THE COINCIDENCE NIELSEN NUMBER FOR COVERING MAPS FOR ORIENTABLE MANIFOLDS

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(Received 14 April 2013; after final revision 6 November 2013;
accepted 10 November 2013)

Let \( f, g : M \to N \) be maps between closed orientable manifolds of the same dimension, and let \( p : \tilde{M} \to M \) and \( p' : \tilde{N} \to N \) be finite regular covering maps. In this paper, we show that the ordinary Nielsen number \( N(f, g) \) of \( f \) and \( g \), under certain conditions, can be computed as a linear combination of the Nielsen numbers of specific lifts of the pair \((f, g)\) (called Reidemeister representatives). In addition, we present an easier method to select such representatives than direct computations.

Key words: Nielsen theory; coincidence theory; covering maps; orientable manifolds; reidemeister representatives.

1. INTRODUCTION

Let \( f : X \to X \) be a map on a topological space \( X \), and let \( \Phi(f) = \{ x \in X \mid f(x) = x \} \) be the set of the fixed points of \( f \). It is not always possible to find the set \( \Phi(f) \) or even its cardinality \( |\Phi(f)| \). One of the fundamental studies of this set has been to find an estimate for its cardinality. The most useful estimates are usually lower bounds of \( |\Phi(f)| \). The closer to \( |\Phi(f)| \) the lower bound is, the better the estimate. The Nielsen number \([11, 12]\) is one method used to find such an estimate. It counts a special type of classes (called Nielsen classes) defined by an equivalence relation (called the Nielsen relation) on the elements of \( \Phi(f) \). The importance of the Nielsen number arises from two facts. The first is that it is homotopy invariant. That is, homotopic maps have the same Nielsen number. The other fact is that it is equal, under certain conditions, to the minimum of the set \( \{ |\Phi(f_1)| \mid f_1 \text{ is homotopic to } f \} \).
A drawback of the Nielsen number is that it is difficult to compute. For this reason, Nielsen Theorists search continuously for methods that help compute the Nielsen number.

Let $X$ be a finite polyhedron, and $H$ be a normal subgroup of $\pi_1(X)$ of finite index. Fix a covering $p : \tilde{X} \rightarrow X$ corresponding to $H$; that is, $p\#(\pi_1(\tilde{X})) = H$. If $f_\#(H) \subseteq H$, then $f$ admits a lift $\tilde{f}$, and hence we have the commutative diagram

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X} \\
p & \downarrow & \downarrow p \\
X & \xrightarrow{f} & X 
\end{array}
$$

(1.1)

In [6], Jezierski gave, under certain conditions, a method that computes the Nielsen number $N(f)$ of $f$ as the following linear combination of the Nielsen numbers of its lifts.

$$
N(f) = \sum_{i=1}^{r} (J_i/I_i) \cdot N(\tilde{f}_i),
$$

(1.2)

where $r$ denotes the number of the nonempty Reidemeister classes represented by the lifts $\tilde{f}_i$ of $f$, and $I_i$ and $J_i$ are the order of specific subgroups of $\frac{\pi_1(X)}{H}$.

Let $f, g : M \rightarrow N$ be maps from a topological space $M$ to a topological space $N$, and let $\Phi(f, g) = \{x \in M \mid f(x) = g(x)\}$ be the set of coincidence points of $f$ and $g$. Coincidence Theory is a natural extension of the Fixed Point Theory. The coincidence Nielsen number $N(f, g)$ of $f$ and $g$ is defined to be a homotopy invariant nonnegative integer which is a lower bound of the set

$$
\{||\Phi(f_1, g_1)|| \mid f_1 \text{ is homotopic to } f \text{ and } g_1 \text{ is homotopic to } g\}.
$$

The Nielsen number $N(f, g)$ is homotopy invariant means that, if $f_1$ is a map homotopic to $f$ and $g_1$ is a map homotopic to $g$, then $N(f_1, g_1) = N(f, g)$. In [10], the author has generalized work of Jezierski in [6] to the Coincidence Theory for smooth manifolds. The Nielsen number used in [10] is the semi-index Nielsen number [3], which requires the smoothness of the involved manifolds and maps. In this paper, we generalize the work done in [10] by removing the smoothness condition and considering orientable manifolds. This is possible because the Nielsen number considered in this paper is the index Nielsen number [1], which is defined by the coincidence index [15], and this index requires only orientability of manifolds regardless the smoothness. Precisely, we study the index Nielsen number (the Nielsen number for short), which is defined for closed manifolds (connected compact manifolds without boundary) which are orientable, in the sense that, given finite regular coverings for which the maps $f$ and $g$ admit lifts, we can, under certain conditions, compute the
coincidence Nielsen number $N(f, g)$ as a linear combination of the Nielsen numbers of lifts of the maps $f$ and $g$.

In the second part of the paper, we present an easier method to select the $H$–Reidemeister representatives than the general method. The general method to find the $H$–Reidemeister representatives is to find the $H$–Reidemeister classes of the given pair $(f, g)$ (see Notation 2.1), and chose any pair (we call this pair an $H$–Reidemeister representative) from each class. This straightforward process might be in many cases very long and involve tiring calculations, especially when the number of sheets of each covering map is big. For this reason, and to make Equation 1.3 easier and more reliable formula for computing $N(f, g)$, we try in this paper to search for a shorter and more flexible way to pick up the $H$–Reidemeister representatives. As we shall see, we could not manage to find a general method to do so, however we succeeded to figure out some solutions for some special cases.

In Section 2, we give the necessary background. We first give some properties of covering spaces. Then, the notions of the $H$–Reidemeister number and the index Nielsen number, along with some of their properties are presented.

In Section 3, given finite regular coverings for which $f$ and $g$ admit lifts, we define the three numbers $J_A$, $I_A$, and $S_A$ for a given Nielsen class $A$ of $f$ and $g$. They generalize $J_A$ and $I_A$ given in [6]. Moreover, they are independent of the orientability of the manifolds. These numbers are used to compute the coefficients in the formulas given in Theorem 4.4, which generalize Theorem 4.2 of [6]. Also, we exhibit geometric and algebraic interpretations of these numbers, and give methods of computing them from the fundamental group $\pi_1(M)$. Also, in this section, we give results that help us to use a single coincidence point to compute the numbers $J$, $I$, and $S$ for the $H$–Reidemeister representatives that appear in the formula given in Theorem 4.4.

In Section 4, we give the relationship between the Nielsen classes in the base space $M$, and those of the covering space $\tilde{M}$ through two equations (see Proposition 4.2) that link the indices of those Nielsen classes. In fact, the equations generalize to coincidences those given in Lemma 3.4 of [6] and in Theorem 3.7 of [14] for orientable manifolds. Then, we use the relationships between Nielsen classes to derive Theorem 4.4, which presents the index Nielsen number $N(f, g)$ as a linear combination of the Nielsen numbers of the lifts of $f$ and $g$. Precisely: Let $M$ and $N$ be closed orientable manifolds of the same dimension, and $(\tilde{M}, p)$ and $(\tilde{N}, p')$ be finite regular coverings which correspond to the normal subgroups $K \subseteq \pi_1(M)$ and $H \subseteq \pi_1(N)$, respectively. Let $f, g : M \to N$ be maps for which there exist lifts $\tilde{f}, \tilde{g} : \tilde{M} \to \tilde{N}$, respectively. Suppose the number $J_A$ (See Definition 3.1) is the same for all Nielsen classes $A$ of $f$ and $g$ lying in the same $H$-Nielsen class.
Then
\[ N(f, g) = \frac{1}{|\text{Aut}(\widetilde{M})|^\delta(f, g) - 1} \sum_{i=0}^{r} N(\tilde{f}_i, \tilde{g}_i), \quad (1.3) \]
where the number \( S(\tilde{f}_i, \tilde{g}_i) \) is defined in Definition 3.18, and \( r \) is the number of the \( H \)-Reidemeister representatives (which is equal to the number of the nonempty \( H \)-Reidemeister classes represented by the lifts \( (\tilde{f}_i, \tilde{g}_i) \) of \( (f, g) \)).

In Section 5, we study the relationship between the \( H \)-Reidemeister representatives and the groups of covering transformations \( \text{Aut}(\tilde{M}) \) and \( \text{Aut}(\tilde{N}) \). We show that in some special cases, we can count the \( H \)-Reidemeister representatives. In spite of that our results in this section show only the number of the \( H \)-Reidemeister representative, they can be used to determine them in farther special cases.

In Section 6, we apply our results given in Section 5 to the special case where the groups \( \text{Aut}(\tilde{M}) \) and \( \text{Aut}(\tilde{N}) \) have prime orders. In this case, we determine the \( H \)-Reidemeister representatives completely and give a more elegant form of the Equation 1.3. In fact, the new formula generalizes Theorem 2.5 of [3] and states the following:

Let \( M \) and \( N \) be connected closed smooth manifolds of the same dimension, and let \( (\tilde{M}, p) \) and \( (\tilde{N}, p) \) be regular coverings corresponding to the normal subgroups \( K \subseteq \pi_1(M) \) and \( H \subseteq \pi_1(N) \) of \( M \) and \( N \) respectively. Assume the coverings are finite and that \( |\text{Aut}(\tilde{M})| \) and \( |\text{Aut}(\tilde{N})| \) are prime numbers. Let \( (f, g) : M \rightarrow N \) be a pair of maps for which there exists a pair of lifts \( (\tilde{f}, \tilde{g}) : \tilde{M} \rightarrow \tilde{N} \). If

i. \( \delta(f, g) = 0 \), or

ii. \( \delta(f, g) = 1 \) with \( C(f_\#, g_\#) p(\bar{x}) \subseteq K(p(\bar{x})) \) for every nonempty Nielsen class \([\bar{x}]\) of \((\tilde{f}, \beta^i \tilde{g})\) with \( 0 \leq i \leq |\text{Aut}(\tilde{N})| - 1 \),

then
\[ N(f, g) = \frac{1}{|\text{Aut}(\tilde{M})|^\delta(f, g)} \sum_{i=0}^{[\text{Aut}(\tilde{N})]^\delta(f, g) - 1} N(\tilde{f}, \beta^i \tilde{g}), \quad (1.4) \]
where the number \( \delta(f, g) \) is defined by Definitions 3.8 and 6.3.

