GENERAL ROTATIONAL SURFACES WITH POINTWISE 1-TYPE GAUSS MAP IN
PESEUDO-EUCLIDEAN SPACE $E^4_2$

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In this paper, we study general rotational surfaces in the 4-dimensional pseudo-Euclidean space $E^4_2$ and obtain a characterization of flat general rotation surfaces with pointwise 1-type Gauss map in $E^4_2$ and give an example of such surfaces.

Key words: Rotation surface, Gauss map, Pointwise 1-type Gauss map, pseudo-Euclidean space.

1. INTRODUCTION

A pseudo-Riemannian submanifold $M$ of the $m-$dimensional pseudo-Euclidean space $\mathbb{E}^m_\sigma$ is said to be of finite type if its position vector $x$ can be expressed as a finite sum of eigenvectors of the Laplacian $\Delta$ of $M$, that is, $x = x_0 + x_1 + ...x_k$, where $x_0$ is a constant map, $x_1, ..., x_k$ are non-constant maps such that $\Delta x_i = \lambda_i x_i, \lambda_i \in \mathbb{R}, i = 1, 2, ..., k$. If $\lambda_1, \lambda_2, ..., \lambda_k$ are all different, then $M$ is said to be of $k-$type. This definition was similarly extended to differentiable maps in Euclidean and pseudo-Euclidean space, in particular, to Gauss maps of submanifolds [6].

If a submanifold $M$ of a Euclidean space or pseudo-Euclidean space has 1-type Gauss map $G$, then $G$ satisfies $\Delta G = \lambda (G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector $C$. Chen and Piccinni made a general study on compact submanifolds of Euclidean spaces with finite type Gauss map and they proved that a compact hypersurface $M$ of $\mathbb{E}^{n+1}$ has 1-type Gauss map if and only if $M$ is a hypersphere in $\mathbb{E}^{n+1}$ [6].

However the Laplacian of the Gauss map of several surfaces and hypersurfaces such as a helicoids of the 1st, 2nd and 3rd kind, conjugate Enneper’s surface of the second kind in 3-dimensional
Minkowski space $E^3$, generalized catenoids, spherical $n$-cones, hyperbolical $n$-cones and Enneper’s hypersurfaces in $E^4_1$ take the form namely,

$$\Delta G = f(G + C)$$  \hspace{1cm} (1)

for some smooth function $f$ on $M$ and some constant vector $C$. A submanifold $M$ of a pseudo-Euclidean space $E^m_s$ is said to have pointwise 1-type Gauss map if its Gauss map satisfies (1) for some smooth function $f$ on $M$ and some constant vector $C$. A submanifold with pointwise 1-type Gauss map is said to be of the first kind if the vector $C$ in (1) is zero vector. Otherwise, the pointwise 1-type Gauss map is said to be of the second kind.

Surfaces in Euclidean space and in pseudo-Euclidean space with pointwise 1-type Gauss map were recently studied in [5, 7, 8, 9, 11-15, 17]. Also Dursun and Turgay in [10] gave all general rotational surfaces in $E^4$ with proper pointwise 1-type Gauss map of the first kind and classified minimal rotational surfaces with proper pointwise 1-type Gauss map of the second kind. Arslan et al. in [2] investigated rotational embedded surface with pointwise 1-type Gauss map. Arslan et al. in [3] gave necessary and sufficient conditions for Vranceanu rotation surface to have pointwise 1-type Gauss map. Yoon in [20] showed that flat Vranceanu rotation surface with pointwise 1-type Gauss map is a Clifford torus and in [19] studied rotation surfaces in the 4-dimensional Euclidean space with finite type Gauss map. Kim and Yoon in [16] obtained the complete classification theorems for the flat rotation surfaces with finite type Gauss map and pointwise 1-type Gauss map. The authors in [1] studied flat general rotational surfaces in the 4-dimensional Euclidean space $E^4$ with pointwise 1-type Gauss map and they showed that flat general rotational surfaces with pointwise 1-type Gauss map is a Lie group if and only if it is a Clifford Torus.

