ON THE DEPTH OF FIBER CONES OF STRETCHED $m$-PRIMARY IDEALS

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In this article, we study certain homological properties of the graded rings associated with stretched $m$-primary ideals in a Cohen-Macaulay local ring $(A, m)$. We compute the $h$-polynomial of the fiber cone and using this expression we show that the fiber cone is Gorenstein is equivalent to being hypersurface under certain assumptions. We obtain some inequalities between the Hilbert coefficients of the fiber cone and obtain sufficient conditions for the equality.

**Key words**: Fiber cone; associated graded ring; Rees algebra; Cohen-Macaulay; Hilbert series; Hilbert coefficients.

1. INTRODUCTION

Let $(A, m)$ be a Cohen-Macaulay local ring of dimension $d$ with $A/m$ infinite. For an ideal $I$ in $A$, let $R(I), G(I)$ and $F(I)$ denote respectively the Rees algebra, the associated graded ring and the fiber cone with respect to the ideal $I$. Abhyankar proved that $\mu(m) \geq e_0(m) + d - 1$, where $\mu(M)$ denote the minimum number of generators for an $A$-module $M$ and $e_0(m)$ denote the Hilbert-Samuel multiplicity of $m$, see [1]. In [16], Sally showed that if equality holds, then $G(m)$ is Cohen-Macaulay. She conjectured that if $\mu(m) = e_0(m) + d - 2$, then $\depth G(m) \geq d - 1$. This conjecture was proved affirmatively by Rossi and Valla (see [12]) and independently by Wang (see [18]). Further, Sally studied local rings $A$ with $\mu(m) = e_0(m) + d - 3$ and proved that if $A$ is Gorenstein, then $G(m)$ is Cohen-Macaulay and the Hilbert series is $HS_A(z) = \frac{1 + (\mu(m) - d)z + z^2 + z^3}{(1 - z)^2}$. Later Rossi

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and Valla studied this class of ideals further and proved that if $A$ is Cohen-Macaulay and the Cohen-Macaulay type $\tau := \dim_A(A/J : m)/J$ is strictly less than $\mu(m) - d$, then depth $G(m) \geq d - 1$. Rossi has asked if $\mu(m) = \epsilon_0(m) + d - 3$, then is depth $G(m) \geq d - 2$. The questions is yet to be answered. It is known that there are two classes of local rings satisfying the property $\mu(m) = \epsilon_0(m) + d - 3$, namely short and stretched, see [14]. Stretched rings were first considered by Sally in [17]. Rossi and Valla extended the definition of stretchedness to $m$-primary ideals and obtained certain sufficient conditions for almost maximal depth of the corresponding associated graded rings, see [1].

**Definition 1.1** — An $m$-primary ideal $I$ in a Noetherian local ring $(A, m)$ is stretched if there exists an ideal $J$ generated by a maximal superficial sequence in $I$ such that $I^2 \cap J = JI$ and $\lambda \left( \frac{I^2 + J}{I^3 + J} \right) = 1$.

Abhyankar’s inequality was generalized in another direction by Chuai to $m$-primary ideals. It was shown in [96] that for an $m$-primary ideal $I$ in a Cohen-Macaulay local ring $A$, $\mu(I) + \lambda \left( \frac{mI}{mJ} \right) = e_0(I) - \lambda \left( \frac{A}{I} \right) + d$. Hence $\mu(I) \leq e_0(I) - \lambda \left( \frac{A}{I} \right) + d$ and the equality occurs if and only if $mI = mJ$. Goto defined an ideal said to have minimal multiplicity if $mI = mJ$ and characterized various properties of the associated graded rings, the Rees algebras and the fiber cones of such ideals (see [6]). In [9], Jayanthan and Verma proved that if $\lambda \left( \frac{mI}{mJ} \right) = 1$ and depth $G(I) \geq d - 2$, then depth $F(I) \geq d - 1$. In [1], Rossi and Valla studied the depth of associated graded rings of stretched $m$-primary ideals with $\lambda \left( \frac{I^2}{J} \right) = 2$.

**Definition 1.2** — Let $L \supset I$. Define $n_L \geq 1$ be the least integer $i$ such that $LI^i \subseteq J$. Let $n_I$ denote the least integer $i$ such that $I^{i+1} \subseteq J$.

Note that $n_L - 1 \leq n_I \leq n_L$ and $n_L = n_I$ if and only if $LI^{n_I} \subseteq J$.

In [13], Rossi and Valla proved that $G(I)$ is Cohen-Macaulay if and only if $I^{n_I + 1} = JI^{n_I}$. They also proved that if $I^{n_I - 1} \subseteq JI^{n_I - 1}$, then depth $G(I) \geq d - 1$ and the numerator of the Hilbert series of $G(I)$, known as $h$-polynomial of $I$, is given by

$$h_I(z) = \lambda \left( \frac{A}{I} \right) + \left[ \lambda \left( \frac{I}{J} \right) - d \lambda \left( \frac{A}{I} \right) \right] z + z^2 + \cdots + z^{n_I - 1} + z^s$$

for some $s$.

In this article we study $m$-primary ideals $I$ satisfying $\lambda \left( \frac{mI}{mJ} \right) = 2$ for a minimal reduction $J$ of $I$. In fact, we prove our results in a slightly more general setting. Let $L$ be an ideal with $I \subseteq L \subseteq m$. Denote by $\mathcal{F}_L$ the filtration $A \supset L \supset LI \supset LI^2 \cdots$. Then $\mathcal{F}_L$ is an $I$-good filtration and $G(\mathcal{F}_L) := A/L \oplus L/LI \oplus LI/LI^2 \oplus \cdots$ is a finitely generated $G(I)$-module with $\dim G(\mathcal{F}_L) = d$. 
Using results of Rossi and Valla [13], generalized to the case of the filtration $\mathcal{F}_L$, we obtain certain properties of the $h$-polynomial and depth of $F_L(I) := A/L \oplus I/L^2 \oplus I^2/L^2 \oplus \cdots$. If $L = m$, then we denote $\mathcal{F}_L$ as $\mathcal{F}$ and $F_L(I)$ as $F(I)$.

If $M$ is an $A$-module and $J \subset I$ be ideals, then $J$ is said to be an $M$-reduction of $I$ if $I^{n+1}M = JI^nM$ for some $n \geq 0$. In this case $r^M_J(I) := \min\{n \mid I^{n+1}M = JI^nM\}$ is called the $M$-reduction number of $I$ with respect to $J$. If $M = A$, then it is known as the reduction number of $I$ with respect to $J$. Note that $n_I \leq r^M_J(I)$ and $n_L \leq r^M_J(I) + 1$.

For $m$-primary ideals $I \subset L$, it is known that $e_0(I) = \lambda \left( \frac{I}{LI} \right) - d\lambda \left( \frac{A}{L} \right) + \lambda(A/I) + K - 1$ for some $K \geq 1$, where $e_0(I)$ is the multiplicity of $I$ (see the proof of Proposition 2.3). If $K = 1, 2$ then the depth and the Hilbert series of $G(\mathcal{F}_L)$ are well-understood (see [15]). We study the case $K \geq 3$. Now we give section wise description of the paper.