In Section 7, we give examples that illustrate and justify the use of Equations 1.3 and 1.4.

2. Preliminaries

In this section, we give the notions of the \( H \)-Reidemeister class and the \( H \)-Reidemeister number, followed by the notions of the \( H \)-Nielsen classes, Nielsen classes, and the index Nielsen number.
Let $M$ and $N$ be path connected, locally path connected topological spaces, and $(\widetilde{M}, p)$ and $(\widetilde{N}, p')$ be regular coverings corresponding to normal subgroups $K \subseteq \pi_1(M)$ and $H \subseteq \pi_1(N)$ of $M$ and $N$, respectively. Let $(f, g) : M \longrightarrow N$ be a pair of maps for which there exists a pair of lifts $(\widetilde{f}, \widetilde{g}) : \widetilde{M} \longrightarrow \widetilde{N}$. Thus, we have the commutative diagram

\[
\begin{array}{ccc}
\widetilde{M} & \xrightarrow{\widetilde{f}, \widetilde{g}} & \widetilde{N} \\
p \downarrow & & \downarrow p' \\
M & \xrightarrow{f, g} & N 
\end{array}
\]  

(2.1)

**Notation 2.1** — In what follows,

- The groups of covering transformations of the covering spaces $(\widetilde{M}, p)$ and $(\widetilde{N}, p')$ are denoted by $\text{Aut}(\widetilde{M})$ and $\text{Aut}(\widetilde{N})$, respectively.

- The sets of lifts of $f$ and $g$ are denoted by $\text{Lift}(f)$ and $\text{Lift}(g)$, respectively.

- The set of lifts of the pair $(f, g)$ is defined by

  \[\text{Lift}(f, g) = \{(\widetilde{f}, \widetilde{g}) | \widetilde{f} \in \text{Lift}(f) \text{ and } \widetilde{g} \in \text{Lift}(g)\}.\]

- The composition of functions $f_1$ and $f_2$ will be denoted by either $f_1 \circ f_2$ or $f_1 f_2$.

- If $\omega$ is a path in the domain of $f$, then the path $f \circ \omega$ is denoted for simplicity by $f(\omega)$.

- To avoid confusion in some parts of this paper, the normal subgroups $K \subseteq \pi_1(M, x)$ and $H \subseteq \pi_1(N, y)$ will be written as $K(x)$ and $H(y)$, respectively.

- To save some ink, an element $\langle w \rangle \in \pi_1(M, x)$, where $w$ is a loop at $x$, will be referred as $w$.

- Let $(\widetilde{f}, \widetilde{g}), (\widetilde{f}', \widetilde{g}') \in \text{Lift}(f, g)$. We say $(\widetilde{f}, \widetilde{g})$ and $(\widetilde{f}', \widetilde{g}')$ are conjugate, if there exist $\alpha \in \text{Aut}(\widetilde{M})$ and $\beta \in \text{Aut}(\widetilde{N})$ such that $(\widetilde{f}', \widetilde{g}') = (\beta \widetilde{f} \alpha, \beta \widetilde{g} \alpha)$. The conjugacy relation on $\text{Lift}(f, g)$ is an equivalence relation. The set of all conjugacy classes (or $H$-Reidemeister classes) of the pair $(f, g)$ is denoted by $R_H(f, g)$. The cardinality of $R_H(f, g)$ (i.e., the $H$–Reidemeister number of $f$ and $g$) is denoted by $R_H(f, g)$.

- If $\omega$ is a path from a point $x$ to a point $y$ in a topological space $X$, then $\omega_\#$ is the well-known isomorphism, induced by $\omega$, from $\pi_1(X, x)$ to $\pi_1(X, y)$. Also, $\overline{\omega}_\#$ is the isomorphism induced by $\omega_\#$ on the quotient group $\pi_1(M, x) / K(x)$. 


• \( f_\# \) denotes the well-known homomorphism, induced by the map \( f \), from \( \pi_1(M) \) to \( \pi_1(N) \).
Also, \( \tilde{f}_\# \) denotes the homomorphism induced by \( f_\# \) on the quotient group \( \frac{\pi_1(M)}{K} \).

Remark 2.2: In the case of fixed points, where \( M = N \) and \( g = 1_M \) is the identity on \( M \), we assume \( \tilde{M} = \tilde{N} \), and of course the covering maps are the same.

Proposition 2.3 [3] — Assume we are given regular coverings as in diagram 2.1. Then,

1. \( \Phi(f, g) = \bigcup_{(\tilde{f}, \tilde{g})} p \Phi(\tilde{f}, \tilde{g}) \), where the index runs over all pairs of lifts.

2. The sets \( p \Phi(\tilde{f}, \tilde{g}) \) and \( p \Phi(\tilde{f}', \tilde{g}) \) are either equal or disjoint.

3. \( p \Phi(\tilde{f}, \tilde{g}) = p \Phi(\tilde{f}', \tilde{g}) \) if and only if \( (\tilde{f}, \tilde{g}) \) and \( (\tilde{f}', \tilde{g}) \) are conjugate.

4. \( \Phi(f, g) = \bigcup_{(\tilde{f}, \tilde{g})} p \Phi(\tilde{f}, \tilde{g}) \) is a disjoint union, where the union takes one \( (\tilde{f}, \tilde{g}) \) from each \( H \) — Reidemeister class.

Definition 2.4 [13] — Let \( H \) be a normal subgroup of \( \pi_1(N) \). Let \( x, y \in \Phi(f, g) \). We say that \( x \) and \( y \) are in the same \( H \)-Nielsen class, and we write \( x \sim_H y \), if there exists a path \( \omega : x \to y \) in \( M \) such that \( f(\omega) \) is homotopic to \( g(\omega) \) relative endpoints (mod \( H \)), which means that \( g(\omega)f(\omega)^{-1} \in H(f(x)) \) (symbolically, \( f(\omega) \sim_H g(\omega) \)).

If \( H = 0 \), then we say that \( x \) and \( y \) are in the same Nielsen class, and we write \( x \sim_0 y \).

For \( x \in \Phi(f, g) \), the symbols \( [x]_H \) and \( [x] \) stand for the \( H \)-Nielsen class and the Nielsen class of \( x \), respectively. Obviously, \( [x] \subseteq [x]_H \).

The next proposition presents an alternative description of the \( H \)-Nielsen classes in terms of the \( H \)-Reidemeister classes.

Proposition 2.5 [3] — Let \( x, y \in \Phi(f, g) \). Then \( x \) and \( y \) belong to the same \( H \)-Nielsen class if and only if there exists a pair \((\tilde{f}, \tilde{g}) \in Lift(f,g)\) such that \( x, y \in p \Phi(\tilde{f}, \tilde{g}) \). Moreover, \((\tilde{f}, \tilde{g})\) is unique up to conjugacy. \( \square \)

Corollary 2.6 [3] — If \( p \Phi(\tilde{f}, \tilde{g}) \neq \emptyset \) for a lift \((\tilde{f}, \tilde{g}) \) of \((f,g)\), then \( p \Phi(\tilde{f}, \tilde{g}) = [x]_H \), for every \( x \in p \Phi(\tilde{f}, \tilde{g}) \). \( \square \)

Next, we define the index Nielsen number. In fact, the concept of the Nielsen number (see [1, 7, 11, 12]) is usually related to the notion of essentiality. However, there are several definitions of essentiality [1, 3, 8]. We focus in this paper on the definition that is related to the notion of index (for the definition of index, see [15]).
Definition 2.7 [13] — Let \((f, g) : M \rightarrow N\) be maps between oriented closed manifolds. A Nielsen class is said to be essential if it has a nonzero index.

The index Nielsen number \(N(f, g)\) (or the Nielsen number for short) of \(f\) and \(g\) is defined to be the number of essential Nielsen classes.

Proposition 2.8 [13] — The Nielsen number \(N(f, g)\) is homotopy invariant. ☐

3. The Numbers \(S, J,\) and \(I\)

In this section, we study three numbers \(S, J\) and \(I\). We generalize the work in [6] related to the numbers \(J\) and \(I\). Then we connect this generalization to our new number \(S\). More precisely, to each Nielsen class \(A \subseteq \Phi(f, g)\), we assign three numbers namely \(J_A, I_A\) and \(S_A\). Under the conditions given, all three numbers are always determined by \(A\) (In fact, \(I_A\) is determined by the \(H\)-Nielsen class containing \(A\)). They have both geometric and algebraic interpretations, and are intimately interrelated. Moreover, they are the major ingredients in the computations of \(N(f, g)\).

Let \(M\) and \(N\) be path connected, locally path connected topological spaces, and \((\widetilde{M}, p)\) and \((\widetilde{N}, p')\) be regular coverings corresponding to the normal subgroups \(K \subseteq \pi_1(M)\) and \(H \subseteq \pi_1(N)\) of \(M\) and \(N\), respectively. Let \((f, g) : M \rightarrow N\) be a pair of maps for which there exists a pair of lifts \((\widetilde{f}, \widetilde{g}) : \widetilde{M} \rightarrow \widetilde{N}\). Consider the commutative diagram 2.1. In [6], in the fixed point case, J. Jezierski used covering spaces to define \(I_A\) and \(J_A\) for a Nielsen class \(A \subseteq \Phi(f)\), and to investigate the relationship between the indices of the Nielsen classes in the base space and in the total space. He showed that essential classes in the total space are mapped onto essential classes in the base space. Our approach is rather to find a similar relationship between the essential classes in both the base and the total spaces. Since the validity of our results requires a nonempty set of coincidences, we assume, without loss of generality, that the set \(\Phi(f, g)\) of coincidence points of \(f\) and \(g\) is nonempty.

Definition 3.1 — Let \(A \subseteq \Phi(f, g)\) and \(\widetilde{A} \subseteq \Phi(\widetilde{f}, \widetilde{g})\) be Nielsen classes such that \(p(\widetilde{A}) = A\), and let \(x \in A\). Define \(J_A\) by \(J_A := |p^{-1}(x) \cap \widetilde{A}|\).

