BALANCING DIRICHLET SERIES AND RELATED L-FUNCTIONS

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We study the lacunary Dirichlet series obtained from the reciprocals of \( s^{th} \) powers of balancing numbers. This function admits an analytic continuation to the entire complex plane. The series converges to irrational numbers at odd negative integral arguments. Finally, we also study the analytic continuation of the balancing \( L \)-function.

**Key words**: Balancing numbers; Dirichlet series; \( L \)-function.

1. INTRODUCTION

A series of the form \( \sum_{n=1}^{\infty} a_n n^{-s} \) where \( a_n, n \in \mathbb{N} \) is a complex sequence and \( s \) is a complex number is called a Dirichlet series. Functions defined by this series very often relate algebraic properties in analytic terms. This mostly happens when \( a_n \) is a multiplicative functions such as number of divisors of \( n \), sum of divisors of \( n \) or the Möbius function and so on. When \( a_n = 1 \) and \( \text{Re}(s) > 1 \), one gets \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \) which is the Riemann zeta function, extensively available in literature [2, 4]. \( \zeta(s) \) can be analytically continued to the whole complex plane, with only one simple pole at \( s = 1 \). Also, \( \zeta(s) \) has an important symmetry around the line \( \text{Re}(s) = \sigma = 1/2 \), in the form of a functional equation. The trivial zeros of \( \zeta(s) \) are located at \(-2, -4, -6, \ldots\) and its values at negative odd integers are rational, and in fact, given by the Bernoulli numbers [9].

The series \( \zeta_F(s) = \sum_{n=1}^{\infty} F_n^{-s} \) where \( F_n \) is the \( n^{th} \) Fibonacci number is a variant of the Riemann zeta function and the analytic continuation of this series was studied by Navas [7]. Unlike \( \zeta(s) \), this series has trivial zeros at \(-2, -6, -10, \ldots\) and simple poles \(-4, -8, \ldots\). Also \( \zeta_F(s) \) takes rational numbers at negative odd integers. This motivates us to consider the analytic continuation of the series

\[ \zeta_B(s) = \sum_{n=1}^{\infty} B_n^{-s}, \quad \text{Re}(s) > 1 \]  

(1.1)
where, as usual, \( s = \sigma + it \in \mathbb{C} \) and \( B_n \) is the \( n \)th balancing number [3]. We call the series \( \zeta_B(s) \) the balancing zeta function. We will show that \( \zeta_B(s) \) has simple poles at \( 0, -2, -4, \ldots \) and can be extended to a meromorphic function on \( \mathbb{C} \). However, the balancing zeta function has no trivial zeros unlike the Riemann zeta function.

2. **Analytic Continuation of Balancing Zeta Function**

Analytic continuation of a series consists of extending the domain of analyticity of the given series. Clearly, the balancing zeta function is analytic in the half plane \( \text{Re}(s) > 1 \). In this section, we will show that the balancing zeta function can be extended to the whole complex plane except at poles.

It is known that the balancing numbers satisfy the recurrence relation \( B_{n+1} = 6B_n - B_{n-1}, \quad n \geq 1 \) with \( B_0 = 0, B_1 = 1 \) and the Binet form is given by \( B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \) where \( \lambda_1 = 3 + 2\sqrt{2} \) and \( \lambda_2 = \frac{1}{\lambda_1} \). For any complex number \( z \)

\[
B_n^z = \left( \frac{\lambda_1^n - \lambda_2^n}{4\sqrt{2}} \right)^z = 2^{-5z/2} \left( \lambda_1^n - \lambda_2^n \right)^z
\]

\[
= 2^{-5z/2} \lambda_1^nz \left( 1 - \left( \frac{\lambda_2}{\lambda_1} \right)^n \right)^z
\]

\[
= 2^{-5z/2} \lambda_1^nz \left( 1 - \left( \frac{1}{\lambda_1^{2n}} \right) \right)^z
\]

\[
= 2^{-5z/2} \lambda_1^nz \sum_{k=0}^{\infty} \binom{z}{k} \lambda_1^{-2nk}
\]

\[
= 2^{-5z/2} \sum_{k=0}^{\infty} \binom{z}{k} \lambda_1^{n(z-2k)}.
\]