In section 2 we prove some properties of $m$-primary ideals $I$ with $LI \cap J = LJ$ and $\lambda \left( \frac{LI + J}{LI^2 + J} \right) = 1$. We give sufficient conditions for $G(I), G(\mathcal{F}_L)$ and $F_L(I)$ to have almost maximal depths. If $I$ is stretched, then we prove that $\lambda \left( \frac{I^2}{JI} \right) \leq \lambda \left( \frac{LI}{LJ} \right)$. We show that if $I$ is stretched with $\lambda(LI/LJ) = 2$ and the $L$-reduction number of $I$ maximum possible, then $F_L(I)$ is Cohen-Macaulay.

In section 3 we investigate the depth of fiber cone of a stretched $m$-primary ideal $I$ satisfying $LI^{mL} \subseteq LJI^{mL-2}$. We compute the $h$-polynomial of $F_L(I)$ and prove that depth $F_L(I) \geq d - 1$ under some mild conditions on the ideals. We also characterize Cohen-Macaulayness of $F_L(I)$ in this case. Finally we prove that if $F(I)$ is Gorenstein, then it has to be a hypersurface.

In section 4 we prove some inequalities on the Hilbert coefficients of $F(I)$ for the class of ideals considered in Section 3. Write the Hilbert polynomial of $F(I)$ as

$$P_{F(I)}(n) = \sum_{i=0}^{d-1} (-1)^i f_i \binom{n + d - i - 1}{d - i - 1}.$$

If $I$ is stretched with $\lambda(mI/mJ) = 2$ and with some other mild conditions, then we prove that $f_i \geq 0$ for all $i \geq 0$. We also obtain certain inequalities among, $f_0, f_1$ and $f_2$ and we obtain sufficient conditions for the equalities.

Throughout the article we assume that $(A, m)$ is a Cohen-Macaulay local ring with $A/m$ infinite, $I$ is an $m$-primary ideal of $A$ with a minimal reduction $J$ and $L \supseteq I$ is any ideal of $A$ such that $LI \cap J = LJ$, unless stated otherwise.
2. Fiber Cone of Stretched $m$-Primary Ideals

First of all, we would like to remark that the results of [13] proved for the $I$-adic filtration remain valid for the $I$-good filtration $F_L$ with proofs being almost verbatim. We will be using the generalized versions of the results with reference to the same article. In this section, we prove some technical lemmas and derive results for $F_L(I)$ analogues to the results in [13] for $G(I)$. We begin with a relation between $\lambda(LI/LJ)$ and $\lambda(LI + J/LL^2 + J)$.

**Lemma 2.1** — If $LI^2 \not\subseteq J$ and $\lambda\left(\frac{LI}{LJ}\right) = 2$, then $\lambda\left(\frac{LI + J}{LL^2 + J}\right) = 1$.

**Proof:** We have $\lambda\left(\frac{LI + J}{LL^2 + J}\right) = \lambda\left(\frac{LI}{LJ}\right) \leq \lambda\left(\frac{LI}{LL^2 + L}J\right) = 2$. Suppose $\lambda\left(\frac{LI + J}{LL^2 + J}\right) = 2$. Then we have $2 = \lambda\left(\frac{LI}{LJ}\right) = \lambda\left(\frac{LI + J}{LL^2 + J}\right) = \lambda\left(\frac{LI}{LL^2 + L}J\right)$. This implies that $LJ = LL^2 + L$. This implies $LL^2 \subseteq J$. This is a contradiction to the hypothesis. Therefore $\lambda\left(\frac{LI + J}{LL^2 + J}\right) \leq 1$. Suppose $\lambda\left(\frac{LI + J}{LL^2 + J}\right) = 0$. Then $\lambda\left(\frac{LI}{LL^2 + L}J\right) = 0$. That is $LI = LL^2 + L$. Then by Nakayama lemma $LI = LJ$ which is a contradiction. Hence $\lambda\left(\frac{LI + J}{LL^2 + J}\right) = 1$. □

It can be seen that the converse of the Lemma 2.1 is not true even for stretched $m$-primary ideals, Example 2.10(1).

**Proposition 2.2** — Assume that $\lambda\left(\frac{LI}{LJ}\right) = 2$ and that there exists $l \in \mathbb{N}$ such that

(i) $LI^l \cap (J : I) \neq LI^l \cap (J : a)$ for some $a \in I \setminus J$.

(ii) $LI^{j+1} \cap J = LJJ^j$ for all $j \leq l - 1$.

Then

(a) $\text{depth} G(F_L) \geq d - 1$;

(b) $\text{depth} G(I) \geq d - 2$ if and only if $\text{depth} F_L(I) \geq d - 1$;

(c) if $I$ is stretched, then $\text{depth} G(I) \geq d - 1$ and $\text{depth} F_L(I) \geq d - 1$.

**Proof:** (a) Since $LI^l \cap (J : I) \neq LI^l \cap (J : a)$, it follows from Lemma 2.4(7) in [13] that $\lambda\left(\frac{LI^{l+1}}{LJJ^l}\right) < \lambda\left(\frac{LI^l}{LJJ^{l-1}}\right) \leq 2$. Therefore by [15, Theorem 4.4], $\text{depth} G(F_L) \geq d - 1$.

(b) The assertion follows directly from (a) and [15, Proposition 5.1].
(e) By (a) we have depth $G(F_L) \geq d - 1$. If $I \cap (J : a) = I \cap (J : I)$, then by intersecting with $LI^3$ we get $LI^3 \cap (J : a) = LI^3 \cap (J : I)$ which is a contradiction to the hypothesis. Therefore $I \cap (J : a) \neq I \cap (J : I)$. By [13, Lemma 2.4] we get $\lambda \left( \frac{I^3}{JI^2} \right) < \lambda \left( \frac{I^3}{JI} \right) = 2$, i.e., $\lambda \left( \frac{I^3}{JI^2} \right) \leq 1$.

Hence from Theorem 4.4 in [15] we get depth $G(I) \geq d - 1$. Therefore from [15, Proposition 5.1], we get depth $F_L(I) \geq d - 1$.

The following proposition gives a connection between $n_L$ and $\lambda \left( \frac{LJ}{IJ} \right)$.

**Proposition 2.3** — If $\lambda \left( \frac{LI + J}{LI^2 + J} \right) = 1$, then $n_L = \lambda \left( \frac{LI}{LJ} \right) + 1$. Furthermore if $I$ is stretched, then $\lambda \left( \frac{J^2}{JI} \right) \leq \lambda \left( \frac{LI}{LJ} \right)$.

**Proof:** From the following diagram,

\[ \begin{array}{c}
& I \\
J & \searrow \\
L & \nearrow J \\
L & \\
\end{array} \]

it follows that

\[
\lambda \left( \frac{I}{LI} \right) + \lambda \left( \frac{LI}{LJ} \right) = \lambda \left( \frac{J}{J} \right) + \lambda \left( \frac{J}{LJ} \right) = e_0(I) - \lambda \left( \frac{A}{L} \right) + d \lambda \left( \frac{A}{L} \right)
\]

This implies that $e_0(I) = \lambda \left( \frac{A}{LI} \right) - d\lambda \left( \frac{A}{L} \right) + \lambda \left( \frac{LI}{LJ} \right)$. From the hypothesis we have

\[
h_{\mathcal{F}_{L/J}}(z) = \lambda \left( \frac{A}{L} \right) + \sum_{n \geq 1} \lambda \left( \frac{L^{n-1} + J}{LI^n + J} \right) z^n = \lambda \left( \frac{A}{L} \right) + \lambda \left( \frac{L}{LI + J} \right) z + z^2 + \cdots + z^{n_L}.
\]