Definition 3.2 — Let \(x \in \Phi(f, g)\). We define

\[C(f_\#, g_\#)_x = \{\gamma \in \pi_1(M, x) | f_\#(\gamma) = g_\#(\gamma)\}.\]

The following two propositions show that \(J_A\) is well defined. Furthermore, the first of them shows that \(J_A\) is the order of a specific subgroup of \(\text{Aut}(\widetilde{M}) \cong \frac{\pi_1(M)}{K}\).

Proposition 3.3 — Let \(A\) be a Nielsen class of \(f\) and \(g\), and let \(x \in A\). Then \(J_A = |j(C(f_\#, g_\#)_x)|\), where \(j : \pi_1(M, x) \rightarrow \frac{\pi_1(M, x)}{K(x)}\) is the natural epimorphism.
PROOF: Let $\tilde{A} \subseteq \Phi(\tilde{f}, \tilde{g})$ be a Nielsen class such that $p(\tilde{A}) = A$, and let $\tilde{x}_0 \in p^{-1}(x) \cap \tilde{A}$. For each $\lambda \in \pi_1(M, x)$, let $\tilde{\lambda} : \tilde{x}_0 \longrightarrow \tilde{\lambda}(1)$ be the unique lift of $\lambda$ which starts at $\tilde{x}_0$. Then, the function $\varphi : j(C(f_\#, g_\#)_x) \longrightarrow p^{-1}(x) \cap \tilde{A} : \lambda \longmapsto \tilde{\lambda}(1)$ is a bijection. Thus, $J_A = |j(C(f_\#, g_\#)_x)|$. \hfill \Box

Remark 3.4: If we change the base point $\tilde{x}_0 \in p^{-1}(x) \cap \tilde{A}$, and follow the same argument as above, we find that $J_A$ is the same. This means that $J_A$ is independent of choosing $\tilde{x}_0 \in p^{-1}(x) \cap \tilde{A}$.

Proposition 3.5 — Let $\tilde{A} \subseteq \Phi(\tilde{f}, \tilde{g})$ be a Nielsen class such that $p(\tilde{A}) = A$, then $J_A$ is independent of the choice of $x \in A$.

PROOF: Let $z$ be another point in $A$ and $\delta : x \longrightarrow z$ be a path in $M$ such that $f(\delta)$ is homotopic to $g(\delta)$ rel. endpoints. The commutativity of the diagram

$$
\begin{array}{ccc}
\pi_1(M, x) & \xrightarrow{\delta_\#} & \pi_1(M, z) \\
j \downarrow & & \downarrow j \\
\pi_1(M, x) & \xrightarrow{\delta_\#} & \pi_1(M, z) \\
K(x) & \xrightarrow{\delta_\#} & K(z)
\end{array}
$$

implies that $j(C(f_\#, g_\#)_x)$ and $j(C(f_\#, g_\#)_z)$ are isomorphic. This means that $J_A$ is independent of the choice of $x \in A$.

Remark 3.6: Since the above argument is the same for each Nielsen class $\tilde{A} \subseteq \Phi(\tilde{f}, \tilde{g})$ with $p(\tilde{A}) = A$, we conclude that $J_A$ depends only on the Nielsen class $A$.

Definition 3.7 — Let $A \subseteq \Phi(f, g)$ and $\tilde{A} \subseteq \Phi(\tilde{f}, \tilde{g})$ be Nielsen classes such that $p(\tilde{A}) = A$, and let $x \in A$. Define $I_A$ by $I_A := |p^{-1}(x) \cap \Phi(\tilde{f}, \tilde{g})|$. Let $(\tilde{f}, \tilde{g})$ be a lift of $(f, g)$, and $\alpha \in Aut(M)$. There exist unique functions $\beta, \hat{\beta} \in Aut(\tilde{N})$ such that $\tilde{f} \circ \alpha = \beta \circ \tilde{f}$ and $\tilde{g} \circ \alpha = \hat{\beta} \circ \tilde{g}$.

Definition 3.8 — We define the number $\delta(\tilde{f}, \tilde{g}; \alpha)$ by

$$\delta(\tilde{f}, \tilde{g}; \alpha) = \begin{cases} 
0 & \text{if } \beta \neq \beta' \\
1 & \text{if } \beta = \beta'. 
\end{cases}$$

Definition 3.9 — Set $\Gamma(\tilde{f}, \tilde{g}) = \left\{ \alpha \in Aut(M) \mid \delta(\tilde{f}, \tilde{g}; \alpha) = 1 \right\}$.

Proposition 3.10 — Let $(\tilde{f}, \tilde{g})$ be a lift of $(f, g)$, and $\beta \in Aut(\tilde{N})$. Then

1. $\Gamma(\tilde{f}, \tilde{g})$ is a subgroup of $Aut(M)$. 

2. \( \Gamma(\tilde{f}, \tilde{g}) = \Gamma(\beta \tilde{f}, \beta \tilde{g}) \).

**Proof:** (1) Obviously, \( 1_{\tilde{g}} \in \Gamma(\tilde{f}, \tilde{g}) \). If \( \alpha_1, \alpha_2, \alpha \in \Gamma(\tilde{f}, \tilde{g}) \), then \( \tilde{f}(\alpha_1 \alpha_2) = (\beta_1 \beta_2) \tilde{f} \), and \( \tilde{g}(\alpha_1 \alpha_2) = (\beta_1 \beta_2) \tilde{g} \). Thus, \( \alpha_1 \alpha_2 \in \Gamma(\tilde{f}, \tilde{g}) \). On the other hand,

\[
\begin{align*}
\tilde{f} \alpha &= \beta \tilde{f} \\
\tilde{g} \alpha &= \beta \tilde{g}
\end{align*}
\]

\[\Rightarrow \begin{cases}
\beta^{-1} \tilde{f} = \tilde{f} \alpha^{-1} \\
\beta^{-1} \tilde{g} = \tilde{g} \alpha^{-1}
\end{cases} \Rightarrow \alpha^{-1} \in \Gamma(\tilde{f}, \tilde{g}).
\]

(2) To prove \( \Gamma(\tilde{f}, \tilde{g}) \subseteq \Gamma(\beta \tilde{f}, \beta \tilde{g}) \), let \( \alpha \in \Gamma(\tilde{f}, \tilde{g}) \) and \( \gamma \in Aut(\tilde{N}) \) such that \( \tilde{f} \alpha = \gamma \tilde{f} \) and \( \tilde{g} \alpha = \gamma \tilde{g} \). Then

\[
\begin{align*}
(\beta \tilde{f}) \alpha &= \beta (\tilde{f} \alpha) = \beta \gamma \tilde{f} \\
(\beta \tilde{g}) \alpha &= \beta (\tilde{g} \alpha) = \beta \gamma \tilde{g}
\end{align*}
\]

\[\Rightarrow \begin{cases}
(\beta \tilde{f}) \alpha = \beta \gamma \beta^{-1} \tilde{f} \\
(\beta \tilde{g}) \alpha = \beta \gamma \beta^{-1} \tilde{g}
\end{cases}
\]

Hence, \( \alpha \in \Gamma(\beta \tilde{f}, \beta \tilde{g}) \). Since the above argument holds for every \( \beta \in Aut(\tilde{N}) \) and \( (\tilde{f}, \tilde{g}) \in \text{Lift}(f, g) \), we get

\[
L(\beta \tilde{f}, \beta \tilde{g}) \subseteq L(\beta^{-1} \tilde{f}, \beta^{-1} \tilde{g}) = \Gamma(\tilde{f}, \tilde{g}).
\]

The next work shows that the number \( I_A \) is well-defined, i.e., it depends only on the \( H \)–Nielsen class that contains \( A \). Further, it shows that \( I_A \) is equal to the order of a particular subgroup of \( Aut(\tilde{M}) \).

**Proposition 3.11** — Let \( A \subseteq \Phi(f, g) \) be a Nielsen class and \( x \in A \). Then \( I_A = |\Gamma(\tilde{f}, \tilde{g})| \).

**Proof:** Fix a point \( \tilde{x}_0 \in p^{-1}(x) \cap \Phi(\tilde{f}, \tilde{g}) \). Consider the bijection \( \xi : p^{-1}(x) \longrightarrow Aut(\tilde{M}) \) given by \( \xi(\tilde{x}) = \alpha \), where \( \alpha \) is the unique covering transformation in \( Aut(\tilde{M}) \) with \( \alpha(\tilde{x}_0) = \tilde{x} \). It follows that the restriction (for simplicity we call it \( \xi \) too)

\[
\xi : p^{-1}(x) \cap \Phi(\tilde{f}, \tilde{g}) \longrightarrow \xi \left( p^{-1}(x) \cap \Phi(\tilde{f}, \tilde{g}) \right) \subseteq Aut(\tilde{M})
\]

is also a bijection. We claim that \( \xi \left( p^{-1}(x) \cap \Phi(\tilde{f}, \tilde{g}) \right) = \Gamma(\tilde{f}, \tilde{g}) \).

