathfrak{g})$ is the pth graded component). Let $\mathcal{E}^p(\mathfrak{g})$ denote the vector space of all smooth p-forms α on P with values $S(\mathfrak{g})$ satisfying for tangent vectors $v_1, v_2, \cdots v_p$ at a point ξ of P,

$$\alpha(dg(v_1), dg(v_2), \cdots dg(v_p)) = Ad\; g^{-1}(\alpha(v_1, v_2 \cdots v_p)).$$

We denote by $\mathcal{E}_0^p(\mathfrak{g})$ the subspace $\{\alpha \in \mathcal{E}_0^p \mid \alpha(v_1, v_2 \cdots v_p) = 0$ if $d\pi(v_i) = 0$ for some i with $1 \leq i \leq p$, for tangent vectors $(v_1, v_2 \cdots v_p)$ at ξ in P and g in $G\}$ of $\mathcal{E}_p(\mathfrak{g})$.

It is clear that $\mathcal{E}^*(\mathfrak{g}) = \coprod_{\{2 \cdot p \;|p \in \mathbb{N}\}} \mathcal{E}^p(\mathfrak{g})$ is a an associative and commutative algebra (under exterior multiplication) of which $\mathcal{E}_0^*(\mathfrak{g}) = \coprod_{\{2 \cdot p \;|p \in \mathbb{N}\}} \mathcal{E}_0^p(\mathfrak{g})$ is a subalgebra. Evidently K_ω is in $\mathcal{E}_0^*$ and since it takes values in $\mathfrak{g}$, K_ω^p takes values in $S^p(\mathfrak{g})$. Suppose now that dim. (M) is even $= 2 \cdot n$, say, so that $\mathfrak{g}$ is the Lie algebra of $(2 \cdot n \times 2 \cdot n)$ skew-symmetric matrices. Now, as is well known, there is a unique $Ad(G)$-invariant polynomial Pf - the Pfaffian - on $\mathfrak{g}$ taking non-negative values such that $Pf^2(X)$ = determinant (X) for all $X \in \mathfrak{g}$. Composing the Pfaffian Pf - a G-invariant linear form on $S^n(\mathfrak{g})$ - with K_ω^n one gets a scalar $2 \cdot n$ form $Pf(K_\omega)$ on P which is zero on any $2 \cdot n$-tuple $v_1, v_2 \cdots v_m$ of tangent vectors at any point ξ of P, if one of the $d\pi(v_i) = 0$. It follows that $Pf(K_\omega)$ is the pull-back from M under π of a $2 \cdot n$-form Pf_ω. Moreover one deduces easily from the fact that $dK_\omega + \omega \wedge \omega = 0$ that $d\, Pf_\omega = 0$. We can now state the Gauss-Bonnet theorem:

***Theorem* 2.1** — *Assume that M is compact and is of dimension $2 \cdot n$. Then*

$$\int_M Pf_\omega \; = (2 \cdot \pi)^n \cdot \chi(M)$$

$\chi(M)$ is the Euler-Poincare characterestic of M.

The theorem says in particular that the integral is independent of the choice of the Riemannian metric and the connection: if g, g' are two Riemannian metrics on M and ω, ω' are connections on the corresponding G-bundles P, P' on M, $Pf_\omega - Pf_{\omega'}$ is exact. We give below a quick proof (due to M S Narasimhan and S Ramanan) of this last statement.

One remarks first that the G-bundles P and P' are isomorphic as $SO(m)$-bundles. We fix one such isomorphism and treat both ω and ω' as connections on the same bundle P. Consider now the G-bundle $P \times I \to M \times I$ where I is the interval $(1-\epsilon, 1+\epsilon)$ with some $\epsilon > 0$. We then define a connection Ω on the G-bundle $P \times I$ by setting $\Omega(\partial/\partial t|_{\xi,t}) = 0$ for all (ξ, t) in $P \times I$ and for v in $T_\xi(P)$, $\Omega((v,0)|_{(\xi,t)}) = t \cdot \omega(v) + (1-t) \cdot \omega'(v)$. If i_0 (resp. i_1) is the inclusion $x \to (x, 0)$ (resp. $x \to (x,1)$) of M in $M \times I$, $i_0^*(Pf_\Omega)$ (resp. $i_1^*(Pf_\Omega)$) = Pf_ω (resp. $Pf_{\omega'}$). As i_0 and i_1 are homotopic maps, the two closed forms Pf_ω and $Pf_{\omega'}$ differ by an exact form and hence (by Stokes theorem) their integrals over M are equal.