Hence $e_0(\mathcal{F}) = h_{\mathcal{F}_{L/J}}(1) = \lambda \left( \frac{A}{L} \right) + \lambda \left( \frac{L}{LI + J} \right) + n_L - 1$. Since $e_0(I) = e_0(\mathcal{F}_L)$, we have

\[
\lambda \left( \frac{A}{LI} \right) - d\lambda \left( \frac{A}{L} \right) + \lambda \left( \frac{LI}{LJ} \right) = \lambda \left( \frac{A}{L} \right) + \lambda \left( \frac{L}{LI + J} \right) + n_L - 1.
\]

Therefore

\[
n_L = \lambda \left( \frac{A}{LI} \right) - d\lambda \left( \frac{A}{L} \right) - \lambda \left( \frac{A}{L} \right) - \lambda \left( \frac{L}{LI + J} \right) + \lambda \left( \frac{LI}{LJ} \right) + 1
\]

\[
= \lambda \left( \frac{A}{LI} \right) - \lambda \left( \frac{J}{LJ} \right) - \lambda \left( \frac{A}{L} \right) - \lambda \left( \frac{L}{LI + J} \right) + \lambda \left( \frac{LI}{LJ} \right) + 1
\]

\[
= \lambda \left( \frac{L}{LI} \right) - \lambda \left( \frac{J}{LJ} \right) - \lambda \left( \frac{L}{LI + J} \right) + \lambda \left( \frac{LI}{LJ} \right) + 1
\]

\[
= \lambda \left( \frac{LI + J}{LI} \right) - \lambda \left( \frac{J}{LJ} \right) + \lambda \left( \frac{LI}{LJ} \right) + 1
\]

\[
= \lambda \left( \frac{LI}{LJ} \right) + 1.
\]
If $I$ is stretched, then we have $n_I = \lambda \left( \frac{I^2}{JI} \right) + 1$. Since $n_I \leq n_L$, $\lambda \left( \frac{I^2}{JI} \right) \leq \lambda \left( \frac{LI}{LJ} \right)$. \hfill \square

**Corollary 2.4** — If $I$ is stretched, $\lambda \left( \frac{LI}{LJ} \right) = 2$ and $LI^{n_I} \not\subseteq J$, then $\lambda \left( \frac{I^2}{JI} \right) = 1$.

**Proof:** Since $I$ is stretched, it follows from [13] (Remark after the proof of Lemma 2.4) that $n_I \geq 2$. This fact along with the hypothesis that $LI^{n_I} \not\subseteq J$ implies that $LI^2 \not\subseteq J$. Therefore from the Lemma 2.1 we get $\lambda \left( \frac{LI + J}{LI^2 + J} \right) = 1$. Hence it follows from Proposition 2.3 that $\lambda \left( \frac{I^2}{JI} \right) \leq \lambda \left( \frac{LI}{LJ} \right) = 2$. Since $LI^{n_I} \not\subseteq J$, we have $n_I < n_L = 3$. Therefore $n_I = 2$. Since $n_I = \lambda \left( \frac{I^2}{JI} \right) + 1$, we have $\lambda \left( \frac{I^2}{JI} \right) = 1$. \hfill \square

Note that in the above corollary $L$ cannot be equal to $I$. The following theorem is a generalization of Theorem 2.6 in [13] to the case of filtrations. We omit the proof as it is pretty much the same as in [13].

**Theorem 2.5** — Let $(A, m)$ be a Cohen-Macaulay local ring and $I$ be an $m$-primary ideal of $A$ with $J \subseteq I$, a minimal reduction of $I$. Let $L \supseteq I$ be any ideal of $A$ such that $LI \cap J = LJ$ and $\lambda \left( \frac{LI + J}{LI^2 + J} \right) = 1$. Then $G(F_L)$ is Cohen-Macaulay if and only if $LI^{n_I} = LJ^{n_I - 1}$.

**Corollary 2.6** — Suppose $I$ is a stretched $m$-primary ideal with $\lambda \left( \frac{LI}{LJ} \right) = 2$ and $LI^{n_I} \not\subseteq J$. If $LI^{n_I + 1} = LJ^{n_I}$, then $F_L(I)$ is Cohen-Macaulay.

**Proof:** From the proof of Corollary 2.4 we have $n_I = 2$. Therefore by using Theorem 4.4 in [13] we obtain $\operatorname{depth} G(I) \geq d - 1$. Hence it follows from Theorem 2.5 that $G(F_L)$ is Cohen-Macaulay if and only if $LI^{n_I} = LJ^{n_I - 1}$. Since $n_L \leq n_I + 1$, $G(F_L)$ is Cohen-Macaulay. Hence by [15, Proposition 5.1], $F_L(I)$ is Cohen-Macaulay. \hfill \square

Now we state a generalization to the filtration case of Corollary 2.11 in [13]. We omit the proof.

**Proposition 2.7** — Let $I$ be an $m$-primary ideal with $\lambda \left( \frac{LI + J}{LI^2 + J} \right) = 1$. If $LI^{n_l} \subseteq LJ^l$ for some $l \leq n_L - 1$, then $h_{F_L}(t) = [\lambda \left( \frac{A}{L} \right) + \lambda \left( \frac{L}{LI + J} \right) t + t^2 + \cdots + t^l] \mod (t^{l+1})$.

Below in (i) we observe that $(J : I) \cap LI \neq LJ$ and in (ii) we compute $\lambda \left( \frac{I}{I^2} \right)$ which will be used in Proposition 2.9.
Remark 2.8: (i) Suppose \( (J : I) \cap LI = LJ \). Since \( LI^{n_L} \subseteq J \), we have \( LI^{n_L-1} \subseteq (J : I) \). This implies \( LI^{n_L-1} \subseteq (J : I) \cap LI = LJ \). That is \( LI^{n_L-1} \subseteq J \) which is a contradiction. Therefore \( (J : I) \cap LI \neq LJ \).

(ii) Suppose \( \lambda \left( \frac{mI}{mJ} \right) = 2 \) and \( mI^{n_m} \subseteq mJJ^2 \). By Corollary 2.1 in [13] we have \( e_0(\mathcal{F}) = h_0(\mathcal{F}) + h_1(\mathcal{F}) + \lambda \left( \frac{mI}{mJ} \right) \) and \( e_0(I) = \lambda \left( \frac{A}{I} \right) - d \lambda \left( \frac{A}{I^2} \right) + \lambda \left( \frac{I^2}{JI} \right) \). From Proposition 2.7 we get \( h_{\mathcal{F}}(z) \equiv \left[ \lambda \left( \frac{A}{I} \right) + \lambda \left( \frac{L}{LI + J} \right) \right] z + z^2 + \cdots + z^l \mod (z^{l+1}) \), if \( LI^{n_L} \subseteq LJJI \) for some \( l \leq n_L - 1 \). This implies that when \( L = m \), we have \( h_0(\mathcal{F}) = 1 \) and \( h_1(\mathcal{F}) = \lambda \left( \frac{A}{I} \right) - d + \mu(I) - 1 \). Since \( e_0(\mathcal{F}) = e_0(I) \) we have \( \lambda \left( \frac{A}{I} \right) - d + \mu(I) + \lambda \left( \frac{mI}{mJ} \right) = \lambda \left( \frac{A}{I} \right) - d \lambda \left( \frac{A}{I^2} \right) + \lambda \left( \frac{I^2}{JI} \right) \). This implies that \( \lambda \left( \frac{I^2}{JI} \right) = \mu(I) + d \lambda \left( \frac{A}{I} \right) + \lambda \left( \frac{mI}{mJ} \right) - d - \lambda \left( \frac{I^2}{JI} \right) \).