In this paper, we study general rotational surfaces in the 4-dimensional pseudo-Euclidean space $E^4_2$ and obtain a characterization for flat general rotation surfaces with pointwise 1-type Gauss map and give an example of such surfaces.

2. Preliminaries

Let $E^m_s$ be the $m$-dimensional pseudo-Euclidean space with signature $(s, m-s)$. Then the metric tensor $g$ in $E^m_s$ has the form

$$g = \sum_{i=1}^{m-s} (dx_i)^2 - \sum_{i=m-s+1}^{m} (dx_i)^2$$

where $(x_1, \ldots, x_m)$ is a standard rectangular coordinate system in $E^m_s$. 
Let $M$ be an $n$-dimensional pseudo-Riemannian submanifold of an $m$-dimensional pseudo-Euclidean space $\mathbb{E}_s^m$. We denote Levi-Civita connections of $\mathbb{E}_s^m$ and $M$ by $\nabla$ and $\nabla_t$, respectively. Let $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ be an adapted local orthonormal frame in $\mathbb{E}_s^m$ such that $e_1, \ldots, e_n$ are tangent to $M$ and $e_{n+1}, \ldots, e_m$ normal to $M$. We use the following convention on the ranges of indices: $1 \leq i, j, k, \ldots \leq n$, $n + 1 \leq r, s, t, \ldots \leq m$, $1 \leq A, B, C, \ldots \leq m$.

Let $\omega_A$ be the dual-1 form of $e_A$ defined by $\omega_A(X) = \langle e_A, X \rangle$ and $\varepsilon_A = \langle e_A, e_A \rangle = \pm 1$. Also, the connection forms $\omega_{AB}$ are defined by

$$d\varepsilon_A = \sum_B \varepsilon_B \omega_{AB} \varepsilon_B, \quad \omega_{AB} + \omega_{BA} = 0.$$ 

Then we have

$$\nabla_{e_k} e_i = \sum_{j=1}^n \varepsilon_{j\omega_{ij}} (e_k) e_j + \sum_{r=n+1}^m \varepsilon_r h^r_{ik} e_r$$

and

$$\nabla_{e_k} \varepsilon_i = -\sum_{j=1}^n \varepsilon_j h^j_{kj} e_j + \sum_{r=n+1}^m \varepsilon_r \omega_{sr} (e_k) e_r, \quad D^e_{ek} = \sum_{r=n+1}^m \varepsilon_r \omega_{sr} (e_k) e_r,$$

where $D$ is the normal connection, $h^r_{ik}$ the coefficients of the second fundamental form $h$.

If we define a covariant differentiation $\nabla h$ of the second fundamental form $h$ on the direct sum of the tangent bundle and the normal bundle $TM \oplus T^\bot M$ of $M$ by

$$\left( \nabla_X h \right)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

for any vector fields $X$, $Y$ and $Z$ tangent to $M$. Then we have the Codazzi equation

$$\left( \nabla_X h \right)(Y, Z) = \left( \nabla_Y h \right)(X, Z)$$

and the Gauss equation is given by

$$\langle R(X, Y)Z, W \rangle = \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle$$

where the vectors $X$, $Y$, $Z$ and $W$ are tangent to $M$ and $R$ is the curvature tensor associated with $\nabla$.

The curvature tensor $R$ associated with $\nabla$ is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$ 

For any real function $f$ on $M$ the Laplacian $\Delta f$ of $f$ is given by

$$\Delta f = -\varepsilon_i \sum_i \left( \nabla_{e_i} \nabla_{e_i} f - \nabla_{\nabla_{e_i} f} \right).$$

