THE LIOUVILLE’S THEOREM OF HARMONIC FUNCTIONS ON ALEXANDROV SPACES WITH NONNEGATIVE RICCI CURVATURE

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(Received 15 October 2012; after final revision 24 October 2013; accepted 4 February 2014)

In this paper, the authors prove the Liouville’s theorem for harmonic function on Alexandrov spaces by heat kernel approach, which extends the Liouville’s theorem of harmonic function from Riemannian manifolds to Alexandrov spaces.

Key words : Harmonic function; Alexandrov space; Liouville’s theorem; heat kernel; Ricci curvature.

1. INTRODUCTION

In this paper we discuss the Liouville properties of harmonic functions on Alexandrov spaces with nonnegative Ricci curvature in the sense of Zhang and Zhu [14, 15]. Recall that the classical Liouville’s theorem states that any bounded (or even just positive) harmonic function on Euclidean spaces is constant. Around 1974, Yau [13] extended the classical Liouville’s theorem to complete noncompact manifolds with nonnegative Ricci curvature. Later, by means of gradient estimate, Cheng and

1Supported by the National Natural Science Foundation of China (Nos. 11171083, 11171084) and the Natural Science Foundation of Zhejiang Province (No. LY13A010018).
Yau [4] proved that any harmonic function with sublinear growth on an open manifold with nonnegative Ricci curvature must be constant.

Recently, Zhang and Zhu [14, 15] introduced a new definition for lower Ricci curvature bounds on Alexandrov spaces. Hence we can generalize the notion of nonnegative Ricci curvature from Riemannian manifolds setting to Alexandrov spaces. As we know, Alexandrov spaces extends the research objects of differential geometry from Riemannian manifolds to singular spaces. In the last few years, plenty of fruitful results on Alexandrov spaces were obtained. Sturm [11] and Lott and Villani [8], independently, introduced curvature-dimension condition \( CD(k, n) \) on metric measure spaces. Kuwae and Shioya [6,7] introduced an infinitesimal version of the Bishop-Gromov comparison condition, denoted by \( BG(k, n) \). The condition \( CD(k, n) \) or \( BG(k, n) \) implies Poincaré inequality and Bishop-Gromov volume comparison and hence doubling property. And for \( n \)-dimensional Riemannian manifolds, it holds

\[
Ric \geq k \text{ (in the classical sense)} \iff CD(k, n) \iff BG(k, n).
\]

By notion of \textquote{\“Ricci curvature has a lower bound \( k \)\”} introduced by Zhang and Zhu, on an \( n \)-dimensional Alexandrov space \( X \), the condition \( Ric \geq k \) implies that \( X \) satisfies conditions \( CD(k, n) \) and \( BG(k, n) \), i.e., for \( n \)-dimensional Alexandrov spaces,

\[
Ric \geq k \text{ (in the sense of Zhang-Zhu)} \Rightarrow CD(k, n) \Rightarrow BG(k, n).
\]

Then a natural question is: for an \( n \)-dimensional Alexandrov space \( X \) satisfies condition \( Ric \geq 0 \) (hence \( CD(0, n) \) and \( BG(0, n) \)), whether the Liouville’s theorem works or not? In the present paper, we will answer this question. Particularly, we show the following result.

\textbf{Theorem 1 (Liouville’s Theorem)} — Let \( X \) be an \( n \)-dimensional complete noncompact Alexandrov space with nonnegative Ricci curvature. Then any bounded harmonic function on \( X \) must be a constant.

More recently, Zhang and Zhu [15] obtained the gradient estimate of positive harmonic functions on Alexandrov spaces. With their result of gradient estimate, one can show the strong Liouville theorem, which states that any positive harmonic function on Alexandrov space with nonnegative Ricci curvature must be a constant. And more generally, as an application of gradient estimate, Zhang and Zhu [15] proved that any sublinear growth harmonic function on Alexandrov space with nonnegative Ricci curvature must be a constant. In particular, any bounded harmonic function must be a constant. Therefore, Theorem 1 is a direct consequence of the result of Zhang and Zhu. We will give an another proof of Theorem 1 by heat kernel method in Section 3.
2. Preliminaries

In this section we shall recall some notions and preliminary results on harmonic functions and Alexandrov spaces. Let $X$ be an $n$-dimensional Alexandrov space and $\text{vol}$ denote the $n$-dimensional Hausdorff measure on $X$. We denote the metric measure space by $(X, | \cdot |, \text{vol})$. Let $\mathbb{M}^n_k$ be the $n$-dimensional complete simply connected space form of constant sectional curvature $k$. We call $\mathbb{M}^2_k$ the $k$-plane. Let $x_0 \in X$, $\text{vol}(B(x_0, r))$ denote the Hausdorff measure of concentric geodesic balls $B(x_0, r) \subset X$. In the sequel, the notations $B(x_0, r)$ and $B_r$ denote geodesic balls in $X$ with radius of $r$.