This expression is valid for any \( z \in \mathbb{C} \) and this binomial series converges since \( \lambda_1 > 1 \). Substituting the final expression for \( B_n^z \) with \( z = -s \) in (1.1) we get,

\[
\sum_{n=1}^{\infty} B_n^{-s} = 2^{5s/2} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \binom{-s}{k} \lambda_1^{-s-2k}.
\] (2.1)

Thus,

\[
\sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \left| \binom{-s}{k} \lambda_1^{-s-2k} \right| = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \left| \binom{-s}{k} \lambda_1^{-s-2k} \right|
\]

\[
\leq \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \left( -1 \right)^k \binom{-s}{k} \lambda_1^{-n(s+2k)}
\]
\[
\zeta_B(s) = \sum_{n=1}^{\infty} B_n^{-s} = 2^{5s/2} \sum_{k=0}^{\infty} \left( -\frac{s}{k} \right) \left( \frac{1}{\lambda_1^{-(s+2k)} - 1} \right)
\]

Thus the series in (2.1) is absolutely convergent and we can interchange the order of summation, i.e.,

\[
\zeta_B(s) = \sum_{n=1}^{\infty} B_n^{-s} = 2^{5s/2} \sum_{k=0}^{\infty} \left( -\frac{s}{k} \right) \left( \frac{1}{\lambda_1^{-(s+2k)} - 1} \right)
\]

This infinite series determines the holomorphic function on \( \mathbb{C} \) except for the poles derived from \( \lambda_1^{(s+2k)} - 1 = 0 \). Let \( k_0 = \max\{1, -\sigma\} \). Then for each \( s \in \mathbb{C} \) and \( k > k_0 \), we have \( |\lambda_1^{(s+2k)} - 1| \geq \lambda_1^{(\sigma+2k)} - 1 > \lambda_1^{(\sigma+k)} \). Hence

\[
\sum_{k=k_0}^{\infty} \left( -\frac{s}{k} \right) \left( \frac{1}{\lambda_1^{(s+2k)} - 1} \right) \leq \lambda_1^{\sigma} \sum_{k=k_0}^{\infty} (-1)^k \left( -\frac{|s|}{k} \right) \lambda_1^{-k} = \lambda_1^{\sigma} (1 - \lambda_1^{-1})^{-|s|} < \infty,
\]

proving that the series in (1.1) converges uniformly and absolutely on compact subsets of \( \mathbb{C} \) which does not contain any poles of the function

\[
f_k(s) = \left( -\frac{s}{k} \right) \frac{1}{\lambda_1^{(s+2k)} - 1}.
\]

The poles of \( f_k(s) \) are given by \( s = -2k + \frac{2\pi i n}{\log \lambda_1} \) for \( k \geq 0 \) and \( n \in \mathbb{Z} \). Thus the poles of the series \( \zeta_B(s) \) lie on the line \( \sigma = -2k \) and spaced in an interval of \( \frac{2\pi i}{\log \lambda_1} \). Hence the series \( \zeta_B(s) \) can be meromorphically continued to the whole complex plane and its simple poles are at \( s_{k,n} = -2k + \frac{2\pi i n}{\log \lambda_1} \). The residue of \( \zeta_B(s) \) at \( s_{k,n} \) is

\[
\operatorname{Res}_{s=s_{k,n}} \zeta_B(s) = 2^{5s_{k,n}} \left( -\frac{s_{k,n}}{k} \right) \lim_{s \to s_{k,n}} \frac{s - s_{k,n}}{\lambda_1^{(s+2k)} - 1}.
\]
By L'Hôpital's rule,
\[ \lim_{s \to s_{k,n}} \frac{s - s_{k,n}}{\lambda_1^{(s+2k)} - 1} = \frac{1}{\log \lambda_1}. \]

The above discussion proves the following theorem.

**Theorem 2.1** — The function \( \zeta_B(s) \) can be meromorphically continued to the whole complex plane and can be expressed as
\[ \zeta_B(s) = 2^{5s/2} \sum_{k=0}^{\infty} \binom{s}{k} \left( \frac{1}{\lambda_1^{(s+2k)} - 1} \right) \]
which is holomorphic except for simple poles at \( s = s_{k,n} = -2k + \frac{2\pi in}{\log \lambda_1} \) and the residue at \( s = s_{k,n} \) is given by \( \frac{2^{5s_{k,n}} \binom{s_{k,n}}{k}}{\log \lambda_1} \).