(6)
Let us now define the Gauss map \( G \) of a submanifold \( M \) into \( G(n, m) \) in \( \wedge^n \mathbb{E}_s^m \), where \( G(n, m) \) is the Grassmannian manifold consisting of all oriented \( n \)-planes through the origin of \( \mathbb{E}_s^m \) and \( \wedge^n \mathbb{E}_s^m \) is the vector space obtained by the exterior product of \( n \) vectors in \( \mathbb{E}_s^m \). Let \( e_{i_1} \wedge \ldots \wedge e_{i_n} \) and \( f_{j_1} \wedge \ldots \wedge f_{j_n} \) be two vectors of \( \wedge^n \mathbb{E}_s^m \), where \( \{e_1, \ldots, e_m\} \) and \( \{f_1, \ldots, f_m\} \) are orthonormal bases of \( \mathbb{E}_s^m \). Define an indefinite inner product \( \langle \cdot, \cdot \rangle \) on \( \wedge^n \mathbb{E}_s^m \) by

\[
\langle e_{i_1} \wedge \ldots \wedge e_{i_n}, f_{j_1} \wedge \ldots \wedge f_{j_n} \rangle = \det \left( \langle e_{i_k}, f_{j_k} \rangle \right).
\]

Therefore, for some positive integer \( t \), we may identify \( \wedge^n \mathbb{E}_s^m \) with some Euclidean space \( \mathbb{E}_t^N \) where \( N = \binom{m}{n} \). The map \( G: M \to G(n, m) \subset \mathbb{E}_t^N \) defined by \( G(p) = (e_1 \wedge \ldots \wedge e_n)(p) \) is called the Gauss map of \( M \), that is, a smooth map which carries a point \( p \) in \( M \) into the oriented \( n \)-plane in \( \mathbb{E}_s^m \) obtained from parallel translation of the tangent space of \( M \) at \( p \) in \( \mathbb{E}_s^m \).

3. Flat Rotation Surfaces with Pointwise 1-Type Gauss Map in \( E_2^4 \)

In this section, we study the flat rotation surfaces with pointwise 1-type Gauss map in the 4-dimensional pseudo-Euclidean space \( E_2^4 \). Let \( M_1 \) and \( M_2 \) be the rotation surfaces in \( E_2^4 \) defined by

\[
\varphi(t, s) = \begin{pmatrix}
\cosh t & 0 & 0 & \sinh t \\
0 & \cosh t & \sinh t & 0 \\
0 & \sinh t & \cosh t & 0 \\
\sinh t & 0 & 0 & \cosh t
\end{pmatrix} \begin{pmatrix}
0 \\
x(s) \\
y(s)
\end{pmatrix},
\]

\[
M_1: \varphi(t, s) = (y(s) \sinh t, x(s) \cosh t, x(s) \sinh t, y(s) \cosh t)
\]

and

\[
\varphi(t, s) = \begin{pmatrix}
\cos t & -\sin t & 0 & 0 \\
\sin t & \cos t & 0 & 0 \\
0 & 0 & \cos t & -\sin t \\
0 & 0 & \sin t & \cos t
\end{pmatrix} \begin{pmatrix}
x(s) \\
y(s)
\end{pmatrix},
\]

\[
M_2: \varphi(t, s) = (x(s) \cos t, x(s) \sin t, y(s) \cos t, y(s) \sin t)
\]

where the profile curve of \( M_1 \) (resp. the profile curve of \( M_2 \)) is unit speed curve, that is, \( (x'(s))^2 - (y'(s))^2 = 1 \). We choose a moving frame \( e_1, e_2, e_3, e_4 \) such that \( e_1, e_2 \) are tangent to \( M_1 \) and \( e_3, e_4 \) are normal to \( M_1 \) and choose a moving frame \( \bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4 \) such that \( \bar{e}_1, \bar{e}_2 \) are tangent to \( M_2 \) and
\( \mathbf{e}_3, \mathbf{e}_4 \) are normal to \( M_2 \) which are given by the following:

\[
\begin{align*}
\mathbf{e}_1 &= \frac{1}{\sqrt{\varepsilon_1 (y^2(s) - x^2(s))}} (y(s) \cosh t, x(s) \sinh t, x(s) \cosh t, y(s) \sinh t) \\
\mathbf{e}_2 &= (y'(s) \sinh t, x'(s) \cosh t, x'(s) \sinh t, y'(s) \cosh t) \\
\mathbf{e}_3 &= (x'(s) \sinh t, y'(s) \cosh t, y'(s) \sinh t, x'(s) \cosh t) \\
\mathbf{e}_4 &= \frac{1}{\sqrt{\varepsilon_1 (y^2(s) - x^2(s))}} (x(s) \cosh t, y(s) \sinh t, y(s) \cosh t, x(s) \sinh t)
\end{align*}
\]

and

\[
\begin{align*}
\bar{\mathbf{e}}_1 &= \frac{1}{\sqrt{\varepsilon_1 (y^2(s) - x^2(s))}} (-x(s) \sin t, x(s) \cos t, -y(s) \sin t, y(s) \cos t) \\
\bar{\mathbf{e}}_2 &= (x'(s) \cos t, x'(s) \sin t, y'(s) \cos t, y'(s) \sin t) \\
\bar{\mathbf{e}}_3 &= (y'(s) \cos t, y'(s) \sin t, x'(s) \cos t, x'(s) \sin t) \\
\bar{\mathbf{e}}_4 &= \frac{1}{\sqrt{\varepsilon_1 (y^2(s) - x^2(s))}} (y(s) \sin t, -y(s) \cos t, x(s) \sin t, -x(s) \cos t)
\end{align*}
\]

where \( \varepsilon_1 (y^2(s) - x^2(s)) > 0, \varepsilon_1 = \pm 1 \). Then it is easily seen that

\[
\langle \mathbf{e}_1, \mathbf{e}_1 \rangle = -\langle \mathbf{e}_4, \mathbf{e}_4 \rangle = \varepsilon_1, \quad \langle \mathbf{e}_2, \mathbf{e}_2 \rangle = -\langle \mathbf{e}_3, \mathbf{e}_3 \rangle = 1
\]

\[
-\langle \bar{\mathbf{e}}_1, \bar{\mathbf{e}}_1 \rangle = \langle \bar{\mathbf{e}}_4, \bar{\mathbf{e}}_4 \rangle = \varepsilon_1, \quad \langle \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_2 \rangle = -\langle \bar{\mathbf{e}}_3, \bar{\mathbf{e}}_3 \rangle = 1
\]

we have the dual 1-forms as:

\[
\omega_1 = \varepsilon_1 \sqrt{\varepsilon_1 (y^2(s) - x^2(s))} dt \quad \text{and} \quad \omega_2 = ds \tag{9}
\]

and

\[
\bar{\omega}_1 = -\varepsilon_1 \sqrt{\varepsilon_1 (y^2(s) - x^2(s))} dt \quad \text{and} \quad \bar{\omega}_2 = ds. \tag{10}
\]

By a direct computation we have components of the second fundamental form and the connection forms as:

\[
\begin{align*}
h^3_{11} &= b(s), \quad h^3_{12} = 0, \quad h^3_{22} = c(s) \\
h^4_{11} &= 0, \quad h^4_{12} = b(s), \quad h^4_{22} = 0 \\
\bar{h}^3_{11} &= -b(s), \quad \bar{h}^3_{12} = 0, \quad \bar{h}^3_{22} = c(s) \\
\bar{h}^4_{11} &= 0, \quad \bar{h}^4_{12} = b(s), \quad \bar{h}^4_{22} = 0
\end{align*} \tag{11}
\]

\[
\begin{align*}
h^3_{11} &= b(s), \quad h^3_{12} = 0, \quad h^3_{22} = c(s) \\
h^4_{11} &= 0, \quad h^4_{12} = b(s), \quad h^4_{22} = 0 \\
\bar{h}^3_{11} &= -b(s), \quad \bar{h}^3_{12} = 0, \quad \bar{h}^3_{22} = c(s) \\
\bar{h}^4_{11} &= 0, \quad \bar{h}^4_{12} = b(s), \quad \bar{h}^4_{22} = 0
\end{align*} \tag{12}
\]
\[
\omega_{12} = \varepsilon_1 a(s) \omega_1, \quad \omega_{13} = \varepsilon_1 b(s) \omega_1, \quad \omega_{14} = b(s) \omega_2 \\
\omega_{23} = c(s) \omega_2, \quad \omega_{24} = \varepsilon_1 b(s) \omega_1, \quad \omega_{34} = \varepsilon_1 a(s) \omega_1
\]