Let \( \alpha \in \xi \left( p^{-1}(x) \cap \Phi(\tilde{f}, \tilde{g}) \right) \), then there exists an \( \tilde{x} \in \tilde{M} \) such that \( \alpha(\tilde{x}_0) = \tilde{x} \), with \( p(\tilde{x}) = x \) and \( \tilde{f}(\tilde{x}) = \tilde{g}(\tilde{x}) \). Hence,

\[
\tilde{f}(\tilde{x}) = \tilde{g}(\tilde{x}) \Rightarrow \tilde{f} \circ \alpha(\tilde{x}_0) = \tilde{g} \circ \alpha(\tilde{x}_0) \Rightarrow \beta \tilde{f}(\tilde{x}_0) = \beta' \tilde{g}(\tilde{x}_0) = \beta' \tilde{f}(\tilde{x}_0) \Rightarrow \beta = \beta'.
\]
Thus $\alpha \in \Gamma(\tilde{f}, \tilde{g})$. Now, let $\alpha \in \Gamma(\tilde{f}, \tilde{g})$. Then

$$
\delta(\tilde{f}, \tilde{g}; \alpha) = 1 \Rightarrow \tilde{f} \alpha = \beta \tilde{f} \text{ and } \tilde{g} \alpha = \beta \tilde{g} \text{ for some } \beta \in \text{Aut}(\tilde{N}).
$$

$$
\Rightarrow \tilde{f} \alpha(\tilde{x}_0) = \beta \tilde{f}(\tilde{x}_0) = \beta \tilde{g}(\tilde{x}_0) = \tilde{g} \alpha(\tilde{x}_0)
$$

$$
\Rightarrow \alpha(\tilde{x}_0) \in \Phi(\tilde{f}, \tilde{g}).
$$

Since \( \tilde{x} = \alpha(\tilde{x}_0) \in p^{-1}(x) \), we get $\alpha \in \xi\left(p^{-1}(x) \cap \Phi(\tilde{f}, \tilde{g})\right)$. Consequently,

$$
\xi\left(p^{-1}(x) \cap \Phi(\tilde{f}, \tilde{g})\right) = \Gamma(\tilde{f}, \tilde{g}), \text{ and } I_A = |\Gamma(\tilde{f}, \tilde{g})|.
$$

\[\square\]

**Remark 3.12**: In the case that $M = N$ and $g = 1_M$ is the identity on $M$, it is easy to see that

$$
\Gamma(\tilde{f}, 1_{\tilde{N}}) = \{ \alpha \in \text{Aut}(\tilde{M}) | \alpha \tilde{f} = \tilde{f} \alpha \} = \Gamma(\tilde{f}),
$$

where $\Gamma(\tilde{f})$ is defined in Lemma 3.3 of [6]. On the other hand, the proof of Proposition 3.11 is independent of the choice of $\tilde{x}_0 \in \Phi(\tilde{f}, \tilde{g})$. Since $|\Gamma(\tilde{f}, \tilde{g})|$ is independent of the choice of $\tilde{x}_0 \in \Phi(\tilde{f}, \tilde{g})$, we have that $I_A$ is independent of the choice of the Nielsen class contained in $\Phi(\tilde{f}, \tilde{g})$. Therefore, for any pair of lifts $(\tilde{f}, \tilde{g})$ of $(f, g)$ we can put $I(\tilde{f}, \tilde{g}) = I_A$ for any $A \subseteq \Phi(\tilde{f}, \tilde{g})$.

Let $x_0$ be a coincidence point of $f$ and $g$ and $y_0 = f(x_0)$. Since $f_#(K(x_0)) \cup g_#(K(x_0)) \subseteq H(f(x_0))$, $f_#$ and $g_#$ induce homomorphisms $\tilde{f}_#$ and $\tilde{g}_#$ which are defined such that the following diagram is commutative:

$$
\begin{array}{ccc}
\pi_1(M, x_0) & \xrightarrow{f_#, g_#} & \pi_1(N, f(x_0)) \\
\downarrow \quad j & & \downarrow j \\
\pi_1(M, x_0) & \xrightarrow{\tilde{f}_#, \tilde{g}_#} & \pi_1(N, f(x_0)) \\
K(x_0) & & H(f(x_0))
\end{array}
$$

**Definition 3.13** — Define $C(\tilde{f}_#, \tilde{g}_#)_{x_0}$ by

$$
C(\tilde{f}_#, \tilde{g}_#)_{x_0} = \left\{ \tilde{a} \in \frac{\pi_1(M, x_0)}{K(x_0)} | \quad \tilde{f}_#(\tilde{a}) = \tilde{g}_#(\tilde{a}) \right\}.
$$

Let $A \subseteq p \Phi(\tilde{f}, \tilde{g})$ be a Nielsen class, and let $x_0 \in A$. We show that $I_A$ is equal to the order of the subgroup $C(\tilde{f}_#, \tilde{g}_#)_{x_0}$. Fix $\tilde{x}_0 \in p^{-1}(x_0) \cap \Phi(\tilde{f}, \tilde{g})$. Let $f(x_0) = g(x_0) = y_0$, and $\tilde{f}(\tilde{x}_0) = \tilde{g}(\tilde{x}_0) = \tilde{y}_0$. Consider the bijections

$$
\frac{\pi_1(M, x_0)}{K(x_0)} \xrightarrow{\nu} \text{Aut}(\tilde{M}) : \quad \tilde{a} \mapsto \alpha_{\tilde{a}},
$$

where $\alpha_{\tilde{a}}(\tilde{x}_0) = \tilde{a}(1)$, and $\tilde{a}$ is the lift of $a$ at $\tilde{x}_0$, and

$$
\frac{\pi_1(M, y_0)}{H(y_0)} \xrightarrow{\nu} \text{Aut}(\tilde{N}) : \quad \tilde{b} \mapsto \beta_{\tilde{b}},
$$

where $\beta_{\tilde{b}}(\tilde{x}_0) = \tilde{b}(1)$.
where $\beta_{\tilde{b}}(\tilde{y}_0) = \tilde{b}(1)$, and $\tilde{b}$ is the lift of $b$ at $y_0$.

**Lemma 3.14** — Let $(\tilde{f}, \tilde{g})$ be a lift of $(f, g)$. Then

$$\tilde{f}\alpha_{\tilde{a}} = \beta_{f(a)} \tilde{f}$$

and

$$\tilde{g}\alpha_{\tilde{a}} = \beta_{g(a)} \tilde{g}.$$  

**Proof:** Since $\tilde{f}(\tilde{a})$ is a lift of $f(a)$, we have $\beta_{f(a)}(\tilde{y}_0) = \tilde{f}(\tilde{a})(1)$. Thus,

$$\tilde{f}\alpha_{\tilde{a}}(\tilde{x}_0) = \tilde{f}(\alpha_{\tilde{a}}(\tilde{x}_0)) = \tilde{f}(\tilde{a}(1)) = \tilde{f}(\tilde{a})(1) = \beta_{f(a)}(\tilde{y}_0) = \beta_{f(a)}(\tilde{f}(\tilde{x}_0)).$$

Since $\tilde{f}\alpha_{\tilde{a}}$ and $\beta_{f(a)} \tilde{f}$ are lifts of $f$, we get $\tilde{f}\alpha_{\tilde{a}} = \beta_{f(a)} \tilde{f}$.

Similarly, $\tilde{g}\alpha_{\tilde{a}} = \beta_{g(a)} \tilde{g}$.

**Proposition 3.15** — Let $(\tilde{f}, \tilde{g})$ be a lift of $(f, g)$. Then, there is a bijection between $\Gamma(\tilde{f}, \tilde{g})$ and $C(\tilde{f}_\#, \tilde{g}_\#)_{x_0}$.

**Proof:** The restriction of the isomorphism $\nu$, which we call $\nu$ too, given just before Lemma 3.14

$$C(\tilde{f}_\#, \tilde{g}_\#)_{x_0} \xrightarrow{\nu} \nu(C(\tilde{f}_\#, \tilde{g}_\#)_{x_0}) \subseteq \text{Aut}(\tilde{M})$$

is also an isomorphism. We claim that $\nu(C(\tilde{f}_\#, \tilde{g}_\#)_{x_0}) = \Gamma(\tilde{f}, \tilde{g})$.

Let $\tilde{a} \in C(\tilde{f}_\#, \tilde{g}_\#)_{x_0}$. We have $\tilde{f}_\nu(\tilde{a}) = \nu(f(a)) \tilde{f}$ and $\tilde{g}_\nu(\tilde{a}) = \nu(g(a)) \tilde{g}$. Since $f(a) = g(a)$, we have $\nu(f(a)) = \nu(g(a))$, i.e., $\delta(\tilde{f}, \tilde{g}; \nu(\tilde{a})) = 1$, which yields $\nu(\tilde{a}) \in \Gamma(\tilde{f}, \tilde{g})$.

On the other hand, assume $\alpha \in \Gamma(\tilde{f}, \tilde{g})$. Hence, there exists $\tilde{a} \in \frac{\pi_1(M, x_0)}{K(x_0)}$, such that $\alpha = \nu(\tilde{a})$. Thus, $\tilde{f}_\nu(\tilde{a}) = \nu(f(a)) \tilde{f}$ and $\tilde{g}_\nu(\tilde{a}) = \nu(g(a)) \tilde{f}$ with $\nu(f(a)) = \nu(g(a))$. Because $\nu$ is an isomorphism, $f(a) = g(a)$ or $\tilde{a} \in C(\tilde{f}_\#, \tilde{g}_\#)_{x_0}$. Therefore, $\alpha \in \nu(C(\tilde{f}_\#, \tilde{g}_\#)_{x_0})$. \(\Box\)

In [6], Jezierski used the fact that every covering map is a local homeomorphism to exhibit the relationship between the indices of the fixed point classes $A$ and $\tilde{A}$ (Lemma 3.4). He used this relationship to derive the formula in Equation 1.2. However, the same idea does not work in the Coincidence Theory. So, to turn around this drawback and derive similar equations that exhibit the relationship between the indices of the Nielsen classes in the base space and those in the total space, we introduce the number $S$, and then exhibit the relationship among the numbers $|\tilde{A}|, |A|, |p_\phi^{-1}(A)|, J_A, J_{\tilde{A}}$, and $S_A$. After that, we are able to derive the equations that connects the indices of the Nielsen classes in the total and the base spaces (Proposition 4.2). The complete picture is given in the following sequence of results.
The following proposition generalizes the last part of Lemma 3.1 of [6].

**Proposition 3.16** — If $M$ and $N$ are connected compact manifolds, and $A$ is a Nielsen class, then the set $p^{-1}(A) \cap \Phi(\tilde{f}, \tilde{g})$, where $(\tilde{f}, \tilde{g}) \in \text{Lift}(f, g)$, is either empty, or splits into a finite union of nonempty Nielsen classes of $(\tilde{f}, \tilde{g})$.

**Proof:** The proof is similar to that of Lemma 3.1 of [6].

The proof of the following lemma is easy.