(i) **Alexandrov space**: A metric space $(X, | \cdot |)$ is called a *length space* if and only if for any two points $x, y \in X$, the distance between $x$ and $y$ is given by

$$|xy| = \inf_{\gamma_{xy}} \text{Length}(\gamma_{xy}),$$

where the infimum is taken over all curves $\gamma_{xy}$ in $X$ which connect $x$ and $y$. A length space $X$ is called a *geodesic space* if and only if, for each pair of points $x, y \in X$, the distance $|xy|$ is realized as the length of a rectifiable curve connecting $x$ and $y$. Such distance-realizing curve, parameterized by arc-length, is called minimal geodesic. (This geodesic is not required to be unique).

Given any $k \in \mathbb{R}$ we say that a length space $X$ *locally has curvature $\geq k$* if and only if each point $p \in X$ has a neighborhood $U(p) \subset X$ such that for each quadruple of points $\{z, x_1, x_2, x_3\} \subset U(p)$,

$$\tilde{Z}_k x_1 x_2 + \tilde{Z}_k x_2 x_3 + \tilde{Z}_k x_3 x_1 \leq 2\pi$$

(2.1)

where $\tilde{Z}_k x_1 x_2, \tilde{Z}_k x_2 x_3$ and $\tilde{Z}_k x_3 x_1$ are the comparison angles in the $k$-plane. That is, $\tilde{Z}_k x_1 x_2$ is the angle at $\tilde{z}$ of a triangle $\triangle \tilde{x} \tilde{z} \tilde{y}$ with side lengths $|\tilde{x}\tilde{z}| = |xz|, |\tilde{z}\tilde{y}| = |zy|$ and $|\tilde{z}\tilde{y}| = |xy|$ in the $k$-plane. We say that a length space $X$ *globally has curvature $\geq k$* if the inequality (2.1) holds for any quadruple of points $\{z, x_1, x_2, x_3\} \subset X$.

**Proposition 2.1** — We note that, for a complete length space, $X$ which locally has curvature $\geq k$ is equivalent to $X$ which globally has curvature $\geq k$. For more details we refer the reader to [2, 11].

**Definition 2.2** — A complete length space $X$ locally (or globally) has curvature $\geq k$ is called to be an *Alexandrov space with curvature $\geq k$* (for short, we say $X$ to be an *Alexandrov space*), if it is locally compact.

**Remark 2.3** : The basic example of Alexandrov space with curvature $\geq k$ is Riemannian manifold without boundary or with locally convex boundary, whose sectional curvature is not less than $k$. 

Remark 2.4: Since a complete and locally compact length space is a geodesic space (refer to [11]), we know that an Alexandrov space is, of course, a geodesic space.

In the last several years, the notion of curvature on Alexandrov spaces have been generalized. Sturm [11] and Lott and Villani [8], independently, introduced curvature-dimension condition $CD(k,n)$ on metric measure spaces. Kuwae and Shioya [6,7] raise the condition of $BG(k,n)$. Based on these two conditions, Zhang and Zhu [14] defined the lower Ricci curvature bounds on Alexandrov spaces. It means that, in some sense, geometers had extended the curvature condition on Alexandrov spaces from sectional curvature to Ricci curvature. Refer to [2, 6-8, 11, 14] for further details.