\[
\bar{\omega}_{12} = \varepsilon_1 a(s) \bar{\omega}_1, \quad \bar{\omega}_{13} = \varepsilon_1 b(s) \bar{\omega}_1, \quad \bar{\omega}_{14} = b(s) \bar{\omega}_2 \\
\bar{\omega}_{23} = c(s) \bar{\omega}_2, \quad \bar{\omega}_{24} = -\varepsilon_1 b(s) \bar{\omega}_1, \quad \bar{\omega}_{34} = -\varepsilon_1 a(s) \bar{\omega}_1.
\]

By covariant differentiation with respect to \(e_1\) and \(e_2\) (resp. \(\bar{e}_1\) and \(\bar{e}_2\)) a straightforward calculation gives:

\[
\tilde{\nabla}_{e_1} e_1 = a(s) e_2 - b(s) e_3 \\
\tilde{\nabla}_{e_2} e_1 = -\varepsilon_1 b(s) e_4 \\
\tilde{\nabla}_{e_1} e_2 = -\varepsilon_1 a(s) e_1 - \varepsilon_1 b(s) e_4 \\
\tilde{\nabla}_{e_2} e_2 = -c(s) e_3 \\
\tilde{\nabla}_{e_1} e_3 = -\varepsilon_1 b(s) e_1 - \varepsilon_1 a(s) e_4 \\
\tilde{\nabla}_{e_2} e_3 = -c(s) e_2 \\
\tilde{\nabla}_{e_1} e_4 = -b(s) e_2 + a(s) e_3 \\
\tilde{\nabla}_{e_2} e_4 = -\varepsilon_1 b(s) e_1
\]

and

\[
\tilde{\nabla}_{\bar{e}_1} \bar{e}_1 = -a(s) \bar{e}_2 + b(s) \bar{e}_3 \\
\tilde{\nabla}_{\bar{e}_2} \bar{e}_1 = \varepsilon_1 b(s) \bar{e}_4 \\
\tilde{\nabla}_{\bar{e}_1} \bar{e}_2 = -\varepsilon_1 a(s) \bar{e}_1 + \varepsilon_1 b(s) \bar{e}_4 \\
\tilde{\nabla}_{\bar{e}_2} \bar{e}_2 = -c(s) \bar{e}_3 \\
\tilde{\nabla}_{\bar{e}_1} \bar{e}_3 = -\varepsilon_1 b(s) \bar{e}_1 + \varepsilon_1 a(s) \bar{e}_4 \\
\tilde{\nabla}_{\bar{e}_2} \bar{e}_3 = -c(s) \bar{e}_2 \\
\tilde{\nabla}_{\bar{e}_1} \bar{e}_4 = -b(s) \bar{e}_2 + a(s) \bar{e}_3 \\
\tilde{\nabla}_{\bar{e}_2} \bar{e}_4 = \varepsilon_1 b(s) \bar{e}_1
\]

where

\[
a(s) = \frac{x(s)x'(s) - y(s)y'(s)}{\varepsilon_1 (y^2(s) - x^2(s))} \\
b(s) = \frac{x(s)y'(s) - x'(s)y(s)}{\varepsilon_1 (y^2(s) - x^2(s))}
\]
\[ c(s) = x''(s)y'(s) - x'(s)y''(s) \] (19)

The Gaussian curvature \( K \) of \( M_1 \) and \( \tilde{K} \) that of \( M_2 \) are respectively given by
\[ K = \varepsilon_1 b^2(s) - b(s)c(s) \] (20)
and
\[ \tilde{K} = b(s)c(s) - \varepsilon_1 b^2(s) \] (21)

If the surfaces \( M_1 \) or \( M_2 \) is flat, then (20) and (21) imply
\[ b(s)c(s) - \varepsilon_1 b^2(s) = 0. \] (22)

Furthermore, after some computations we obtain Gauss and Codazzi equations for both surfaces \( M_1 \) and \( M_2 \)
\[ \varepsilon_1 a^2(s) - a'(s) = b(s)c(s) - \varepsilon_1 b^2(s) \] (23)
and
\[ b'(s) = 2\varepsilon_1 a(s)b(s) - a(s)c(s) \] (24)
respectively.