**Lemma 3.17** — Assume we are given finite regular coverings as in Diagram 2.1. Then the following are equivalent

1. $\Phi(f, g)$ is finite.
2. $\Phi(\tilde{f}, \beta \cdot \tilde{g})$ is finite, for each $\beta \in Aut(\tilde{N})$.
3. $\Phi(\tilde{f}, \tilde{g})$ is finite, for each lift $(\tilde{f}, \tilde{g})$ of $(f, g)$.

**Definition 3.18** — Assume we are given finite regular coverings as in Diagram 2.1.1. Let $A \subseteq p \phi(\tilde{f}, \tilde{g})$ be a Nielsen class. We define $S_A$ to be the number of Nielsen classes $\tilde{A} \subseteq \phi(\tilde{f}, \tilde{g})$ such that $p(\tilde{A}) = A$.

The next proposition gives important relationships among the numbers $|\tilde{A}|$, $|A|$, $|p^{-1}(A)|$, $J_A$, $I_A$, and $S_A$.

**Proposition 3.19** — Assume that $\Phi(f, g)$ is finite. Let $A \subseteq \Phi(f, g)$ and $\tilde{A} \subseteq \Phi(\tilde{f}, \tilde{g})$ be Nielsen classes such that $p(\tilde{A}) = A$. Then

1. $|\tilde{A}| = J_A \cdot |A|$.
2. $|p^{-1}(A) \cap \Phi(\tilde{f}, \tilde{g})| = I_A \cdot |A|$.
3. $S_A = \frac{I_A}{J_A}$.

**Proof:** (1) Since the family $\{\tilde{A} \cap p^{-1}(x)\}$ for all $x \in \tilde{A}$ is a partition of $\tilde{A}$, and $J_A = |p^{-1}(x) \cap \tilde{A}|$, we have

$$|\tilde{A}| = \sum_{x \in A} |\tilde{A} \cap p^{-1}(x)| = \sum_{x \in A} J_A = J_A \cdot |A|.$$

(2) $|p^{-1}(A) \cap \Phi(\tilde{f}, \tilde{g})| = \sum_{x \in A} \frac{|p^{-1}(x) \cap \Phi(\tilde{f}, \tilde{g})|}{I_A} = \sum_{x \in A} I_A = I_A \cdot |A|$.
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(3) Assume \( p^{-1}(A) \cap \Phi(\tilde{f}, \tilde{g}) = \bigcup_{j=1}^{S_A} \tilde{A}_j \), where \( \tilde{A}_j \) is a Nielsen class of \( \tilde{f} \) and \( \tilde{g} \) such that \( p(\tilde{A}_j) = A \), for every \( j \) with \( 1 \leq j \leq S_A \). Condition (1) yields

\[
|p^{-1}(A) \cap \Phi(\tilde{f}, \tilde{g})| = \sum_{j=1}^{S_A} |\tilde{A}_j| = \sum_{j=1}^{S_A} J_A \cdot |A| = S_A \cdot J_A \cdot |A|,
\]

which implies by (2) that \( S_A = \frac{I_A}{J_A} \). \( \square \)

Remark 3.20: Since the number \( I \) depends only on the \( H \)-Nielsen class, by Proposition 3.19, \( J \) only depends on the \( H \)-Nielsen class if and only if the number \( S \) does. In this case, we write \( J_A = J(\tilde{f}, \tilde{g}) \) and \( S_A = S(\tilde{f}, \tilde{g}) \), where \( A \) is any Nielsen class in \( p \Phi(\tilde{f}, \tilde{g}) \).

So far, we have seen that the numbers \( J, I, \) and \( S \) depend on the Nielsen class or the \( H \)-Nielsen class. Therefore, to compute any of these numbers for a Nielsen class, we should pick out a coincidence point in that class and use Propositions 3.3, 3.11, 3.15 or 3.19. So we expect to do such computations several times as many as the Nielsen classes of \( f \) and \( g \), which we don’t know in the first place. For this reason, we show in the reminder of this section that one coincidence point is sufficient to compute those numbers, for all Nielsen classes (and of course for all \( H \)-Nielsen classes).

Definition 3.21 — Let \( x \) and \( z \) be coincidence points and \( \omega : x \to z \) be a path. We denote the loop \( g(\omega) f(\omega)^{-1} \) by \( h_\omega \), and define \( C(f^h_\#, g^h_\#)_x \) by

\[
C(f^h_\#, g^h_\#)_x = \left\{ \lambda \in \pi_1(M,x) \mid f^h_\#(\lambda) = g^h_\#(\lambda) \right\}
\]

\[
= \left\{ \lambda \in \pi_1(M,x) \mid f_\omega f(\lambda) = g(\lambda) h_\omega \right\},
\]

where \( f^h_\# = h_\omega \circ f_\# \circ h_\omega^{-1} \). We also define \( C(\tilde{f}^h_\#, \tilde{g}^h_\#)_x \) by

\[
C(\tilde{f}^h_\#, \tilde{g}^h_\#)_x = \left\{ \bar{\lambda} \in \pi_1(M,x) \mid \tilde{f}^h_\#(\bar{\lambda}) = \tilde{g}^h_\#(\bar{\lambda}) \right\}
\]

\[
= \left\{ \bar{\lambda} \in \pi_1(M,x) \mid \tilde{h}_\omega \tilde{f}(\bar{\lambda}) = \tilde{g}(\bar{\lambda}) \tilde{h}_\omega \right\},
\]

The proof of the next lemma is simple.

Lemma 3.22 — Let \( x_0 \in \Phi(f,g) \). Then, \( f_\#(C(f_\#, g_\#)_{x_0}) = g_\#(C(f_\#, g_\#)_{x_0}) \). \( \square \)

The next proposition generalizes Lemma 3.10 of [6]. It also shows that \( C(f_\#, g_\#) \) and \( C(\tilde{f}_\#, \tilde{g}_\#) \), and hence \( I, J \) and \( S \), can be computed using a single coincidence point.

Proposition 3.23 — Let \( x_0 \) and \( x \) be coincidence points, and \( \omega : x_0 \to x \) be a path. Then
1. $\omega_\# \left(C(f^{h\omega}_\#, g\#)_{x_0}\right) = C(f_\#, g\#)_x$;

2. $C(f^{h\omega}_\#, g\#)_{x_0} = C(f_\#, g\#)_{x_0}$ if and only if $h_\omega$ commutes with $f_\# (C(f_\#, g\#)_{x_0})$; and

3. $\overline{\omega}_\# \left(C(f^{h\omega}_\#, \overline{g\#})_{x_0}\right) = C(f_\#, \overline{g\#})_x$;

**Proof:** (1) Let $\sigma \in C(f_\#, g\#)_x$ and $\lambda \in \pi_1(M, x_0)$ such that $\sigma = \omega_\#(\lambda)$. Then

$$
\sigma \in C(f_\#, g\#)_x \iff f(\omega^{-1} \lambda \omega) = g(\omega^{-1} \lambda \omega) \\
\iff g(\omega) f(\omega^{-1}) f(\lambda) = g(\lambda) g(\omega) f(\omega^{-1}) \\
\iff h_\omega f(\lambda) = g(\lambda) h_\omega \\
\iff \lambda \in C(f^{h\omega}_\#, g\#)_{x_0} \\
\iff \sigma \in \omega_\# \left(C(f^{h\omega}_\#, g\#)_{x_0}\right).
$$

(2) Suppose that $h_\omega$ commutes with $f_\# (C(f_\#, g\#)_{x_0})$. We have

$$
\lambda \in C(f_\#, g\#)_{x_0} \iff f(\lambda) = g(\lambda) \\
\iff f(\lambda) h_\omega = g(\lambda) h_\omega \\
\iff h_\omega f(\lambda) = g(\lambda) h_\omega \\
\iff \lambda \in C(f^{h\omega}_\#, g\#)_{x_0}.
$$

The rest of the proof is easy to carry out.

(3) Let $\bar{a} \in \pi_1(M, x_0)$, $K(x_0)$, and $e$ be the identity of $H(f(x_0))$. We have

$$
\bar{b} = \omega_#(\bar{a}) \text{ and } \bar{a} \in C(f^{h\omega}_\#, \overline{g\#})_{x_0} \iff \bar{b} = \omega_#(\bar{a}) \text{ and } \overline{h_\omega}_# \overline{f_#}(\bar{a}) = \overline{g_#}(\bar{a}) \overline{h_\omega} \\
\iff \overline{h_\omega}_# \overline{f_#}(\omega^{-1}_#(\bar{b})) = \overline{g_#}(\omega^{-1}_#(\bar{b})) \overline{h_\omega} \\
\iff \overline{h_\omega}_# \overline{f_#}(\omega^{-1}_#(\bar{b})) = \overline{g_#}(\omega^{-1}_#(\bar{b})) \overline{h_\omega} \\
\iff \overline{h_\omega}_# \overline{f_#}(\omega^{-1}_#(\bar{b})) = g_#(\omega^{-1}_#(\bar{b})) \overline{h_\omega} \\
\iff g(\omega) f(\omega^{-1}) f(\omega) f(b) f(\omega^{-1}) = g(\omega) g(b) g(\omega^{-1}) g(\omega) f(\omega^{-1}) \\
\iff g(\omega) f(\omega^{-1}) f(\omega) f(b) f(\omega^{-1}) = g(\omega) g(b) g(\omega^{-1}) g(\omega) f(\omega^{-1}) \\
\iff g(\omega) f(b) f(\omega^{-1}) = g(\omega) g(b) f(\omega^{-1}) \\
\iff (g(\omega) g(b) f(\omega^{-1}))^{-1} g(\omega) f(b) f(\omega^{-1}) = e
$$
\[ \Leftrightarrow f(\omega) g(b)^{-1} f(b) f(\omega^{-1}) = e \]
\[ \Leftrightarrow f(\omega) g(b)^{-1} f(b) f(\omega^{-1}) = h \in H(f(x_0)) \]
\[ \Leftrightarrow g(b)^{-1} f(b) = f(\omega)^{-1} h f(\omega) \in H(f(x)) \]
\[ \Leftrightarrow \tilde{g}(b) = \tilde{f}(b) \iff b \in C(f_\#, \tilde{g}_\#)_\times. \]

**Corollary 3.24** — Let \( x_0 \) and \( x \) be coincidence points and \( \omega: x_0 \to x \) be a path. Then, \( \omega_\# (C(f_\#, g_\#)_{x_0}) = C(f_\#, g_\#)_x \) if and only if \( h_\omega \) commutes with \( f_\# (C(f_\#, g_\#)_{x_0}) \).