Now we compute \( \mu(I^2) \).

Proposition 2.9 — Assume that \( \lambda \left( \frac{mI + J}{mI^2 + J} \right) = 1 \) and \( mI^{n_m} \subseteq mJJ^2 \). Then

\[
\mu(I^2) = d \mu(I) + \lambda \left( \frac{I^2}{JI} \right) + 1 - \lambda \left( \frac{mI}{mJ} \right) - \left( \begin{array}{c} d \\ 2 \end{array} \right).
\]

Proof: From Proposition 2.7, we get

\[
h_{\mathcal{F}}(z) \equiv 1 + \left[ e_0(I) - 1 - \lambda \left( \frac{mI}{mJ} \right) \right] z + z^2 \mod (z^3).
\]

Now the Hilbert Series of \( G(\mathcal{F}) \) is given by

\[
HS_{\mathcal{F}}(t) = \frac{h_{\mathcal{F}}(t)}{(1 - t)^d} = 1 + \left( e_0(I) - 1 - \lambda \left( \frac{mI}{mJ} \right) \right) t + t^2 + \cdots
\]

\[
= \left( 1 + \left( e_0(I) - 1 - \lambda \left( \frac{mI}{mJ} \right) \right) t + t^2 + \cdots \right) \left( \sum_{j \geq 0} \left( \begin{array}{c} d + j - 1 \\ j \end{array} \right) t^j \right).
\]

Therefore the coefficient of \( t^2 \) is

\[
H_{\mathcal{F}}(2) = \left( \begin{array}{c} d + 1 \\ 2 \end{array} \right) + d \left[ e_0(I) - 1 - \lambda \left( \frac{mI}{mJ} \right) \right] + 1
\]

\[
= \left( \begin{array}{c} d + 1 \\ 2 \end{array} \right) + d \left[ \lambda \left( \frac{A}{I} \right) - d + \mu(I) - 1 \right] + 1.
\]
Now from the exact sequences,

\[ 0 \to mG(I) \to G(I) \to F(I) \to 0 \quad (1) \]
\[ 0 \to F(I) \to G(F) \to mG(I)(-1) \to 0 \quad (2) \]

we have \( H_{F(I)}(n) = H_{G(I)}(n) - H_{mG(I)}(n) \) and \( H_{mG(I)}(n) = H_{G(F)}(n+1) - H_{F(I)}(n+1) \). From these we get \( H_{F(I)}(n + 1) = H_{G(F)}(n + 1) - H_{G(I)}(n) + H_{F(I)}(n) \). Now put \( n = 1 \) and substitute for \( H_{G(F)}(2) \) to get

\[
\mu(I^2) = H_{F(I)}(2) \\
= \binom{d+1}{2} + d\left[\varepsilon_0(I) - 1 - \lambda\left(\frac{mI}{mJ}\right)\right] + 1 - H_{G(I)}(1) + H_{F(I)}(1) \\
= \binom{d+1}{2} + d\left[\lambda\left(\frac{A}{I}\right) - d + \mu(I) - 1\right] + 1 - \lambda\left(\frac{I^2}{I^2}\right) + \mu(I) \\
= \binom{d+1}{2} + d\left[\lambda\left(\frac{A}{I}\right) - d + \mu(I) - 1\right] + 1 \\
- \left[\mu(I) + d\lambda\left(\frac{A}{I}\right) + \lambda\left(\frac{mI}{mJ}\right) - d - \lambda\left(\frac{I^2}{IJ}\right)\right] + \mu(I).
\]

Last equality follows by substituting \( \lambda\left(\frac{I}{I^2}\right) \) from Remark 2.8(ii). This implies that

\[
H_{F(I)}(2) = \mu(I^2) = d\mu(I) + \lambda\left(\frac{I^2}{IJ}\right) + 1 - \lambda\left(\frac{mI}{mJ}\right) - \binom{d}{2}
\]
as desired. \( \square \)

The first part of the example below shows that the converse of Lemma 2.1 is not true. The second part example gives a stretched m-primary ideal with \( n_I = 5 \), \( \lambda\left(\frac{mI}{mJ}\right) = 4 \) and \( mI^5 \subseteq mJJ^3 \). It shows that the reduction number can be strictly bigger than \( n_I \).

**Example 2.10:** Let \( A = \mathbb{Q}[[X,Y,Z,W]]/(X^4 - YZ, Y^5 - Z^2) \), \( L = m = (x,y,z,w) \). Then \( (A,m) \) is a two dimensional Cohen-Macaulay local ring. Let \( I \) be an \( m \)-primary ideal and \( J \) be a minimal reduction of \( I \).

1. Let \( I = (x, y, w) \) and \( J = (x, w) \). Then it can be seen that \( \lambda\left(\frac{mI + J}{mI^2 + J}\right) = 1 \) and \( \lambda\left(\frac{mI}{mJ}\right) = 4 \). Note that \( I \) is stretched with \( n_I = 5 \) and \( mI^5 \subseteq mJJ^3 \).

2. Let \( I = (x, y, z^2, w), J = (x, w) \). Then \( I \) is an \( m \)-primary ideal and \( J \) is a minimal reduction of \( I \) of reduction number 6. It can be seen that \( I \) is a stretched ideal with \( n_I = 5 \), \( \lambda\left(\frac{mI + J}{mI^2 + J}\right) = \lambda\left(\frac{mI}{mJ}\right) = 4 \).
1 and \( I^5 \not\subseteq J I^4 \). Also \( m I^5 \subseteq J, m I^5 \neq m J I^4 \) and \( \lambda \left( \frac{m I}{m J} \right) = 4 \). Therefore \( G(I) \) and \( G(\mathcal{F}) \) are not Cohen-Macaulay. But \( m I^5 \subseteq m J I^3 \). Therefore \( \text{depth } G(\mathcal{F}) = 1 \). Therefore \( \text{depth } F(I) \geq 1 \). It can also be easily verified that \( F(I) \) is in fact Cohen-Macaulay.

3. H-Polynomial and Depth of Fiber Cone of Stretched Ideals

From Theorem 2.5, it follows that if \( I \) is stretched with \( \lambda \left( \frac{LI + J}{LI^2 + J} \right) = 1 \), then \( G(\mathcal{F}_L) \) is Cohen-Macaulay if and only if \( LI^{n_L} = L JJ^{n_L - 1} \). Therefore, in this case, one can see that \( F_L(I) \) is Cohen-Macaulay if \( G(I) \) is Cohen-Macaulay and if \( G(I) \) is not Cohen-Macaulay, then \( \text{depth } F_L(I) = \text{depth } G(I) + 1 \). In this section, we study the properties of \( F_L(I) \) for the next best case namely \( LI^{n_L} \subseteq L J J^{n_L - 2} \). First of all, we remark that if \( LI \cap J = LJ \) and \( \lambda \left( \frac{LI + J}{LI^2 + J} \right) = 1 \), then one can generalize Theorem 3.1 of [13] to get the following two results (proof being pretty much similar):

**Remark 3.1:** (1) Suppose \( LI^{n_I} \not\subseteq J \). Then \( LI^{n_I + 1} \subseteq J J^{n_I - 1} \) if and only if \( \lambda \left( \frac{LI^{n_I}}{L J J^{n_I - 1}} \right) = 1 \) and when this is the case \( \text{depth } G(\mathcal{F}_L) \geq d - 1 \) and for some \( s \geq n_I + 1 \)

\[
h_{\mathcal{F}_L}(z) = \lambda(A/L) + \lambda(L/L I + J)z + z^2 + \cdots + z^{n_I - 1} + \left[ \lambda \left( \frac{LI^{n_I - 1}}{L J J^{n_I - 2}} \right) \right] z^{n_I} + z^s.
\]