Remark 2.2 : Note that the proof holds for a G-bundle on a manifold for *any* G and *any* G-invariant polynomial on $\mathfrak{g}$.

3. Morse Theory

In this section we summarise some basic results (without proofs) from Morse theory that are needed in the sequel. Milnor [2] has a beautiful account of Morse theory where all the results below can be found. Let M be a smooth manifold of dimension m and $f : M \to \mathbb{R}$ a smooth function. A point $b \in M$ is a *critical* point of f iff in some (and hence in any) co-ordinate system, $(x_i)_{1 \le i \le m}$ in a neghbourhood of b, $(\partial f/\partial x_i)(b) = 0$ for $1 \le i \le m$. A critical point p of f is *non-degenarate* if in some (therefore any) coordinate system $(x_i)_{1 \le i \le m}$ in a neighbourhood of b, the (symmetric square) matrix $S = (\partial^2 f/\partial x_i \partial x_j)(b)_{1 \le i\ j \le m}$ is non-singular. The number of *negative* eigen values of S (which is independent of the choice of the coordinate system) is the *index* of the critical point. With these definitions we have

Lemma 3.1 — Let $f : U \to \mathbb{R}$ a smooth function on a smooth m-dimensional manifold M and b a non-degenerate critical point for f. Then there is an open set W in $\mathbb{R}^m$ with $0 \in W$ and a smooth diffeomorhism Φ of W onto an open set $V \subset M$ such that $\Phi(0) = b$ and

$$f \circ \Phi(x) = f(b) + \sum_{1 \le i \le p} x_i^2 - \sum_{p < i \le m} x_i^2$$

for all $x = (x_i)_{1 \le i \le m} \in W$ where p is the number of positive eigen values of S.

Corollary 3.2 — If b is a non-degenerate critical point of a smooth function f on M then there is a neighbourhood U of b in M such that b is the only critical point of f in U.

Proposition 3.3 — Let $a \in \mathbb{R}$. On any connected smooth paracompact manifold M there is a proper smooth map $f : M \to [a, \infty) \subset \mathbb{R}$ all of whose critical points are degenerate.

Definition 3.4 — A smooth function on a connected paracompact manifold M as in the above proposition is a Morse function.

***Theorem* 3.5** — *Let M be a connected paracompact manifold and $f : M \to [a, \infty)$ a Morse function. Then M is of the homotopy type of a C-W complex with one cell of dimension q for each critical point of f of index q.*

Corollary 3.6 — Let M be a connected compact manifold of dimension n and $f : M \to [a, \infty)$ be a Morse function on M. For $1 \leq i \leq m$ let n_i be the number of crtical points of index i. Then $\sum_{1 \leq i \leq n} (-1)^i \cdot n_i = \chi(M)$, the Euler-Poincare characterestic of M.

Proposition 3.7 — Let M be a connected paracompact manifold and $f : M \to [d, \infty)$ a Morse function. Let $a, b \in (d, \infty)$ with $a < b$ be such that there are no critcal points of f in $f^{-1}([a, b])$. Then given q with $0 \leq q < n$, there is a Morse function f' on M such that $f = f'$ on $M \backslash f^{-1}([a, b])$, $f^{-1}([a, b]) = f'^{-1}([a, b])$ and f' has exactly two critical points of index q and $q + 1$ in $f^{-1}([a, b])$.