3. VALUES OF \( \zeta_B(s) \) AT INTEGRAL ARGUMENTS

### 3.1 Values at Negative Integers

In this section, we discuss the values of \( \zeta_B(s) \) at negative integers. We have already verified that \( 0, -2, -4, -6, \ldots \) are simple poles of \( \zeta_B(s) \). The following theorem shows the irrationality of \( \zeta_B(s) \) at odd negative integer.

**Theorem 3.1** — If \( m \) is an odd natural number, then \( \zeta_B(-m) \) is a irrational number.

**Proof**: Let \( m \geq 0 \) be an integer which is not a multiple of 2. Then
\[ \zeta_B(-m) = 2^{-5m/2} \sum_{k=0}^{\infty} \binom{m}{k} \left( \frac{1}{\lambda_1^{m+2k} - 1} \right) \]
and since all terms with \( k > m \) are zero, it is a finite sum belonging to \( \mathbb{Q}(\sqrt{2}) \). Let \( \sigma_k = \binom{m}{k} (\lambda_1^{-m+2k} - 1)^{-1} \) and \( \alpha_k = \sigma_k + \sigma_{m-k} \) so that \( \alpha_k = \alpha_{m-k} \) and
\[ \zeta_B(-m) = \frac{1}{2} 2^{-5m/2} \sum_{k=0}^{m} \alpha_k. \]

Now
\[ \alpha_k = \binom{m}{k} (\lambda_1^{-m+2k} - 1)^{-1} + \binom{m}{m-k} (\lambda_1^{m-2k} - 1)^{-1} \]
\[ = \binom{m}{k} \frac{1}{\lambda_1^{-m+2k} - 1} + \binom{m}{m-k} \frac{1}{\lambda_1^{m-2k} - 1} \]
(m \choose k) \left[ \frac{1}{\lambda_1^{-m-2k-1}} + \frac{1}{\lambda_1^{-2k-1}} \right] \]

\[= (m \choose k) \left[ \frac{1}{\lambda_1^{-m+2k-1}} + \frac{1}{\lambda_2^{-2k-m-1}} \right] \]

\[= (m \choose k) \left[ \frac{\lambda_1^{-m+2k-1} + \lambda_2^{-m+2k-1}}{(\lambda_1^{-m+2k-1})(\lambda_2^{-2k-m-1})} \right] = -(m \choose k). \]

So, if \( m \not\equiv 0 \pmod{2} \), then

\[
\zeta_B(-m) = -2^{-5m/2} \sum_{k=0}^{m} (m \choose k)
\]

\[= -2^{-5m/2} \frac{1}{\sqrt{2} 2^{3m+2}} \]

which is irrational. \( \square \)

3.2 Values at Positive Integers

We know that \( B_n^z = 2^{-5z/2} \sum_{k=0}^{\infty} z \choose k \lambda_1^{n(z-2k)} \) and we also have

\[ (-1)^k \left( \begin{array}{c} -s \\ k \end{array} \right) = \left( \begin{array}{c} s+k-1 \\ k \end{array} \right). \]

For \( m \in \mathbb{N} \)

\[ B_n^{-m} = 2^{5m/2} \sum_{k=0}^{\infty} (m \choose k) \lambda_1^{n(-m-2k)} \]

\[= 2^{5m/2} \sum_{k=0}^{\infty} (-1)^k \left( m+k-1 \choose k \right) \lambda_1^{-n(m+2k)} \]

\[= 2^{5m/2} \sum_{k=0}^{\infty} (-1)^k \left( m+k-1 \choose m-1 \right) \lambda_1^{-n(m+2k)}. \]

Taking \( d = m + 2k \) and \( S_m = \{ d \geq m : d \equiv m \pmod{2} \} \), we get

\[ B_n^{-m} = 2^{5m/2} \sum_{d \in S_m} (-1)^{d-m} \left( \frac{d+m-2}{2} \frac{d+m-2}{m-1} \right) \lambda_1^{-nd}. \]