(2) \( LI^{n_L} \subseteq L J J^{n_L - 2} \) if and only if \( \lambda \left( \frac{LI^{n_L - 1}}{L J J^{n_L - 2}} \right) = 1 \) and when this is the case \( \text{depth } G(\mathcal{F}_L) \geq d - 1 \) and for some \( s \geq n_L \)

\[
h_{\mathcal{F}_L}(z) = \lambda(A/L) + \lambda(L/L I + J)z + z^2 + \cdots + z^{n_L - 2} + \left[ \lambda \left( \frac{LI^{n_L - 1}}{L J J^{n_L - 3}} \right) \right] z^{n_L - 1} + z^s.
\]

Throughout this section, we will assume that \( I^{n_I + 1} \subseteq J J^{n_I - 1} \). This ensures high depth for \( G(I) \) as well as nice expression for its \( h \)-polynomial. We first study the case \( n_I = n_L - 1 \), equivalently \( LI^{n_I} \not\subseteq J \).

**Theorem 3.2** — Suppose \( I \) is stretched and \( \lambda \left( \frac{LI + J}{LI^2 + J} \right) = 1 \). If \( I^{n_I + 1} \subseteq J J^{n_I - 1} \) and \( LI^{n_I} \not\subseteq J \), then

(i) the \( h \)-polynomial of \( F_L(I) \) is

\[
h_{F_L(I)}(z) = \lambda \left( \frac{A}{L} \right) + \lambda \left( \frac{I}{L I + J} \right) z + z^2 + \cdots + z^{n_I - 1} + z^s + \cdots + z^r
\]

where \( s = \text{deg } h_{\mathcal{F}_L} \geq n_I + 1 \) and \( r = r_J(I) \);
(ii) depth $F_L(I) = d - 1$.

PROOF: Since $I^{n+1} \subseteq JJ^{n-1}$ and $LI^n \notin J$, it follows from [13, Proposition 3.1] and Remark 3.1(1) that depth $G(I) \geq d - 1$ and depth $G(F_L) \geq d - 1$. Therefore depth $F_L(I) \geq d - 1$. Let $x_1, \ldots, x_d$ be a minimal generating set for $J$ such that $x_1^*, \ldots, x_d^*$ is a superficial sequence in $G(I)$ and $x_1^0, \ldots, x_d^0$ is a superficial sequence in $F_L(I)$. Let "\mod" denote modulo the ideal $(x_1, \ldots, x_{d-1})$. Then $G(I)/(x_1^*, \ldots, x_{d-1}^*) \cong G(I)$ and $F_L(I)/(x_1^0, \ldots, x_{d-1}^0) \cong F_L(I)$. Therefore to prove (a) and (b) it is enough to prove the assertion for $d = 1$. Therefore assume that $d = 1$. Then deg $h_{G(I)} = \deg h_{F_L(I)} = r(I)$ and deg $h_{F_L} \leq r(I)$, the reduction number of $I$.

From the exact sequences (1) and (2) we get the relation between Hilbert series

\[(1 - z)HS_{F_L(I)}(z) = HS_{G(F_L)}(z) - zHS_{G(I)}(z)\]

This implies that $(1 - z)h_{F_L(I)}(z) = h_{G(F_L)}(z) - zh_{G(I)}(z)$. Let $h_{F_L(I)}(z) = h_0 + h_1z + \cdots + h_rz^r$. By Theorem 3.1 in [13] we have $h_{G(I)}(z) = \lambda\left(A\left(\frac{I}{J}\right)\right) + hz + z^2 + \cdots + z^{n-I-1} + z^r$, where $r \geq n_I$ and $h = e(I) - \lambda\left(A\left(\frac{I}{J}\right)\right) - n_I + 1$. Suppose $LI^n \notin J$. From Remark 3.1(1), we get

\[h_{F_L}(z) = \lambda\left(A\left(\frac{I}{J}\right)\right) + \lambda\left(LJ\left(\frac{LJ}{LI+J}\right)\right)z + z^2 + \cdots + z^{n-I-1} + \left[\lambda\left(\frac{LI^{n-1}}{LI^{n-1}}\right) - 1\right]z^n + z^s\]

for some $s \geq n_I + 1$. Therefore

\[(1 - z)h_{F_L(I)}(z) = \lambda\left(A\left(\frac{I}{J}\right)\right) + \left[\lambda\left(L\left(\frac{L}{LI+J}\right)\right) - \lambda\left(A\left(\frac{I}{J}\right)\right)\right]z + \left(1 - h\right)z^2 - \left(2 - \lambda\left(LI^{n-I-1}\right)\right)z^n + z^s - z^{r+1}\]

with $r \geq n_I$ and $s \geq n_I + 1$. This implies that

\[h_0 + (h_1 - h_0)z + (h_2 - h_1)z^2 + \cdots + (h_r - h_{r-1})z^r - h_rz^{r+1}\]

\[= \lambda\left(A\left(\frac{I}{J}\right)\right) + \left[\lambda\left(L\left(\frac{L}{LI+J}\right)\right) - \lambda\left(A\left(\frac{I}{J}\right)\right)\right]z + (1 - h)z^2\]

\[\quad - \left(2 - \lambda\left(LI^{n-I-1}\right)\right)z^n + z^s - z^{r+1}\]

with $r \geq n_I, s \geq n_I + 1$. By comparing the coefficients from both sides, we get:

$h_0 = \lambda\left(A\left(\frac{I}{J}\right)\right), h_1 = \lambda\left(\frac{L}{LI+J}\right), h_2 = n_I - \lambda(LI/LJ) = 1 = h_3 = \cdots = h_{n_I-1};$

$h_{n_I} = \lambda\left(LI^{n-I-1}\right) - 1 = h_{n_I+1} = \cdots = h_{s-2} = h_{s-1};$
\[ h_{s-1} + 1 = h_s = h_{s+1} = \cdots = h_r = 1 \text{ so that } \lambda \left( \frac{LI^{n_I-1}}{LIJ^{n_I-2}} \right) = 1 \text{ and hence } h_i = 0 \text{ for } i = n_I, \ldots, s-1. \] This proves (i).

Since \( s \geq n_I + 1 \), we get \( h_{n_I} = 0 \) and hence \( F_L(I) \) is not Cohen-Macaulay. Therefore depth \( F_L(I) = d - 1 \).

The previous result implies that in the case of our main interest, \( L = \mathfrak{m} \) and \( \lambda(\mathfrak{m}I/\mathfrak{m}J) = 2 \), if \( \mathfrak{m}I^{n_I} \not\subseteq J \), then the fiber cone cannot be Cohen-Macaulay. But still it possesses a nice \( h \)-polynomial.

Now we compute the \( h \)-polynomial of \( F_L(I) \) if \( I^{n_I+1} \subseteq JI^{n_I-1} \) and \( LI^{n_I} \subseteq LIJ^{n_I-2} \). Since \( LI^{n_I+1} \subseteq J \) and the case \( LI^{n_I} \not\subseteq J \) has already been discussed, we may as well assume that \( LI^{n_I} \subseteq J \), i.e., \( n_I = n_L \).