SKETCH OF PROOF 3.8 : We assume as we may without loss of generality that $d < -1$, $a = -1$ and $b = 1$. Let $u : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $u(t) = 0$ for $|t| \geq 1$, $u(t) = 1$ for $|t| \leq 1/2$ and $u(t) > 0$ for $t \in (-1, 1)$. Set $f^{-1}(0) = V$. Then since f has no critical points in $f^{-1}(-1 - \delta, 1 + \delta)$ for $\delta > 0$ sufficiently small, one has a diffeomorphism Φ of $f^{-1}(-1 - \delta, 1 + \delta)$ on $V \times (-1 - \delta, 1 + \delta)$ such that the cartesian projection $\pi_2 : V \times (-1 - \delta, 1 + \delta) \to (-1 - \delta, 1 + \delta)$ composed with the diffeomorphism is the restriction of f to $f^{-1}(-1 - \delta, 1 + \delta)$. Let U be an open subset of V admitting a diffeomorphism Ψ on to the disc $D = \{(x, y) \in \mathbb{R}^{(m-1-q)} \times \mathbb{R}^q = \mathbb{R}^{(m-1)} | (\sum_{1 \leq i \leq (m-1-q)} x_i^2) + (\sum_{1 \leq i \leq q} y_i^2) < (1 + \delta)\}$. Let $Q_q : D \to \mathbb{R}$ be the quadratic function defined by setting for $(x, y) \in D$, $Q_q(x, y) = |x|^2 - |y|^2$. Let $Q : V \to \mathbb{R}$ be the function on V defined as follows: for $p \in U$, $Q(p) = u(p) \cdot Q_q(\Psi(p))$ and for $p \notin U$, $Q(p) = 0$. We then define f' on M as follows: let π_1 be the cartesian projection of $V \times (-\delta - 1, 1 + \delta)$ on V. Then $f' = f$ outside $f^{-1}[-1, 1]$ while for $p \in f^{-1}(-\delta - 1, 1 + \delta)$,

$$f'(p) = u(f(p)) \cdot Q(\pi_1 \circ \Phi(p)) + f(p)^3/3 - f(p)/4.$$

Then one checks easily that the critical points of f are all critical points of f' and f' has exactly two more critical points, both non-degenerate viz. $\Phi^{-1}(\Psi^{-1}(0), -1/2)$ of index $(q+1)$ and $\Phi^{-1}(\Psi^{-1}(0), 1/2)$ of index q. Hence the proposition.

4. Proof of the Theorem

Let $D(r)$ denote the open disc in $\mathbb{R}^m$ of radius r around the origin:

$$D(r) = \{z \in \mathbb{R}^m ||z| = (\sum_{1 \leq i \leq m} z_i^2)^{1/2} < r\}.$$

Let $q \geq 0$ be an integer $\leq (m-1)$ and let $p = m - q$. We identify $\mathbb{R}^m$ with the product $\mathbb{R}^p \times \mathbb{R}^q$ so that elements z of $\mathbb{R}^m$ are pairs (x, y) with $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$. Let $F(q) : \mathbb{R}^m \to \mathbb{R}$ is the quadratic function defined by setting for $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$,

$$F(q)(x, y) = |x|^2 - |y|^2 = (\sum_{1 \leq i \leq p} x_i^2) - (\sum_{1 \leq j \leq q} y_j^2).$$

Let $X(q)$ be the vector field

$$[(\sum_{1 \leq i \leq p} x_i \cdot \partial/\partial x_i) - (\sum_{1 \leq j \leq q} y_i \cdot \partial/\partial y_j)]/(|x|^2 + |y|^2)^{1/2}$$

on $\mathbb{R}^m$. For every $z \in (\mathbb{R}^m \backslash 0)$, $X(q)(z)$ is a unit tangent vector and the tangent space T_z at z decomposes as an othogonal direct sum $T_z = \mathbb{R} \cdot X(q)_z \oplus E(q)$ where $E(q) = \{v \in T_z | < X(q), v >= 0\}$ where T_z is identified with $\mathbb{R}^m = \mathbb{R}^p \times \mathbb{R}^q$ through the basis $\{\partial/\partial x_i | 1 \leq i \leq p\} \cup \{\partial/\partial y_j | 1 \leq j \leq q\}$. We see thus that we have a reduction of the structure group of the tangent bundle of $\mathbb{R}^m \backslash 0$ to $SO(m-1)$. We fix a $SO(m-1)$-connection ω_q on the tangent bundle of $\mathbb{R}^m \backslash 0$ compatible with this reduction of strucure group. Let $u(r) : \mathbb{R} \to [0, 1]$ be a C^∞ function such that $u(r)(t) = 0$ if $|t| \geq 2r$ and $u(r)(t) = 1$ if $|t| \leq r$. Let β denote the flat Riemannian connection on $\mathbb{R}^m$ and let $\omega_r(q)$ be the connection on R^m defined as follows: over $D(r)$, $\omega_r(q) = \beta$; over $\mathbb{R}^m \backslash 0$, $\Omega_r(q) = u(r) \cdot \beta + (1 - u(r)) \cdot \omega_r(q)$