Let \( S_m^+ = \{ d \geq m : d \equiv m \pmod{4} \} \) and \( S_m^- = \{ d \geq m + 2 : d \equiv m \pmod{4} \} \). Then

\[ B_n^{-m} = 2^{5m/2} \left( \sum_{d \in S_m^+} \left( \frac{d+m-2}{m-1} \right) \lambda_1^{-nd} + (-1) \sum_{d \in S_m^-} \left( \frac{d+m-2}{m-1} \right) \lambda_1^{-nd} \right). \quad (3.2) \]
To take sum over \( n \), we will collect like powers \( l = nd \), so that \( l \) runs over all natural numbers and we restrict to \( d \mid l \). We thus have

\[
\sum_{n=1}^{\infty} B_{n}^{-m} = 2^{5m/2} \sum_{l=1}^{\infty} \left( \sum_{d \mid l} \frac{d + m - 2}{m - 1} \right) \lambda_{1}^{-l}. \tag{3.3}
\]

The above discussion proves the following theorem.

**Theorem 3.2** — \( \zeta_B(m) = 2^{5m/2} \sum_{l=1}^{\infty} a_{l} \lambda_{1}^{-l} \) for \( m \in \mathbb{N} \), where the coefficients \( a_{l} \) are combinations of sums of the powers of divisors of \( l \).

When \( m = 1 \), we have \( S_{1}^{+} = \{ d \geq 1 : d \equiv 1 \pmod{4} \} = \{ 1, 5, 9, \cdots \} \) and \( S_{1}^{-} = \{ d \geq 1 : d \equiv 3 \pmod{4} \} = \{ 3, 7, 11, \cdots \} \). Also all the binomial coefficients reduce to 1 and then

\[
\sum_{n=1}^{\infty} B_{n}^{-1} = 2^{5/2} \sum_{l=1}^{\infty} \left( \sum_{d \mid l} 1 + (-1) \sum_{d \mid l} 1 \right) \lambda_{1}^{-l}
\]

\[
= 2^{5/2} \sum_{l=1}^{\infty} (d_{1}(l) - d_{3}(l)) \lambda_{1}^{-l}
\]

where

\[
d_{i}(n) = \sum_{\substack{d \mid n \\lvert d \equiv i \pmod{4}}} 1.
\]

Hence we have the following identities:

\[
\sum_{n=1}^{\infty} B_{2n}^{-1} = 2^{5/2} \sum_{l \equiv 0 \pmod{2}} (d_{1}(l) - d_{3}(l)) \lambda_{1}^{-l},
\]

\[
\sum_{n=1}^{\infty} B_{2n+1}^{-1} = 2^{5/2} \sum_{l \equiv 1 \pmod{2}} (d_{1}(l) - d_{3}(l)) \lambda_{1}^{-l}.
\]

Using the above equation, the sum \( \sum_{n=1}^{\infty} B_{2n+1}^{-1} \) can be expressed as special values of classical theta function. In this connection, we have the following result.

**Theorem 3.3** — For \( \delta = \frac{i\pi}{\log \lambda_{1}} \) we have

\[
\sum_{n=1}^{\infty} \frac{1}{B_{2n+1}} = \frac{2^{5/2}}{4} \left[ \Theta\left( -\frac{1}{\delta} \right)^{2} - \Theta\left( -\frac{2}{\delta} \right)^{2} \right]
\]
where $\Theta(q) = \sum_{n \in \mathbb{Z}} q^{n^2}, q = e^{\pi i z}$ denotes the Jacobi's theta function.

**Proof**: The proof easily follows from $\Theta(q)^2 = \sum_{l=0}^{\infty} r_2(l) q^l$, where $r_2(l)$ is the number of representation of $l$ as sum of two integer squares and $r_2(l) = 4(d_1(l) - d_3(l))$. \[\square\]

It is interesting to observe the special values of $\zeta_B(s)$ when $s$ is a natural number. André-Jeannin [1] proved that the Fibonacci Dirichlet series $\sum_{n=1}^{\infty} \frac{1}{F_n}$ is an irrational number. Also Duverney et al., [5] proved that $\sum_{n=1}^{\infty} \frac{1}{F_{n^2k}}$ is transcendental for $k = 1, 2, \cdots$. Similarly one can prove that $\zeta_B(2k)$ is transcendental for $k = 1, 2, \cdots$, which is, indeed, a particular case of transcendence of binary linear recurrence sequences proved in [6]. Ram Murty [8] also proved the same results of the Fibonacci Dirichlet series using q-series, where he used q-exponential for irrationality of $\sum_{n=1}^{\infty} \frac{1}{F_n^s}$ at $s = 1$ and q-logarithm for transcendence of the series for $s = 2k$ and establish a connection to Ramanujan’s mock-theta function.