By using (6), (15), (16) and straight-forward computations, the Laplacians \( \Delta G \) and \( \Delta \tilde{G} \) of the Gauss map \( G \) and \( \tilde{G} \) can be expressed as
\[ \Delta G = - (3b^2(s) + c^2(s)) \left( e_1 \wedge e_2 \right) + (2a(s)b(s) - \varepsilon_1 a(s)c(s) + c'(s)) \left( e_2 \wedge e_4 \right) + 2 \left( \varepsilon_1 b(s)c(s) - b^2(s) \right) \left( e_3 \wedge e_4 \right) \] (25)
\[ \Delta \tilde{G} = - (3b^2(s) + c^2(s)) \left( \tilde{e}_1 \wedge \tilde{e}_2 \right) + (2a(s)b(s) - \varepsilon_1 a(s)c(s) + c'(s)) \left( \tilde{e}_2 \wedge \tilde{e}_4 \right) + 2 \left( \varepsilon_1 b(s)c(s) - b^2(s) \right) \left( \tilde{e}_3 \wedge \tilde{e}_4 \right). \] (26)

Now we investigate the flat rotation surfaces in \( E^4 \) with the pointwise 1-type Gauss map satisfying (1).

Suppose that the rotation surface \( M_1 \) given by the parametrization (7) is a flat rotation surface. From (20), we obtain that \( b(s) = 0 \) or \( \varepsilon_1 b(s) - c(s) = 0 \). We assume that \( \varepsilon_1 b(s) - c(s) \neq 0 \). Then \( b(s) \) is equal to zero and (24) implies that \( a(s)c(s) = 0 \). Since \( \varepsilon_1 b(s) - c(s) \neq 0 \), it implies that \( c(s) \) is not equal to zero. Then we obtain as \( a(s) = 0 \). In that case, by using (17) and (18) we obtain that
\( \alpha(s) = (0, x(s), 0, y(s)) \) is a constant vector. This is a contradiction. Therefore \( \varepsilon_1 b(s) = c(s) \) for all \( s \). From (23), we get
\[
\varepsilon_1 a^2(s) - a'(s) = 0
\]
whose the trivial solution and non-trivial solution
\[
a(s) = 0
\]
and
\[
a(s) = \frac{1}{-\varepsilon_1 s + c},
\]
respectively. We assume that \( a(s) = 0 \). By (24) \( b = b_0 \) is a constant and \( c = \varepsilon_1 b_0 \). In that case by using (17), (18) and (19), \( x \) and \( y \) satisfy the following differential equations
\[
x^2(s) - y^2(s) = \mu \quad \mu \text{ is a constant},
\]
\[
x(s)y'(s) - x'(s)y(s) = -\varepsilon_1 b_0 \mu,
\]
\[
x''(s)y'(s) - x'(s)y''(s) = \varepsilon_1 b_0.
\]
From (28) we may put
\[
x(s) = \frac{1}{2} \varepsilon \left( \mu_2 e^{\theta(s)} + \mu_1 e^{-\theta(s)} \right), \quad y(s) = \frac{1}{2} \varepsilon \left( \mu_2 e^{\theta(s)} - \mu_1 e^{-\theta(s)} \right),
\]
where \( \theta(s) \) is some smooth function, \( \varepsilon = \pm 1 \) and \( \mu = \mu_1 \mu_2 \). Differentiating (31) with respect to \( s \), we have
\[
x'(s) = \theta'(s)y(s), \quad y'(s) = \theta'(s)x(s).
\]
By substituting (31) and (32) into (29), we get
\[
\theta(s) = -\varepsilon_1 b_0 s + d, \quad d = \text{const}.
\]
And since the curve \( \alpha \) is a unit speed curve, we have
\[
b_0^2 \mu = -1.
\]
Since \( \mu = -\frac{1}{b_0^2} \), \( y^2(s) - x^2(s) > 0 \). In that case we obtain that \( \varepsilon_1 = 1 \). Then we can write components of the curve \( \alpha \) as:
\[
x(s) = \frac{1}{2} \varepsilon \left( \mu_2 e^{(-b_0 s + d)} + \mu_1 e^{(-b_0 s + d)} \right),
\]
\[
y(s) = \frac{1}{2} \varepsilon \left( \mu_2 e^{(-b_0 s + d)} - \mu_1 e^{(-b_0 s + d)} \right), \quad \mu_1 \mu_2 = -\frac{1}{b_0^2}.
\]
On the other hand, by using (25) we can rewrite the Laplacian of the Gauss map $G$ with $a(s) = 0$ and $b = c = b_0$ as follows:

$$
\Delta G = -4b_0^2 \left( e_1 \wedge e_2 \right)
$$

that is, the flat surface $M$ is pointwise 1-type Gauss map with the function $f = -4b_0^2$ and $C = 0$. Even if it is a pointwise 1-type Gauss map of the first kind.

Now we assume that $a(s) = \frac{1}{-\varepsilon_1 s + c}$. By using $c(s) = \varepsilon_1 b(s)$ and (24) we get

$$
b'(s) = \varepsilon_1 a(s)b(s) \quad (34)
$$

or we can write

$$
\frac{b'(s)}{b(s)} = \frac{\varepsilon_1}{-\varepsilon_1 s + c},
$$

whose the solution

$$
b(s) = \frac{\lambda}{-\varepsilon_1 s + c}, \quad \lambda \text{ is a constant.} \quad (35)
$$

By using (25) we can rewrite the Laplacian of the Gauss map $G$ with the equations $c(s) = \varepsilon_1 b(s)$, $b'(s) = \varepsilon_1 a(s)b(s)$ and $a'(s) = \varepsilon_1 a^2(s)$

$$
\Delta G = -4b^2(s) \left( e_1 \wedge e_2 \right) + 2a(s)b(s) \left( e_1 \wedge e_3 \right) + 2a(s)b(s) \left( e_2 \wedge e_4 \right). \quad (36)
$$

We suppose that the flat rotational surface $M_1$ has pointwise 1-type Gauss map. From (1) and (36), we get

$$
-4\varepsilon_1 b^2(s) = f \varepsilon_1 + f \left< C, e_1 \wedge e_2 \right> \quad (37)
$$

$$
-2\varepsilon_1 a(s)b(s) = f \left< C, e_1 \wedge e_3 \right> \quad (38)
$$

$$
-2\varepsilon_1 a(s)b(s) = f \left< C, e_2 \wedge e_4 \right> \quad (39)
$$

Then, we have

$$
\left< C, e_1 \wedge e_4 \right> = 0, \quad \left< C, e_2 \wedge e_3 \right> = 0, \quad \left< C, e_3 \wedge e_4 \right> = 0 \quad (40)
$$

By using (38) and (39) we obtain

$$
\left< C, e_1 \wedge e_3 \right> = \left< C, e_2 \wedge e_4 \right> \quad (41)
$$

By differentiating the first equation in (40) with respect to $e_1$ and by using the third equation in (40) and (41), we get

$$
2a(s) \left< C, e_1 \wedge e_3 \right> - b(s) \left< C, e_1 \wedge e_2 \right> = 0 \quad (42)
$$
Combining (37), (38) and (42) we then have
\[ f = 4 \left( a^2(s) - b^2(s) \right) \]
that is, a smooth function \( f \) depends only on \( s \). By differentiating \( f \) with respect to \( s \) and by using (34) and (27), we get
\[ f' = 2 \varepsilon_1 a(s)f \]
(43)

By differentiating (38) with respect to \( s \) and by using (15), (27), (34), (37) and (38) we have
\[ a^2b = 0 \]
or from (35) we can write
\[ \lambda a^3 = 0 \]

Since \( a(s) \neq 0 \), it follows that \( \lambda = 0 \). Then we obtain that \( b = c = 0 \). Then the surface \( M_1 \) is a totally geodesic.