Suppose now that we are given a Morse function f on the smooth oriented compact manifold M of even dimension $m = 2 \cdot n$. Let $C(f)$ denote the (finite) set of critical points of f and for $c \in C(f)$, let $q(c)$ be the index of c. Then we can find an $r : C(f) \to \mathbb{R}^+$, open neighbourhoods $U(c)$ of c, $c \in C(f)$ and diffeomorphisms Φ_c of $U(c)$ onto $D(3 \cdot r(c))$ such that $F(q(c)) \circ \Phi_c = f|_{U(c)} - f(c)$. Let g be a Riemannian metric on M such that Φ_c is a Riemannian isometry of $U'(c) = \Phi^{-1}(D(5 \cdot r(c)/2))$

on $D(5 \cdot r(c)/2)$ with the standard euclidean metric. Let $M' = M \backslash C(f)$. The metric g defines an isomorphism of the space of differentials at any point $b \in M$ with the tangent space T_b of M at b. Let V_0 be the vector field that corresponds to the 1-form df under this isomorphism and let $V = V_0 / \parallel V \parallel$. We then have a decomposition of $T_b = \mathbb{R} \cdot V(b) \oplus V(b)^{\perp}$ of T_b for all $b \in M'$ yielding a reduction of structure group to $SO(m-1)$ of the tangent bundle of M. In view of our choice of the metric, we see that the diffeomorphism Φ_c is compatible with the two reductions of the structure group over $U'(c)$ and (its image under Φ_c, $D(5 \cdot r(c)/2)$. Let $u_c : M \to [0,1]$ be the C^{∞} function defined as follows: $u_c(b) = 0$ if b is not in $\Phi_c^{-1}(D(2 \cdot r(c)))$ and $u_c(\Phi_c^{-1}(z)) = u(r(c))(|z|)$. Let Ω_c be the pull back under Φ_c to $U(c)$ of the connection $\Omega_{q(c)}$ on $\mathbb{R}^m$. Let η be a $SO(m-1)$ connection on M' extending the connections $\{\Omega_c | c \in C(f)\}$ on the $M' \cap U'(c)$. Then $\sum_{c \in C(f)} u_c \cdot \Omega_c + (1 - \sum_{c \in C(f)} u_c) \cdot \eta$ defines a connection Ω_f on the tangent bundle M which when restricted to $M(f) = M \backslash \cup_{c \in C(f)} \Phi^{-1}(B(2 \cdot r(c))$ is an $SO(m-1)$ connection.

Now the determinant of a skew symmetric matrix in odd dimensions is zero and thus the Pfaffian on $\mathfrak{g}$ (the Lie algebra of $m \times m$ skew symmetric matrices) restricted to $\mathfrak{g}'$ (= subalgebra of $(m-1) \times (m-1)$-matrices in $\mathfrak{g}$) is zero: hence the form Pf_{Ω_f} vanishes on $M(f)$. It follows that

$$\int_M Pf_{\Omega_f} = \sum_{c \in C(f)} \int_{B(2 \cdot r(c))} Pf_{\Omega_{r(c)}(q(c))}.$$

Let f' be a Morse function on M which equals f outside $E = f^{-1}[a,b]$ for some $a, b \in \mathbb{R}$ and having exactly two crtical points of index q and $q+1$ for some $0 \leq q < 2 \cdot m$: such a function f' exists (Proposition 3.7). Then we have:

$$\sum_{c \in C(f)} \int_{B(2 \cdot r(c))} Pf_{\Omega_{r(c)}(q(c))} = \int_M Pf_{\Omega_f} = \int_M Pf_{\Omega'_f} = \sum_{c \in C(f')} \int_{B(2 \cdot r(c))} Pf_{\Omega_{r(c)}(q(c))}$$

We see thus that

$$\int_{B(2 \cdot r)} Pf_{\Omega_r(q)} = - \int_{B(2 \cdot r)} Pf_{\Omega_r(q+1)}.$$

If we set $I(q) = \int_{B(2 \cdot r)} Pf_{\Omega_r(q)}$ we have then

$$\int_M \Omega_f = I(0) \cdot \sum_{c \in C(f)} (-1)^{q(c)} = I(0) \cdot \chi(M).$$

Thus to prove the theorem it suffices to show that $I(0) = (2 \cdot \pi)^n$ and this is easily checked by evaluating the integral for the product of n copies of $\mathbb{S}^2$ equipped with the Riemannian product connection, each factor $\mathbb{S}^2$ being given the standard (constant curvature) metric.

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