4. **Balancing L-Function**

Let $\chi$ be a Dirichlet character with modulo $p$. The Dirichlet $L$-function is defined as $L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$. We define the balancing $L$ function as

$$L_B(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{B_n^s}, \quad \text{Re}(s) = \sigma > 1. \quad (4.1)$$

We know from [2] that $\zeta(s)$ and $L(s, \chi)$ can be unified using the Hurwitz zeta function $\zeta(s, a) = \sum_{n=1}^{\infty} \frac{1}{(n + a)^s}$. Similarly, we can write the balancing $L$-function in terms of balancing zeta function using balancing zeta function in arithmetic progression which we define as

$$\zeta_B(s, (r, p)) := \sum_{n \equiv r \pmod{p}}^{\infty} \frac{1}{B_n^s}. \quad (4.2)$$

Thus, $\zeta_B(s, (1, 1)) = \zeta_B(s)$. Further,

$$\zeta_B(s, (r, p)) = 4^s 2^{s/2} \sum_{n=0}^{\infty} \left(\lambda_1^{pn+r} - \lambda_2^{pn+r}\right)^{-s}$$

$$= 2^{5s/2} \sum_{n=0}^{\infty} \lambda_1^{-(pn+r)s} \left(1 - \left(\frac{\lambda_2}{\lambda_1}\right)^{pn+r}\right)^{-s}$$

$$= 2^{5s/2} \sum_{n=0}^{\infty} \lambda_1^{-(pn+r)s} \sum_{k=0}^{\infty} \frac{(-s)^k}{k!} \left(\frac{\lambda_2}{\lambda_1}\right)^{(pn+r)k}$$
\[
2^{5s/2} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{-s}{k} \right) \lambda_1^{-(pn+r)(s+2k)} 
= 2^{5s/2} \sum_{k=0}^{\infty} \left( \frac{-s}{k} \right) \frac{\lambda_1^{-(s+2k)r}}{1 - \lambda_1^{-(s+2k)p}}.
\]

This shows that the function \( \zeta_B(s, (r, p)) \) can be meromorphically continued to the whole complex plane except for the simple poles at \( s = -2k + \frac{2n\pi i}{p \log \lambda_1} \).

The following theorem, which uses the notion of Gauss sum [2], establishes the analytic continuation of the balancing \( L \)-function. For any Dirichlet character \( \chi \) modulo \( k \) the sum
\[
G(n, \chi) = \sum_{m=1}^{k} \chi(m) \exp \left( \frac{2\pi inm}{k} \right)
\]
is called the Gauss sum associated with \( \chi \).

**Theorem 4.1** — The function \( L_B(s, \chi) \) can be meromorphically continued to the whole complex plane which is holomorphic except for simple poles at \( s = s_{k,n} = -2k + \frac{2n\pi i}{p \log \lambda_1} \) and the residue at \( s = s_{k,n} \) is given by
\[
\text{Res}_{s=s_{k,n}} L_B(s, \chi) = 2^{5s_{k,n}} \left( \frac{-s_{k,n}}{k} \right) \frac{1}{p \log \lambda_1} \chi(-1) G(n, \chi).
\]

**Proof:** Observe that
\[
L_B(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{B_n^s} 
= \sum_{r=1}^{p} \chi(r) \zeta_B(s, (r, p)).
\]

It is clear from equation (4.6) that the poles of \( \zeta_B(s, (r, p)) \) give the poles of the balancing \( L \)-function \( L_B(s, \chi) \). Therefore the balancing \( L \)-function can be analytically continued to the whole complex plane. For residue of the balancing \( L \)-function at the poles, we first find out the residue of \( \zeta_B(s, (r, p)) \).
\[
\text{Res}_{s=s_{k,n}} \zeta_B(s, (r, p)) = 2^{5s_{k,n}} \left( \frac{-s_{k,n}}{k} \right) \lim_{s \to s_{k,n}} \frac{\lambda_1^{(s+2k)r}}{1 - \lambda_1^{(s+2k)p}} 
= 2^{5s_{k,n}} \left( \frac{-s_{k,n}}{k} \right) \exp \left( \frac{-2n\pi i}{p} \right) \lim_{s \to s_{k,n}} \frac{(s - s_{k,n})}{1 - \lambda_1^{(s+2k)p}}.
\]
\[ = 2^{5s_{k,n}} \left( -\frac{s_{k,n}}{k} \right) \exp \left( -\frac{2\pi i}{p} \right)^{nr} \frac{1}{p \log \lambda_1}. \]