**Proof:** Apply conditions (1) and (2) of Proposition 3.23. \( \square \)

**Remark 3.25** — If, in Proposition 3.23, \( x \) and \( x_0 \) belong to the same \( H \)-Nielsen class, then \( h_\omega = 1 \), and hence \( C(f_\#, \tilde{g}_\#)_{x_0} = C(f_\#, \tilde{g}_\#)_x \).

The next definition allows us to change from the covering space approach to the fundamental group approach.

**Definition 3.26** — Let \( \tilde{\Phi}(f, g) \) be the set of all nonempty Nielsen classes of \( f \) and \( g \). Fix \( x_0 \in \Phi(f, g) \). For every \( x \in \Phi(f, g) \), define \( \omega \) and \( h_\omega \) as in Definition 3.21. Consider the injection
\[ \rho: \tilde{\Phi}(f, g) \to \mathcal{R}(f_\#, g_\#) : [x] \mapsto [h_\omega]. \]

We define
\[ J_{[h_\omega]} = |j(C(f_\#, g_\#)_x)| = |j(\omega_\#(C(f_\#, g_\#)_{x_0}))| \]
and
\[ I_{[h_\omega]} = |C(f_\#, \tilde{g}_\#)_x| = |\omega_\#(C(f_\#, \tilde{g}_\#)_{x_0})|. \]

4. **Computation of the Nielsen Number** \( N(f, g) \).

Let \( (f, g): M \to N \) be a pair of maps between orientable closed manifolds of the same dimension. In this section, we show that if the number \( J \) is the same for all Nielsen classes that lie in the same \( H \)-Nielsen class, then \( N(f, g) \) is a linear combination of the Nielsen numbers of the lifts of \( f \) and \( g \). In addition, we show how Theorem 4.4 generalizes Theorem 4.2 of 4.2, and Theorem 5.7 of [10] for the class of non-smooth orientable closed manifolds.

Let \( M \) and \( N \) be orientable closed manifolds of the same dimension \( n \), and \( (\tilde{M}, \tilde{p}) \) and \( (\tilde{N}, \tilde{p}^\prime) \) finite orientable regular coverings corresponding to the normal subgroups \( K \subseteq \pi_1(M) \) and \( H \subseteq \pi_1(N) \) of \( M \) and \( N \), respectively. Let \( (f, g): M \to N \) be a pair of maps which admits a pair of
lifts \((\tilde{f}, \tilde{g}) : \tilde{M} \to \tilde{N}\). Recall the commutative diagram 2.1. Since we can homotope the pair \((f, g)\) to a pair with finite set of coincidences (see Theorem 2 of [13]), without loss of generality, we may assume that \(\Phi(f, g)\) is finite. By Lemma 3.17, each coincidence point of either \((f, g)\) or \((\tilde{f}, \tilde{g})\) is isolated. We refer to the proof of Corollary 5.7 of [9] for the proof of the following Lemma.

**Lemma 4.1** — Let \(x \in \Phi(f, g)\), and \(\tilde{x} \in p^{-1}(x) \cap \Phi(\tilde{f}, \tilde{g})\). Then

\[
\text{index}(\tilde{f}, \tilde{g}; \tilde{x}) = \text{index}(f, g; x),
\]

where \(\text{index}(f, g; x)\) stands for the index of \(f\) and \(g\) at \(x \in \Phi(f, g)\). Similarly, \(\text{index}(\tilde{f}, \tilde{g}; \tilde{x})\) carries the same meaning for the corresponding maps and point. □

The following proposition explains the relationship between the indices of the Nielsen classes in the total space and those in the base space. It generalizes Lemma 3.4 of [6].

**Proposition 4.2** — Let \((f, g) : M \to N\) be a pair of maps between the given orientable manifolds, and let \(A \subseteq \Phi(f, g)\) and \(\tilde{A} \subseteq \Phi(\tilde{f}, \tilde{g})\) be Nielsen classes such that \(p(\tilde{A}) = A\). Then

1. \(\text{index}(\tilde{f}, \tilde{g}; \tilde{A}) = J_A \cdot \text{index}(f, g; A)\), where \(\text{index}(f, g; A)\) stands for the index of \(f\) and \(g\) at the set \(A\). Similar meaning does \(\text{index}(\tilde{f}, \tilde{g}; \tilde{A})\) carry for the corresponding maps and set.

2. \(\text{index}(\tilde{f}, \tilde{g}; p^{-1}(A) \cap \Phi(\tilde{f}, \tilde{g})) = I_A \cdot \text{index}(f, g; A)\).

**Proof**: 1. We have

\[
\text{index}(\tilde{f}, \tilde{g}; \tilde{A}) = \sum_{x \in A} \sum_{\tilde{x} \in p^{-1}(x) \cap \tilde{A}} \text{index}(\tilde{f}, \tilde{g}; \tilde{x}) = \sum_{x \in A} J_A \cdot \text{index}(f, g; x) = J_A \cdot \text{index}(f, g; A).
\]

2. Assume that \(p^{-1}(A) \cap \Phi(\tilde{f}, \tilde{g}) = \bigcup_{i=1}^{S_A} \tilde{A}_i\), where \(\tilde{A}_i\) is a Nielsen class of the pair \((\tilde{f}, \tilde{g})\), and \(p(\tilde{A}_i) = A\), for each \(1 \leq i \leq S_A\). Then

\[
\text{index}(\tilde{f}, \tilde{g}; p^{-1}(A) \cap \Phi(\tilde{f}, \tilde{g})) = \sum_{i=1}^{S_A} \text{index}(\tilde{f}, \tilde{g}; \tilde{A}_i) = J_A \cdot \sum_{i=1}^{S_A} \text{index}(f, g; A) = I_A \cdot \text{index}(f, g; A).
\]

**Corollary 4.3** — Let \(A\) be a Nielsen class of \((f, g)\), and \(\tilde{A}\) a Nielsen class of \((\tilde{f}, \tilde{g})\) such that \(p(\tilde{A}) = A\). Then, \(A\) is essential if and only if \(\tilde{A}\) is essential.
PROOF: Apply part (1) of Proposition 4.3.

Let \((\tilde{f}_1, \tilde{g}_1), \ldots, (\tilde{f}_{R_\{H(f,g)\}}, \tilde{g}_{R_\{H(f,g)\}})\) be the representatives of the \(H\)-Reidemeister classes of the pair \((f, g)\), and let \(r\) be the number of nonempty \(H\)-Nielsen classes of \(f\) and \(g\). Without loss of generality, let \((\tilde{f}_1, \tilde{g}_1), \ldots, (\tilde{f}_r, \tilde{g}_r)\) be the representatives of the \(H\)-Reidemeister classes of the pair \((f, g)\) corresponding to the nonempty \(H\)-Nielsen classes. On the other hand, let \(\tilde{\Phi}(f, g)\) be the set of the Nielsen classes of the corresponding pair, and let \(\tilde{\Phi}_E(f, g)\) be the set of the essential Nielsen classes of the corresponding pair. Also, let \(p \tilde{\Phi}(\tilde{f}, \tilde{g})\) denote the set of Nielsen classes in the \(H\)-Nielsen class \(p \Phi(f, g)\), and \(p \tilde{\Phi}_E(\tilde{f}, \tilde{g})\) the set of the essential Nielsen classes that lie in the \(H - \text{Nielsen class} p \Phi(\tilde{f}, \tilde{g})\). We are ready now to prove our main theorem of this paper, which shows that \(N(f, g)\) is a linear combination of the Nielsen numbers of lifts of \((f, g)\).

**Theorem 4.4** — Let \(M\) and \(N\) be closed orientable manifolds of the same dimension, and \((\tilde{M}, p)\) and \((\tilde{N}, p')\) be finite regular coverings which correspond to the normal subgroups \(K \subseteq \pi_1(M)\) and \(H \subseteq \pi_1(N)\), respectively. Let \(f, g : M \to N\) be maps such that \(\Phi(f, g)\) is finite, and for which there exist lifts \(\tilde{f}, \tilde{g} : \tilde{M} \to \tilde{N}\), respectively. Suppose the number \(J_A\) is the same for all Nielsen classes \(A\) of \(f\) and \(g\) lying in the same \(H\)-Nielsen class. Then

\[
N(f, g) = \sum_{i=1}^{r} \frac{N(\tilde{f}_i, \tilde{g}_i)}{S(\tilde{f}_i, \tilde{g}_i)}.
\]  

(4.1)

PROOF: Define the function \(\chi : Z \to \{0, 1\}\) by

\[
\chi(m) = \begin{cases} 
0 & \text{if } m = 0, \\
1 & \text{otherwise}.
\end{cases}
\]

The number of the essential Nielsen classes that lie in the \(H - \text{Nielsen class} p \Phi(\tilde{f}_i, \tilde{g}_i)\) is given by

\[
|p \tilde{\Phi}_E(\tilde{f}_i, \tilde{g}_i)| = \sum_{A \in p \Phi(\tilde{f}_i, \tilde{g}_i)} \chi(\text{index}(f, g; A)).
\]

Because \(J_A = J_B\) for all Nielsen classes \(A, B \subseteq \Phi(\tilde{f}_i, \tilde{g}_i)\),

\[
N(\tilde{f}_i, \tilde{g}_i) = \sum_{A \in \Phi(\tilde{f}_i, \tilde{g}_i)} \chi(\text{index}(\tilde{f}_i, \tilde{g}_i; \tilde{A}))
\]

\[
= \sum_{A \in \Phi(\tilde{f}_i, \tilde{g}_i)} \sum_{\tilde{A} \in p^{-1}(A) \cap \Phi(\tilde{f}_i, \tilde{g}_i)} \chi(\text{index}(\tilde{f}_i, \tilde{g}_i; \tilde{A}))
\]

\[
= \sum_{A \in p \Phi(\tilde{f}_i, \tilde{g}_i)} S(\tilde{f}_i, \tilde{g}_i) \cdot \chi(\text{index}(f, g; A))
\]

\[
= S(\tilde{f}_i, \tilde{g}_i) \cdot |p \tilde{\Phi}_E(\tilde{f}_i, \tilde{g}_i)|.
\]

\]
Therefore, 
\[
|p \Phi_E(\tilde{f}_i, \tilde{g}_i)| = \frac{N(\tilde{f}_i, \tilde{g}_i)}{S(\tilde{f}_i, \tilde{g}_i)} .
\]  
(4.2)

for each \(1 \leq i \leq r\). Now, we have 
\[
N(f, g) = \sum_{i=1}^{r} |p \Phi_E(\tilde{f}_i, \tilde{g}_i)| = \sum_{i=1}^{r} \frac{N(\tilde{f}_i, \tilde{g}_i)}{S(\tilde{f}_i, \tilde{g}_i)}
\]
as required.  
\[
\]

Remark 4.5: If we define \(S(\tilde{f}, \tilde{g}) = 1\), for an empty H-Nielsen class \(p \Phi(\tilde{f}, \tilde{g})\), then we can replace \(r\) in equation 1.3 by \(R_H(f, g)\).