**Theorem 3.3** — Suppose \( I \) is stretched with \( \lambda \left( \frac{LI + J}{LI^2 + J} \right) = 1 \), \( I^{n_I+1} \subseteq JI^{n_I-1} \) and \( LI^{n_I} \subseteq LIJ^{n_I-2} \). Then

(i) The \( h \)-polynomial of \( F_L(I) \) is

\[
h_{F_L(I)}(z) = \lambda \left( \frac{A}{L} \right) + \lambda \left( \frac{I}{LI + J} \right) z + z^2 + \cdots + z^{n_I-1} + z^s + \cdots + z^r
\]

where \( s \geq n_I \) and \( r = r_J(I) \);

(ii) depth \( F_L(I) \geq d - 1 \);

(iii) \( F_L(I) \) is Cohen-Macaulay if and only if \( s = n_I \) if and only if \( G(F_L) \) is Cohen-Macaulay.

**Proof:** Since \( I^{n_I+1} \subseteq JI^{n_I-1} \) and \( LI^{n_I} \subseteq LIJ^{n_I-2} \), depth \( G(F_L) \geq d - 1 \), depth \( G(I) \geq d - 1 \) and hence depth \( F_L(I) \geq d - 1 \). This proves (ii). Therefore, as before, we may assume that \( d = 1 \) for the rest of the proof.

From Remark 3.1(2) we get

\[
h_{F_L}(z) = \lambda \left( \frac{A}{L} \right) + \lambda \left( \frac{L}{LI + J} \right) z + z^2 + \cdots + z^{n_L-2} + \left[ \lambda \left( \frac{LI^{n_L-2}}{LIJ^{n_I-3}} \right) - 1 \right] z^{n_L-1} + z^s
\]

for some \( s \geq n_L \). It also follows that \( h_{G(I)}(z) = \lambda \left( \frac{A}{I} \right) + h z + z^2 + \cdots + z^{n_I-1} + z^r \), where \( r \geq n_I \) and \( h = e_0(I) - \lambda \left( \frac{A}{I} \right) - n_I + 1 \). Therefore

\[
(1 - z)h_{F_L(I)}(z) = \lambda \left( \frac{A}{L} \right) + \left[ \lambda \left( \frac{L}{LI + J} \right) - \lambda \left( \frac{A}{I} \right) \right] z + (1 - h)z^2 + \left( \lambda \left( \frac{LI^{n_L-2}}{LIJ^{n_I-3}} \right) - 2 \right) z^{n_L-1} - z^{n_I} + z^s - z^{r+1}
\]
with \( r \geq n_I \) and \( s \geq n_I \). By making a comparison of coefficients from both the ends, we obtain:

\[
h_0 = \lambda(A/L), \quad h_1 = \lambda \left( \frac{I}{LI + J} \right), \quad h_2 = n_I - \lambda \left( \frac{LI}{LJ} \right) = 1 = h_3 = \cdots = h_{n_I - 2};
\]

\[
h_{n_I} = \lambda \left( \frac{LI^n - 2}{LJ^n - 3} \right) - 1, \quad h_{n_I} = h_{n_I - 1} - 1 = h_{n_I + 1} = \cdots = h_{s - 1};
\]

\[
h_s = h_{s - 1} + 1 = h_{s + 1} = \cdots = h_r = 1 \quad \text{so that} \quad \lambda \left( \frac{LI^n - 2}{LJ^n - 3} \right) = 2. \quad \text{Hence} \quad h_{n_I - 1} = 1 \quad \text{and} \quad h_i = 0 \quad \text{for} \quad i = n_I, \ldots, s - 1. \quad \text{This proves (i).}
\]

From Theorem 2.1 in [5], it follows that \( F_L(I) \) is Cohen-Macaulay if and only if

\[
1 + \left( \frac{L}{LI + J} \right) z + z^2 + \cdots z^{n_I - 2} + z^{n_I - 1} + 2z^r = \sum_{i=0}^{r} \lambda \left( \frac{i^i}{JJ - 1 + Li^i} \right) z^i
\]

if and only if \( s = n_I \). Since \( s - 1 \) is the \( L \)-reduction number of \( I \) we have \( LI^s = LJJ^{s - 1} \). Therefore \( F_L(I) \) is Cohen-Macaulay if and only if \( LI^n = LJJ^{n - 1} \). From Theorem 2.5, this happens if and only if \( G(F_L) \) is Cohen-Macaulay.

We conclude this section by proving a necessary and sufficient condition for Gorenstein property of \( F(I) \).

**Corollary 3.4** — Suppose \( I \) is stretched such that \( \lambda \left( \frac{mI + J}{m^2 + J} \right) = 1 \), \( F(I) \) is Cohen-Macaulay, 

\( I^{n_I + 1} \subseteq JJ^{n_I - 1} \) and \( m^{n_I} \subseteq mJJ^{n_I - 2} \). Then the following statements are equivalent.

(a) \( F(I) \) is a hypersurface;

(b) \( F(I) \) is Gorenstein;

(c) \( \mu(I) = d + 1 \).

**Proof** The statement (a) clearly implies (b).

Assume that \( F(I) \) is Gorenstein. Then \( h_{F(I)}(t) \) is symmetric. Therefore from Theorem 3.3(ii), we get \( \mu(I) = d + 1 \). This proves (b) implies (c).

(c) \( \Rightarrow \) (a) This implication is proved in (11, Proposition 4.3).

Putting \( I = m \) in the above corollary we get:

**Corollary 3.5** — If \( m \) is a stretched and \( G(m) \) is Cohen-Macaulay, then \( G(m) \) is Gorenstein if and only if emb.dim.\( (A) = d + 1 \).

**Example 3.6** : Let \( A = \mathbb{Q}[[X, Y, Z, W]]/(XZ - YZ, XZ + Y^3 - Z^2), \quad m = (x, y, z, w) \). Then \( (A, m) \) is a two dimensional Cohen-Macaulay local ring. Let \( I = (x, y, w), \quad J = (x, w) \). Then \( I \) is an \( m \)-primary ideal with \( J \) a minimal reduction of \( I \) of reduction number 3. By using [3] we can check
that \( I \) is a stretched ideal with \( n_L = n_I = 3 \) and \( \lambda \left( \frac{mI}{mJ} \right) = 2 \). Since \( I^4 = JI^3 \), by Theorem 2.5, \( G(I) \) is Cohen-Macaulay. Since \( m^3 = mJI^2 \), from Theorem 2.5, we get \( G(F) \) is Cohen-Macaulay. Therefore \( F(I) \) is Cohen-Macaulay. Since \( \mu(I) = d + 1 = 3 \), by Corollary 3.4, \( F(I) \) is Gorenstein.

4. SOME INEQUALITIES ON THE HILBERT COEFFICIENTS OF \( F(I) \)

In this section we give a lower bound for the degree of \( h_F(z) \) in terms of Hilbert coefficients of \( G(F) \). For the class of ideals we considered in section 3 we prove that \( f_i \geq 0 \) for all \( i \geq 0 \). We prove certain relations between the Hilbert coefficients of the fiber cone.

Let \( d = 1 \). Recall from [15]:

**Remark 4.1** : Let \( v_j := \lambda \left( \frac{mI^{j+1}}{mJI^j} \right) \) for \( j \geq 0 \) and \( h_F(z) = h_0 + h_1 z + \cdots + h_s z^s \) with \( h_i \in \mathbb{Z} \) be the \( h \)-polynomial of \( G(F) \). Then

(a) \( h_i = H_F(i) - H_F(i - 1) \);

(b) Let \( J = (x) \) be a minimal reduction of \( I \). Then \( H_F(n) = e_0(I) - v_{n-1} \).