(ii) Harmonic functions: Here we mean harmonic functions on Alexandrov spaces in the sense of Petrunin [9]. Let $X$ be an $n$-dimensional Alexandrov space without boundary and $\Omega$ be a bounded open domain in $X$. Given a point $x \in X$, we denote by $T_x X$ the tangent cone at $x$. We call elements of the tangent cone $T_x X$ the tangent vectors at $x$. The origin $o = oz$ of $T_x X$ plays the role of a zero vector. A point $x$ in an $n$-dimensional Alexandrov space $X$ is called to be regular if its tangent cone $T_x X$ is isometric to Euclidean space $\mathbb{R}^n$ with standard metric. We denote by $\sum_x \subset T_x X$ the set of unit vectors in $T_x X$. For any function $f : \Omega \to \mathbb{R}$, Petrunin [10] defined, $d_x f : T_x \to \mathbb{R}$, the differential of $f$. Then the gradient, $\nabla_x f$, of $f$ is defined as $d_x f (\xi_x) \cdot \xi_x$, where $\xi_x \in \sum_x$ is the maximal value point of $d_x f : \sum_x \to \mathbb{R}$.

Let $x \in \Omega$ be a regular point, we say that a function $f$ is differentiable at $x$, if there exists a vector in $T_x X$, denoted by $\nabla f(x)$, such that for all geodesic $\gamma(t) : [0, \epsilon) \to \Omega$ with $\gamma(0) = x$ we have

$$f(\gamma(t)) = f(x) + t \cdot \langle \nabla f(x), \gamma'(0) \rangle + o(t).$$

According to Rademacher theorem, which was proved by Cheeger [3] in the framework of general metric measure spaces with a doubling measure and a Poincaré inequality for uppergradients and was proved by Bertrand [1] in Alexandrov space, a locally Lipschitz function $f$ is differentiable almost everywhere in $X$. Hence the vector $\nabla f(x)$ is well defined almost everywhere in $X$ for a locally Lipschitz function $f$. In [15], Zhang and Zhu showed that $|\nabla f(x)| = |\nabla_x f|$ if $f$ is differentiable at $x$.

We denote by $Lip_{loc}(\Omega)$ the set of locally Lipschitz continuous functions on $\Omega$ and by $Lip_0(\Omega)$ the set of Lipschitz continuous functions on $\Omega$ with compact support in $\Omega$. The closer of set of locally Lipschitz continuous functions $Lip_{loc}(\Omega)$ in norm

$$\|u\|_{W^{1,2}(\Omega)} := \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}.$$

is called Sobolev space $W^{1,2}(\Omega)$, where $\|u\|_{L^2(\Omega)} = \int_{\Omega} u^2$ and $\|\nabla u\|_{L^2(\Omega)} = \int_{\Omega} |\nabla u|^2$. We denote
the closure of \( \text{Lip}_0(\Omega) \) by \( W^{1,2}_0(\Omega) \). As we know, a canonical Dirichlet form \( \mathcal{E} \) is defined by

\[
\mathcal{E}(u, v) := \int_\Omega \langle \nabla u, \nabla v \rangle, \quad \text{for} \quad u, v \in W^{1,2}(\Omega). 
\]

For a Lipschitz function \( u \in W^{1,2}(\Omega) \) when \( \Omega \subset X \), the Laplacian of \( u \) as a signed-Radon measure is defined by

\[
\int_\Omega \phi \Delta u = -\int_\Omega \langle \nabla \phi, \nabla u \rangle = -\mathcal{E}(u, \phi) \quad (2.2)
\]

for all Lipschitz function \( \phi \) with compact support in \( \Omega \).

**Definition 2.5** — Let \( u \in W^{1,2}(\Omega) \), we say that \( u \) is a harmonic (subharmonic, superharmonic) function if for any positive function \( \phi \in W^{1,2}_0(\Omega) \), it holds that

\[
\mathcal{E}(u, \phi) = \int_\Omega \langle \nabla u, \nabla \phi \rangle = 0 (\leq 0, \geq 0). \quad (2.3)
\]

**Remark 2.6** : According to this definition, \( u \) is a harmonic (subharmonic, superharmonic) function means \( \int_\Omega \phi \Delta u = 0 (\geq 0, \leq 0) \). It can be understood in the sense of weak Laplacian. Then this definition of harmonic (subharmonic, superharmonic) function coincides with the classical one.

3. PROOF OF THEOREM 1

In the sequel, we will give the proof of Theorem 1 by two lemmas. We rely heavily on the heat kernel method investigated by Kuwae, Machigashira and Shioya in [5].