Next, we generalize Theorem 4.4 to the case where \(f\) and \(g\) possess infinite coincidence points.

**Theorem 4.6** — Assume the conditions of Theorem 4.4 except that \(\Phi(f, g)\) is infinite. Then, 
\[
N(f, g) = \sum_{i=1}^{r} \frac{N(\tilde{f}_i, \tilde{g}_i)}{S(\tilde{f}_i, \tilde{g}_i)} .
\]

**Proof:** Minimum Theorem for the Coincidence Nielsen number allows us to homotope the pair \((f, g)\) to a pair \((f', g')\) with finite number of fixed points. The result follows from the following facts:

- The covering is regular and the numbers \(I, J,\) and \(S\) are homotopy invariant (Proposition 4.5 of [10]).
- Each Reidemeister representative of \((f, g)\) is homotopic to a Reidemeister representative of the pair \((f', g')\), and vice versa.
- The coincidence Nielsen number is homotopy invariant.  

The proof of the next corollary follows directly from Theorem 4.4.

**Corollary 4.7** — If in Theorems 4.4 and 4.6 we further have the condition "\(S(\tilde{f}, \tilde{g})\) is equal to a constant number \(q\) for every lifting pair of \((f, g)\)", then 
\[
N(f, g) = \frac{1}{q} \cdot \sum_{i=1}^{r} N(\tilde{f}_i, \tilde{g}_i) .
\]  
(4.3)

Next, Theorem 4.8 shows that Theorem 4.4 generalizes Theorem 4.2 of [6].
Theorem 4.8 — Let $M$ be a closed orientable manifold, $(\tilde{M}, p)$ be a finite regular covering of $M$, and $f : M \to M$ be a map for which there exists a lift $\tilde{f} : \tilde{M} \to \tilde{M}$. Assume that all Nielsen fixed point classes that lie in the same $H$-Nielsen class have the same number $J$. Then

$$N(f) = \sum_{i=1}^{r} \frac{J(\tilde{f}_{i})}{I(\tilde{f}_{i})} N(\tilde{f}_{i}) .$$

where $r$ is the number of nonempty $H$-Reidemeister classes of $f$, and $\tilde{f}_{i}$ is a collection of one representative from each of these classes.

**Proof** : Apply Theorem 4.4 for $M = N$, $(\tilde{M}, p) = (\tilde{N}, p')$, and $g = 1_M$. Also, recall the fact that the fixed point index coincides with the index of the pair $(f, 1_M)$. □

**Remark 4.9** : One can immediately see that Theorem 5.7 of [10], applied to orientable closed manifolds, agrees with Theorem 4.4.

In order to give applications of Theorems 4.4 and 4.6, we need to impose the condition "the number $J$ is the same for all Nielsen classes that lie in the same $H$-Nielsen class". Proposition 4.12 below gives a sufficient condition for this to hold. Actually, Proposition 4.12 is a generalization of Lemma 3.5 of [6]. The proof of the next lemma is simple.

**Lemma 4.10** — Let $x_0$ and $x$ be coincidence points, and $\omega : x_0 \to x$ be a path. Then

$$h_{\omega^{-1}} = (f(\omega))_\#(h_\omega^{-1}) = (g(\omega))_\#(h_\omega^{-1})$$

□

**Corollary 4.11** — Let $x_0$ and $x$ be coincidence points, and $\omega : x_0 \to x$ be a path. Then, $h_\omega$ commutes with $f_\#(C(f_\#, g_\#)_x)$ if and only if $h_{\omega^{-1}}$ commutes with $f_\#(C(f_\#, g_\#)_x)$.

**Proof** : Apply Corollary 3.24 for $h_\omega$ and $h_{\omega^{-1}}$. □

Proposition 4.12 — Let $x_0$ and $x$ be in the same $H$-Nielsen class, and $\omega : x_0 \to x$ be a path that establishes the $H$-Nielsen relation. If $h_{\omega}$ commutes with $f_\#(C(f_\#, g_\#)_x)$, then $J_{[x_0]} = J_{[x]}$.

**Proof** : By Corollary 3.24, the isomorphism $\omega_\# : C(f_\#, g_\#)_x \to C(f_\#, g_\#)_x$ induces the isomorphism

$$\overline{\omega}_\# : j(C(f_\#, g_\#)_x) \to j(C(f_\#, g_\#)_x) : \overline{a} \to \overline{\omega}_\#(a).$$

By Proposition 3.3, $J_{[x_0]} = J_{[x]}$. □

In what follows, two subgroups of a given group are said to commute, if each element in the former commutes with each element of the latter.
Proposition 4.13 — Let $x_0$ and $x$ be in the same $H$-Nielsen class. Then, $H(f(x_0))$ commutes with $f_\#(C(f_\#, g_\#)_x)$ if and only if $H(f(x))$ commutes with $f_\#(C(f_\#, g_\#)_x)$.

Proof: Let $\omega : x_0 \to x$ be a path that establishes the $H$-Nielsen relation. Consider the commutative diagram

$$
\begin{array}{ccc}
\pi_1(M, x_0) & \xrightarrow{\omega_\#} & \pi_1(M, x) \\
\downarrow f_\# & & \downarrow f_\#
\end{array}
\Rightarrow
\begin{array}{ccc}
\pi_1(N, f(x_0)) & \xrightarrow{(f(\omega))_\#} & \pi_1(N, f(x))
\end{array} \tag{4.4}
$$

We need only to show that if $f_\#(C(f_\#, g_\#)_{x_0})$ commutes with $H(f(x_0))$ then $H(f(x))$ commutes with $f_\#(C(f_\#, g_\#)_x)$. So, assume that $f_\#(C(f_\#, g_\#)_{x_0})$ commutes with $H(f(x_0))$, and let $h \in H(f(x))$ and $\delta \in C(f_\#, g_\#)_x$. Then

$$
h f(\delta) = f(\omega)^{-1} f(\omega) h f(\omega)^{-1} \bigg|_{\in H(f(x_0))} \bigg|_{\in f_\#(C(f_\#, g_\#)_{x_0})} \bigg|_{\in f_\#(C(f_\#, g_\#)_x)}
$$

Thus, $H(f(x))$ commutes with $f_\#(C(f_\#, g_\#)_x)$.

\[ \square \]

Remark 4.14: Proposition 4.13 states that the commutativity of $H(f(x))$ with $f_\#(C(f_\#, g_\#)_x)$ is independent of the choice of $x$ within its $H$-Nielsen class.

Corollary 4.15 — Let $x_0$ belong to some $H$-Nielsen class $p \Phi(\tilde{f}, \tilde{g})$. If $H(f(x_0))$ commutes with $f_\#(C(f_\#, g_\#)_{x_0})$, then $J_A = J_B$ for all Nielsen classes $A, B \subseteq p \Phi(\tilde{f}, \tilde{g})$.

Proof: Apply Proposition 4.12 and Lemma 4.13. \[ \square \]

5. COUNTING THE $H$—REIDEMEISTER REPRESENTATIVES

In this section, we discuss to what extent we can characterize the $H$—Reidemeister representatives of $f$ and $g$. The main result in this section (Corollary 5.21) determines, in some special cases, the number of these representatives. Although our method counts these representatives, it will be useful to find them in some farther special cases.

Let $M$ and $N$ be path connected, locally path connected topological manifolds, $(\tilde{M}, p)$ and $(\tilde{N}, p)$ be regular coverings corresponding to the normal subgroups $K \subseteq \pi_1(M)$ and $H \subseteq \pi_1(N)$ of finite index on $M$ and $N$, respectively. Let $(f, g): M \to N$ be a pair of maps for which there exists a pair of lifts $(\tilde{f}, \tilde{g}): \tilde{M} \to \tilde{N}$. Consider the commutative diagram 2.1. Recall that the homomorphism
\[ \tilde{f}_\#: \frac{\pi_1(M)}{K} \cong \text{Aut}(\tilde{M}) \longrightarrow \text{Aut}(\tilde{N}) \cong \frac{\pi_1(N)}{H} \] induced by the map \( f \) relative to the lift \( \tilde{f} \) satisfies that \( \tilde{f}_\#(\alpha) = \beta \) if and only if \( \tilde{f} \alpha = \beta \tilde{f} \). The homomorphism induced by the map \( f \) relative to a lift of \( f \) is different from the homomorphism induced by the map \( f \) relative to another lift of \( f \), but both homomorphisms carry the same notation \( \tilde{f}_\# \). To avoid confusion, we introduce the following notation.