Hence the residue of balancing \( L \)-function at the requisite pole is

\[
\text{Res}_{s=s_{k,n}} \mathcal{L}_B(s, \chi) = \sum_{r=1}^{p} \chi(r) \text{Res}_{s=s_{k,n}} \zeta_B(s, (r, p)) \\
= 2^{5s_{k,n}} \left( -\frac{s_{k,n}}{k} \right) \frac{1}{p \log \lambda_1} \sum_{r=1}^{p} \chi(r) \exp \left( -\frac{2\pi i}{p} \right)^{nr}. \]

Using (4.4) we get

\[
\text{Res}_{s=s_{k,n}} \mathcal{L}_B(s, \chi) = 2^{5s_{k,n}} \left( -\frac{s_{k,n}}{k} \right) \frac{1}{p \log \lambda_1} G(-n, \chi) \\
= 2^{5s_{k,n}} \left( -\frac{s_{k,n}}{k} \right) \frac{1}{p \log \lambda_1} \chi(-1) G(n, \chi).
\]

This completes the proof of the theorem. \( \square \)

The following result gives the value of the balancing \( L \)-function at negative integers.

**Theorem 4.2** — Let \( \chi \) be any non-principal character modulo \( p \). Then \( \mathcal{L}_B(-m, \chi) = 0 \) for all \( m \in \mathbb{N} \).

**Proof:** Using (4.3) and (4.6) we get

\[
\mathcal{L}_B(s, \chi) = \sum_{r=1}^{p} \chi(r) 2^{5s/2} \sum_{k=0}^{\infty} \left( -\frac{s}{k} \right) \frac{\lambda_1^{-(s+2k)r}}{1 - \lambda_1^{-(s+2k)p}}.
\]

Hence for \( s = -m \), we have

\[
\mathcal{L}_B(-m, \chi) = \sum_{r=1}^{p} \chi(r) 2^{-5m/2} \sum_{k=0}^{\infty} \left( \frac{m}{k} \right) \frac{\lambda_1^{(m-2k)r}}{1 - \lambda_1^{(m-2k)p}}.
\]

Proceeding like the proof of Theorem 3.1, we get

\[
\sum_{k=0}^{\infty} \left( \frac{m}{k} \right) \frac{\lambda_1^{(m-2k)r}}{1 - \lambda_1^{(m-2k)p}} = \frac{1}{2} \sum_{k=0}^{m} \left[ \left( \frac{m}{k} \right) \frac{\lambda_1^{(m-2k)r}}{1 - \lambda_1^{(m-2k)p}} + \left( \frac{m}{m-k} \right) \frac{\lambda_1^{(-m+2k)r}}{1 - \lambda_1^{(-m+2k)p}} \right]
\]

\[
= \frac{1}{2} \sum_{k=0}^{m} \left( \frac{m}{k} \right) \frac{\lambda_1^{(m-2k)r}}{1 - \lambda_1^{(m-2k)p}} + \frac{\lambda_1^{(-m+2k)r}}{1 - \lambda_1^{(-m+2k)p}}
\]

\[
= \frac{1}{2} \sum_{k=0}^{m} \left( \frac{m}{k} \right) \frac{\lambda_1^{(m-2k)r}}{1 - \lambda_1^{(m-2k)p}} + \frac{\lambda_2^{(m-2k)r}}{1 - \lambda_2^{(m-2k)p}} = -2^{m-1}.
\]
Therefore,
\[ \mathcal{L}_B(-m, \chi) = -2^{-(3m+2)/2} \sum_{r=1}^{p} \chi(r) = 0, \quad \text{as} \quad \sum_{r=1}^{p} \chi(r) = 0. \]

This completes the proof of the theorem. \(\square\)

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**References**