**Lemma 3.1** — Let \( X \) be an \( n \)-dimensional complete noncompact Alexandrov space with nonnegative Ricci curvature. If \( u \) is a bounded superharmonic function on \( X \), then

\[
\lim_{r \to \infty} \frac{1}{\text{vol}(B(p, r))} \int_{B(p, r)} u(x) dx = \inf_X u. \quad (3.1)
\]

**Proof** : We define the function \( v_0(x) = u(x) - \inf_X u \), then \( v_0(x) \geq 0 \) and for any positive function \( \phi \in W^{1,2}_0(X) \), it holds that

\[
\int_X \phi \Delta v_0 = -\int_X \langle \nabla \phi, \nabla v_0 \rangle = -\mathcal{E}(v_0, \phi) \leq 0.
\]

We only need to show that

\[
\lim_{r \to \infty} \frac{1}{\text{vol}(B(p, r))} \int_{B(p, r)} v_0(x) dx = 0.
\]

We denote \( P(x, y, t) \) to be the heat kernel on \( X \) and set \( v(x, t) = \int_X P(x, y, t)v_0(y)dy \). Let us solve the following heat equation

\[
\left( \Delta - \frac{\partial}{\partial t} \right) v(x, t) = 0
\]
in the sense of
\[
\int \left\{ \mathcal{E}(v(x, t), \phi) + \left( \frac{\partial}{\partial t} v(x, t), \phi \right) \right\} dt = 0
\]
with initial condition \(v(x, 0) = v_0(x)\). A direct computation shows
\[
\frac{\partial}{\partial t} v(x, t) = \frac{\partial}{\partial t} \int_X P(x, y, t)v_0(y)dy
\]
\[
= \int_X P(x, y, t)\Delta v_0(y)dy
\]
\[
\leq 0,
\]
we know the function \(v(x, t)\) is monotonically nonincreasing with respect to \(t\). The maximum principle implies that \(v(x, t) \geq 0\), \(v(x, t)\) must converge uniformly to a nonnegative harmonic function on \(X\), and therefore it is a constant. By definition we get
\[
\inf_X v_0(x) = \inf_X \{u(x) - \inf_X u(x)\} = 0,
\]
then the limit must be identically 0. Now we get
\[
v(x, t) \to 0, \; (\text{as } t \to \infty \text{ for any } x \in X).
\]
Furthermore, by the definition of the function \(v(x, t)\), we have
\[
\int_{B(x, \sqrt{t})} P(x, y, t)v_0(y)dy \leq \int_X P(x, y, t)v_0(y)dy
\]
\[
= v(x, t).
\]  \hspace{1cm} (3.2)

Applying results on heat kernel obtained by [5] and [12], we have the following lower bounds estimate for heat kernel
\[
\frac{C}{\text{vol}(B(x, \sqrt{t}))} \leq P(x, y, t).
\]  \hspace{1cm} (3.3)

A combination of (3.2) and (3.3) derives that
\[
\frac{C}{\text{vol}(B(x, \sqrt{t}))} \int_{B(x, \sqrt{t})} v_0(y)dy \leq v(x, t).
\]  \hspace{1cm} (3.4)

Since \(v(x, t) \to 0\) (as \(t \to \infty\)), taking the limit \(t \to \infty\) in inequality (3.4) we complete the proof.

We can obtain the following result by slightly modifying the arguments in the proof of Lemma 3.1.
Lemma 3.2 — Let $X$ be an $n$-dimensional complete noncompact Alexandrov space with nonnegative Ricci curvature. If $u$ is a bounded subharmonic function on $X$, then
\[
\lim_{r \to \infty} \frac{1}{\text{vol}(B(p, r))} \int_{B(p, r)} u(x) \, dx = \sup_X u.
\]

PROOF: We only need to change the auxiliary function $v_0(x) = u(x) - \inf_X u$ in the proof of Lemma 3.1 into $v_0(x) = \sup_X u - u(x)$. Then one can complete the proof of Lemma 3.2 just by repeating the procedures in the proof of Lemma 3.1.

Combining Lemma 3.1 and Lemma 3.2, we show that
\[
\inf_X u = \lim_{r \to \infty} \frac{1}{\text{vol}(B(p, r))} \int_{B(p, r)} u(x) \, dx = \sup_X u
\]
for any bounded harmonic function $u$ on $X$. Hence $u$ must be a constant. Theorem 1 is proved.

ACKNOWLEDGEMENT

The authors are grateful to the referees for their valuable comments and helpful suggestions, which contribute to improve the quality of the paper.

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