Thus we can give the following theorems.

**Theorem 1** — Let \( M_1 \) be the flat rotation surface given by the parametrization (7). Then \( M_1 \) has pointwise 1-type Gauss map if and only if \( M_1 \) is either totally geodesic or parametrized by
\[
\varphi(t,s) = \left( \begin{array}{c}
\frac{1}{2} \varepsilon \left( \mu_2 e^{-(b_0 s+d)} - \mu_1 e^{-(b_0 s+d)} \right) \sinh t,
\frac{1}{2} \varepsilon \left( \mu_2 e^{-(b_0 s+d)} + \mu_1 e^{-(b_0 s+d)} \right) \cosh t,
\frac{1}{2} \varepsilon \left( \mu_2 e^{-(b_0 s+d)} + \mu_1 e^{-(b_0 s+d)} \right) \sinh t,
\frac{1}{2} \varepsilon \left( \mu_2 e^{-(b_0 s+d)} - \mu_1 e^{-(b_0 s+d)} \right) \cosh t,
\end{array} \right), \quad \mu_1 \mu_2 = -\frac{1}{b_0^2}, \quad (44)
\]
where \( b_0, \mu_1, \mu_2 \) and \( d \) are real constants.

**Example 1** — Let \( M_1 \) be the flat rotation surface with pointwise 1-type Gauss map given by the parametrization (44). If we take as \( b_0 = -1, \mu_1 = -1, \mu_2 = 1, \) \( d = 0 \) and \( \varepsilon = 1 \), then we obtain a surface as follows:
\[
\varphi(t,s) = (\cosh s \sinh t, \sinh s \cosh t, \sinh s \sinh t, \cosh s \cosh t).
\]
This surface is the product of two plane hyperbolas.

**Theorem 2** — Let \( M_2 \) be the flat rotation surface given by the parametrization (8). Then \( M_2 \) has pointwise 1-type Gauss map if and only if \( M_2 \) is either totally geodesic or parametrized by
\[
\varphi(t,s) = \left( \begin{array}{c}
\frac{1}{2} \varepsilon \left( \mu_2 e^{-(b_0 s+d)} + \mu_1 e^{-(b_0 s+d)} \right) \cos t,
\frac{1}{2} \varepsilon \left( \mu_2 e^{-(b_0 s+d)} + \mu_1 e^{-(b_0 s+d)} \right) \sin t,
\frac{1}{2} \varepsilon \left( \mu_2 e^{-(b_0 s+d)} - \mu_1 e^{-(b_0 s+d)} \right) \cos t,
\frac{1}{2} \varepsilon \left( \mu_2 e^{-(b_0 s+d)} - \mu_1 e^{-(b_0 s+d)} \right) \sin t,
\end{array} \right), \quad \mu_1 \mu_2 = -\frac{1}{b_0^2}, \quad (45)
\]
Example 2: Let $M_2$ be the flat rotation surface with pointwise 1-type Gauss map given by the parametrization (45). If we take as $b_0 = -1, \mu_1 = -1, \mu_2 = 1, d = 0$ and $\varepsilon = 1$, then we obtain a surface as follows:

$$\varphi(t, s) = (\sinh s \cos t, \sinh s \sin t, \cosh s \cos t, \cosh s \sin t).$$

This surface is the product of a plane circle and a plane hyperbola.

Corollary 1 — Let $M$ be non-totally geodesic flat general rotation surface given by the parametrization (7) or (8). If $M$ has pointwise 1-type Gauss map then the Gauss map $G$ on $M$ is of 1-type.

REFERENCES