(c) Since \( h_i = H_F(i) - H_F(i - 1) \), we have \( h_i = e_0(I) - v_{i-1} - [e_0(I) - v_{i-2}] = v_{i-2} - v_{i-1} \) for \( i \geq 2 \). Therefore we obtain

\[
e_1(F) = \sum_{j=0}^{s} jh_j = h_1 + 2h_2 + \cdots + sh_s
\]

\[
= h_1 + 2(v_0 - v_1) + \cdots + (s - 1)(v_{s-3} - v_{s-2}) + s(v_{s-2} - v_{s-1})
\]

\[
= h_1 + 2v_0 + v_1 + \cdots + v_{s-2}
\]

\[
= \sum_{i=0}^{s-2} v_i + (h_1 + v_0).
\]

**Proposition 4.2** — Let \( (A, m) \) be a Cohen-Macaulay local ring of dimension \( d > 0 \). Suppose \( I \) is a stretched \( m \)-primary ideal with \( J \) a minimal reduction of \( I \) such that \( \lambda \left( \frac{mI}{mJ} \right) = 2 \) and \( mI^{n_I} \subseteq mJI \). Then \( \deg h_F \geq \frac{e_1(F) - e_0(I) + 1}{2} \).

**Proof** : Since the degree of the \( h \)-polynomial does not change while going modulo a regular sequence in \( G(F_L) \), we may as well assume that \( d = 1 \). From Corollary 2.7, we get \( h_1 = \lambda \left( \frac{A}{I} \right) - d + \mu(I) - 1 \). It follows from the diagram in page 5 that \( \lambda \left( \frac{A}{I} \right) - d + \mu(I) - 1 = e_0(I) - 1 - \lambda \left( \frac{mI}{mJ} \right) \).
and hence we have $h_1 = e_0(I) - 1 - \lambda \left( \frac{mI}{mJ} \right) = e_0(I) - 1 - v_0$. Substituting the value of $h_1$ in Remark (4.1)(c), we get $e_1(\mathcal{F}) = \sum_{i=0}^{\lambda} v_i + (e_0(I) - 1)$. Since $v_i \leq 2$, this implies that $e_1(\mathcal{F}) \leq 2s + e_0(I) - 1$. That is $s \geq \frac{e_1(\mathcal{F}) - e_0(I) + 1}{2}$ as required.

The example below shows that the $I^{n_{r+1}} \subset J$ doesn’t ensure that $mI^{n_r} \subset mJJ$. We give two examples, one which has this property and one which does not have.

**Example 4.3:** Let $A = \mathbb{Q}[t^7, t^8, t^{12}, t^{13}, t^{18}]$, where $t$ is an indeterminate over $\mathbb{Q}$ and $m = (t^7, t^8, t^{12}, t^{13})$. Then $(A, m)$ is a one dimensional Cohen-Macaulay local ring.

1. Let $I = (t^7, t^8, t^{24}, t^{26})$ and $J = (t^7)$. Then $I$ is an $m$-primary ideal with $J$ a minimal reduction of reduction number 6. It can be easily verified that $I$ is stretched with $\lambda \left( \frac{mI}{mJ} \right) = 2$ and $n_I = 3$. Also $I^4 \not\subseteq JJ^2$, $mI^3 \neq mJJ^2$ and $mI^3 \subseteq mJJ$.

2. Let $I = (t^7, t^8, t^{18})$ and $J = (t^7)$. Then $I$ is an $m$-primary ideal with $J$ a minimal reduction with reduction number 6. It can be easily verified that $I$ is stretched with $\lambda \left( \frac{mI}{mJ} \right) = 2$ and $n_I = 3$. Also $I^4 \not\subseteq JJ^2$, $mI^3 \subseteq J$ and $mI^3 \not\subseteq mJJ$.

Now we prove main results of this section.

**Theorem 4.4** — Suppose $I$ is stretched with $\lambda \left( \frac{mI + J}{mI^2 + J} \right) = 1, n_I \geq 2, I^{n_{r+1}} \subseteq JJ^{n_{r-1}}$ and $mI^{n_r} \not\subseteq J$. Then

(i) $f_i \geq 0$ for all $i \geq 0$;

(ii) $f_1 \geq f_0 + 1$;

(iii) $f_2 \geq f_1 - f_0 + 3$.

If equality holds either in (ii) or in (iii), then $r = s = 3$, $n_I = 2$ and $\lambda(I^2/JI) = 1$.

**Proof:** Assume that $I^{n_{r+1}} \subseteq JJ^{n_{r-1}}$ and $mI^{n_r} \not\subseteq J$. From Theorem 3.2 it follows that $h_{F(I)}(t) = 1 + (\mu(I) - d)t + t^2 + \cdots + t^{n_{r-1}} + t^s + \cdots + t^r$ with $s \geq n_I + 1$. Therefore

$$f_0 = \mu(I) - d + n_I + r - s$$

$$f_1 = \mu(I) - d - 1 + \begin{pmatrix} n_I \\ 2 \end{pmatrix} + \begin{pmatrix} r + 1 \\ 2 \end{pmatrix} - \begin{pmatrix} s \\ 2 \end{pmatrix}$$

and for $i \geq 2$,

$$f_i = \frac{h_{F(I)}^{(i)}(1)}{i!} = 1 + \begin{pmatrix} i + 1 \\ i \end{pmatrix} + \cdots + \begin{pmatrix} n_I - 1 \\ i \end{pmatrix} + \begin{pmatrix} s \\ 2 \end{pmatrix} + \cdots + \begin{pmatrix} r \\ 2 \end{pmatrix} \geq 0.$$
Therefore, we have
\[
f_1 - f_0 = \left[ \mu(I) - d - 1 + \binom{n_I}{2} + \binom{r+1}{2} - \binom{s}{2} \right] - [\mu(I) - d + n_I + r - s] \\
= \left[ \binom{r}{2} - \binom{s}{2} \right] + \left[ \binom{n_I}{2} - n_I + 1 \right] + s - 2.
\]

Since \( r \geq s \geq n_I + 1 \geq 3 \), we get \( \binom{n_I}{2} - n_I + 1 \geq 0 \) and hence \( f_1 - f_0 \geq 1 \). Suppose \( f_1 = f_0 + 1 \). Then we have \( \binom{r}{2} - \binom{s}{2} + \binom{n_I}{2} - n_I + 1 + s - 2 = 1 \). This implies that \( r = s = 3 \) and \( \binom{n_I}{2} - n_I + 1 = 0 \). This gives that \( n_I = 2 \) and \( r = s = n_I + 1 = 3 \). Since \( n_I = \lambda(\frac{I^2}{JI}) \), we get \( \lambda(\frac{I^2}{JI}) = 1 \). Now consider
\[
f_2 - [f_1 - f_0 + 1] = \left[ 1 + \binom{3}{2} + \cdots + \binom{n_I - 1}{2} + \binom{s}{2} + \cdots + \binom{r}{2} \right] \\
- \left[ \binom{r}{2} - \binom{s}{2} + \binom{n_I}{2} - n_I + 1 \right] + s - 1 \\
= \left[ \binom{2}{2} + \binom{3}{2} + \cdots + \binom{n_I - 1}{2} + (n_I - 1) - \binom{n_I}{2} \right] \\
+ \binom{s}{2} + \cdots + \binom{r}{2} - \binom{s}{2} + \binom{s}{2} - s + 1.
\]

This implies that
\[
f_2 - [f_1 - f_0 + 1] \geq 1 + \binom{s}{2} + \cdots + \binom{r}{2} - \binom{s}{2} - s + 1.
\]
\[
\geq 2 \quad (\text{since} \quad r \geq s \geq n_I + 1 \geq 3).
\]

Therefore \( f_2 \geq f_1 - f_0 + 3 \). For the equality, one can argue in a similar manner as in the case of equality in (ii). \( \square \)

Note that in the above case, the fiber cone cannot be Cohen-Macaulay. It can also be seen that if the equality holds in (ii) and (iii) of the previous theorem, then \( n_m = 3 \) which implies that \( \lambda(\frac{mI}{J}) = 2 \). Now we prove a similar expression when \( mI^{n_I} \subseteq J \). Note first that in this case \( n_I = n_m \). If \( n_I = n_m = 2 \), then \( \lambda(mI/mJ) = 1 = \lambda(\frac{I^2}{JI}) \). In this case the fiber cone is completely studied (see [10]). Therefore, we may assume that \( n_I \geq 3 \).

**Theorem 4.5** — Suppose \( I \) is stretched with \( \lambda(\frac{mI + J}{mI^2 + J}) = 1 \), \( n_I \geq 3 \), \( I^{n_I+1} \subseteq J I^{n_I-1} \) and \( mI^{n_I} \subseteq mJ I^{n_I-2} \). Then

(i) \( f_i \geq 0 \) for all \( i \geq 0 \);
(ii) $f_1 \geq f_0 + 2$.

(iii) $f_2 \geq f_1 - f_0 + 1$.

If equality holds either in (ii) or in (iii), then $n_I = 3$ and both $F(I)$ and $G(F)$ are Cohen-Macaulay.

**Proof**: Assume that $I^{n_I+1} \subseteq JJ^{n_I-1}$ and $mI^{n_I} \subseteq mJ^{n_I-2}$. Then by Theorem 3.3 we have $h_F(t) = 1 + (\mu(I) - d) + t^2 + \cdots + t^{n_I-2} + t^{n_I-1} + t^s + \cdots + t^r$. Therefore

$$f_0 = h_F(1) = \mu(I) - d + n_I + r - s,$$

$$f_1 = \mu(I) - d + n_I - 2 + \binom{n_I - 1}{2} + \binom{r + 1}{2} - \binom{s}{2}$$

and for $i \geq 2$,

$$f_i = \frac{h_F^{(i)}(1)}{i!} = 1 + \binom{i + 1}{i} + \cdots + \binom{n_I - 2}{i} + \binom{r}{i} + \binom{s}{i} + \cdots + \binom{r}{i} \geq 0.$$

Thus (i) follows. From the above equations, we have

$$f_1 - f_0 = \left[ \mu(I) - d + n_I - 2 + \binom{n_I - 1}{2} + \binom{r + 1}{2} - \binom{s}{2} \right] - \left[ \mu(I) - d + n_I + r - s \right]$$

$$= \left[ \binom{r}{2} - \binom{s}{2} \right] + \left[ \binom{n_I - 1}{2} - n_I + 2 \right] + s + n_I - 4$$

$$\geq 2n_I - 4 \quad \text{(since } r \geq s \geq n_I \geq 3)$$

$$\geq 2 \quad \text{(since } n_I \geq 3).$$

Thus $f_1 \geq f_0 + 2$. Suppose $f_1 = f_0 + 2$. Then $2n_I - 4 = 2$, i.e., $n_I = 3$. Also we have $\binom{n_I}{2} - \binom{s}{2} + s - 1 = 2$ which implies $r = s = 3$. Thus we have $s = n_I$. Hence by Theorem 3.3(iii) we have $F(I)$ and $G(F)$ are Cohen-Macaulay. Now consider

$$f_2 - [f_1 - f_0 + 1] = \left[ 1 + \binom{3}{2} + \cdots + \binom{n_I - 2}{2} + \binom{n_I - 1}{2} + \binom{s}{2} + \cdots + \binom{r}{2} \right]$$

$$- \left[ \binom{r}{2} - \binom{s}{2} + \binom{n_I - 1}{2} + s + n_I - 1 \right]$$

$$= \left[ 1 + \binom{3}{2} + \cdots + \binom{n_I - 2}{2} + n_I - 2 \right]$$

$$+ \left[ \binom{s}{2} + \cdots + \binom{r}{2} \right] - \left[ \binom{r}{2} - \binom{s}{2} \right] - s - n_I + 3$$

$$\geq \binom{s}{2} + \cdots + \binom{r}{2} - \binom{r}{2} + \binom{s}{2} - 2s + 3 \quad \text{(since } n_I \leq s \text{ we have } -n_I \geq -s)$$

$$\geq 0 \quad \text{(for } n_I \geq 3).$$
If \( f_2 = f_1 - f_0 + 1 \), then as in the case of equality in (ii), one can conclude that \( r = s = n_I = 3 \) and that \( F(I) \) and \( G(F) \) are Cohen-Macaulay. \( \square \)

In [4], the authors prove that if \( I \) is an ideal with \( \dim F(I) = 2 \) and \( a(I) < 0 \), then \( f_1 \leq f_0 - 1 \). As a consequence of Theorems 4.5 and 4.4 we have:

**Corollary 4.6** — Let \((A, m)\) be a two dimensional Cohen-Macaulay local ring with \( A/m \) infinite. Suppose \( I \) is a stretched \( m \)-primary ideal and \( J \) is a minimal reduction of \( I \) with \( \lambda \left( \frac{mI + J}{mI^2 + J} \right) = 1 \) and \( I^{n_I+1} \subseteq J I^{n_I-1} \). Assume that one of the following holds.

(a) \( \lambda \left( \frac{mI^{n_I-2}}{mJ I^{n_I-3}} \right) \leq 3 \) and \( mI^{n_I} \subseteq mJ I^{n_I-2} \);

(b) \( \lambda \left( \frac{mI^{n_I-1}}{mJ I^{n_I-2}} \right) \leq 2 \) and \( mI^{n_I} \nsubseteq J \).

Then

(i) \( a(I) \geq 0 \);

(ii) \( R(I) \) is not Cohen-Macaulay.

**Proof:** Suppose \( \lambda \left( \frac{mI^{n_I-2}}{mJ I^{n_I-3}} \right) \leq 3 \) and \( mI^{n_I} \subseteq mJ I^{n_I-2} \).

(i) Suppose \( a(I) < 0 \). Then by ([4], Theorem 1) we have \( f_1 \leq f_0 - 1 \). This contradicts Theorem 4.5. Therefore \( a(I) \geq 0 \).

(ii) If \( R(I) \) is Cohen-Macaulay, then by Goto-Shimoda Theorem [7, Theorem 3.1], \( a(I) < 0 \) which is a contradiction to (i). Therefore \( R(I) \) is not Cohen-Macaulay.

In a similar manner one can prove the assertion assuming (b) and by using Theorem 4.4. \( \square \)

**Remark 4.7** — Let \((A, m)\) be a two dimensional Cohen-Macaulay local ring and \( I \) is a stretched \( m \)-primary ideal such that \( \lambda \left( \frac{mI}{mJ} \right) = 2 \) and \( I^4 \subseteq J I^2 \). Suppose \( mI^3 \nsubseteq J \) or \( mI^3 \subseteq mJJ \). Then by Corollary 4.6, \( a(I) \geq 0 \) and \( R(I) \) is not Cohen-Macaulay.

**References**