**Notation 5.1** — The homomorphism \( \tilde{f}_\#: \frac{\pi_1(M)}{K} \cong \text{Aut}(\tilde{M}) \longrightarrow \text{Aut}(\tilde{N}) \cong \frac{\pi_1(N)}{H} \) induced by the map \( f \) relative to the lift \( \tilde{f} \) is denoted by \( [\tilde{f} : \cdot ] \). So, according to this notation we write \( [\tilde{f} : \alpha] = \beta \) if and only if \( \tilde{f} \alpha = \beta \tilde{f} \).

**Proposition 5.2** — For every \( \alpha \in \text{Aut}(\tilde{M}) \) and \( \beta \in \text{Aut}(\tilde{N}) \), \( [\beta \tilde{f} : \alpha] = \beta \cdot [\tilde{f} : \alpha] \cdot \beta^{-1} \).

**Proof:** Let \( \alpha \in \text{Aut}(\tilde{M}) \) and \( \beta \in \text{Aut}(\tilde{N}) \). Then

\[
(\beta \cdot \tilde{f}) \cdot \alpha = \beta \cdot (\tilde{f} \cdot \alpha) \\
= \beta \cdot ([\tilde{f} : \alpha] \cdot \tilde{f}) \\
= \beta \cdot [\tilde{f} : \alpha] \cdot \beta^{-1} \cdot \beta \cdot \tilde{f} \\
= \beta \cdot [\tilde{f} : \alpha] \cdot \beta^{-1} \cdot (\beta \cdot \tilde{f}) .
\]

Thus,

\[
[\beta \cdot \tilde{f} : \alpha] = \beta \cdot [\tilde{f} : \alpha] \cdot \beta^{-1} .
\]

**Definition 5.3** — We define the group \( G(\tilde{f}) \) to be the image of the homomorphism \( [\tilde{f} : \cdot ] \) in \( \text{Aut}(\tilde{N}) \). That is,

\[
G(\tilde{f}) = \{ [\tilde{f} : \alpha] \mid \alpha \in \text{Aut}(\tilde{M}) \} .
\]

From now on, we fix a lift \( (\tilde{f}, \tilde{g}) \) of \( (f, g) \).

**Proposition 5.4** — The lift \( (\tilde{f}, \beta_1 \cdot \tilde{g}) \) is conjugate to \( (\tilde{f}, \beta_2 \cdot \tilde{g}) \) if and only if there exists \( \alpha \in \text{Aut}(\tilde{M}) \) such that

\[
\beta_1 = [\tilde{f} : \alpha] \cdot \beta_2 \cdot [\tilde{g} : \alpha^{-1}] .
\]

**Proof:** Suppose that \( (\tilde{f}, \beta_1 \cdot \tilde{g}) \) is conjugate to \( (\tilde{f}, \beta_2 \cdot \tilde{g}) \). Thus, there exist \( \alpha \in \text{Aut}(\tilde{M}) \) and \( \gamma \in \text{Aut}(\tilde{N}) \) such that

\[
\begin{align*}
\gamma \cdot \tilde{f} \cdot \alpha &= \tilde{f} , \\
\gamma \cdot \beta_1 \cdot \tilde{g} \cdot \alpha &= \beta_2 \cdot \tilde{g} .
\end{align*}
\]
Hence,
\[
\begin{align*}
\gamma \cdot [\tilde{f} : \alpha] \cdot \tilde{f} &= \tilde{f}, \\
\gamma \cdot \beta_1 \cdot [\tilde{g} : \alpha] \cdot \tilde{g} &= \beta_2 \cdot \tilde{g}.
\end{align*}
\]
(5.2)

The first line in Equation 5.2 implies that
\[
\gamma = [\tilde{f} : \alpha^{-1}].
\]
(5.3)

Thus, the second line in Equation 5.3 along with Equation 5.3 implies that
\[
\beta_1 = \gamma^{-1} \cdot \beta_2 \cdot [\tilde{g} : \alpha]^{-1} = [\tilde{f} : \alpha] \cdot \beta_2 \cdot [\tilde{g} : \alpha^{-1}].
\]

For the converse, we need to show that Equation 5.1 implies that \((\tilde{f}, \beta_2 \cdot \tilde{g})\) is conjugate to \((\tilde{f}, \beta_1 \cdot \tilde{g})\).

\[
[\tilde{f} : \alpha] \cdot (\tilde{f}, \beta_2 \cdot \tilde{g}) \cdot \alpha^{-1}
\]
\[
= \left( [\tilde{f} : \alpha] \cdot \tilde{f} \cdot \alpha^{-1}, [\tilde{f} : \alpha] \cdot \beta_2 \cdot \tilde{g} \cdot \alpha^{-1} \right)
\]
\[
= \left( [\tilde{f} : \alpha] \cdot [\tilde{f} : \alpha^{-1}] \cdot \tilde{f}, [\tilde{f} : \alpha] \cdot \beta_2 \cdot [\tilde{g} : \alpha^{-1}] \cdot \tilde{g} \right)
\]
\[
= (\tilde{f}, \beta_1 \cdot \tilde{g}).
\]

\[\square\]

**Remark 5.5** : Proposition 5.4 states that the set of the covering transformations \([\tilde{f} : \alpha] \cdot \beta_2 \cdot [\tilde{g} : \alpha^{-1}]\), for \(\alpha \in \text{Aut}(\tilde{M})\), is closely related to the set of the lifts \((\tilde{f}, \beta_1 \cdot \tilde{g})\) that lie in the Reidemeister class represented by (i.e., conjugate to) \((\tilde{f}, \beta_2 \cdot \tilde{g})\) and vice versa. As we will see, it is not necessarily that the two sets be in one to one correspondence with each other. So, our next job is to farther investigate the relationship between them.

**Definition 5.6** — Let \(\beta \in \text{Aut}(\tilde{N})\). The set \(\widehat{G}(\tilde{f}, \beta \cdot \tilde{g}) \subseteq \text{Aut}(\tilde{N})\) is defined by
\[
\widehat{G}(\tilde{f}, \beta \cdot \tilde{g}) = \left\{ [\tilde{f} : \alpha] \cdot \beta \cdot [\tilde{g} : \alpha^{-1}] \mid \alpha \in \text{Aut}(\tilde{M}) \right\}.
\]

**Lemma 5.7** — Let \(\beta_1, \beta_2 \in \text{Aut}(\tilde{N})\). Then, \((\tilde{f}, \beta_1 \cdot \tilde{g})\) is conjugate to \((\tilde{f}, \beta_2 \cdot \tilde{g})\) if and only if \(\widehat{G}(\tilde{f}, \beta_1 \cdot \tilde{g}) \cap \widehat{G}(\tilde{f}, \beta_2 \cdot \tilde{g}) \neq \emptyset\).

**Proof** : Suppose \((\tilde{f}, \beta_1 \cdot \tilde{g})\) is conjugate to \((\tilde{f}, \beta_2 \cdot \tilde{g})\). By Proposition 5.4, there exists \(\alpha \in \text{Aut}(\tilde{M})\) such that \(\beta_1 = [\tilde{f} : \alpha] \cdot \beta_2 \cdot [\tilde{g} : \alpha^{-1}]\). Thus, \(\beta_1 \in \widehat{G}(\tilde{f}, \beta_2 \cdot \tilde{g})\). Since \(\beta_2 \in \widehat{G}(\tilde{f}, \beta_2 \cdot \tilde{g})\), we get that \(\widehat{G}(\tilde{f}, \beta_1 \cdot \tilde{g}) \cap \widehat{G}(\tilde{f}, \beta_2 \cdot \tilde{g}) \neq \emptyset\).
Conversely, assume that \( \tilde{G}(\tilde{f}, \beta_1 \cdot \tilde{g}) \cap \tilde{G}(\tilde{f}, \beta_2 \cdot \tilde{g}) \neq \emptyset \). This means that there exist \( \alpha_1, \alpha_2 \in Aut(\tilde{M}) \) such that
\[
[\tilde{f} : \alpha_1] \cdot \beta_1 \cdot [\tilde{g} : \alpha_1^{-1}] = [\tilde{f} : \alpha_2] \cdot \beta_2 \cdot [\tilde{g} : \alpha_2^{-1}] .
\]
Hence,
\[
\beta_1 = [\tilde{f} : \alpha_1^{-1}] \cdot [\tilde{f} : \alpha_2] \cdot \beta_2 \cdot [\tilde{g} : \alpha_2^{-1}] \cdot [\tilde{g} : \alpha_1] .
\]
\[
= [\tilde{f} : \alpha_1^{-1} \cdot \alpha_2] \cdot \beta_2 \cdot [\tilde{g} : \alpha_2^{-1} \cdot \alpha_1] .
\]
\[
= [\tilde{f} : \alpha_1^{-1} \cdot \alpha_2] \cdot \beta_2 \cdot [\tilde{g} : (\alpha_1^{-1} \cdot \alpha_2)^{-1}] .
\]
By Proposition 5.4, \((\tilde{f}, \beta_1 \cdot \tilde{g})\) is conjugate to \((\tilde{f}, \beta_2 \cdot \tilde{g})\).

**Definition 5.8** — We define the subset \( \Delta(\beta) \) of \( Lift(f, g) \) by
\[
\Delta(\beta) = \left\{ \mu (\tilde{f}, \beta \tilde{g}) \in Lift(f, g) \mid \mu \in Aut(\tilde{N}) \right\} .
\]

**Lemma 5.9** — The following are true:

1. \( \Delta(\beta) = \Delta(\hat{\beta}) \) if and only if \( \beta = \hat{\beta} \). Moreover, \( \Delta(\beta) \cap \Delta(\hat{\beta}) = \emptyset \) if and only if \( \beta \neq \hat{\beta} \).

2. \( Lift(f, g) = \bigcup_{\beta \in Aut(\tilde{N})} \Delta(\beta) . \) Thus, the family \( \Delta = \left\{ \Delta(\beta) \mid \beta \in Aut(\tilde{N}) \right\} \) is a partition of \( Lift(f, g) \).

3. The set \( \Delta(\beta) \) is a subset of the conjugacy class which includes \((\tilde{f}, \beta \tilde{g})\). Furthermore, each conjugacy class is a union of some of these \( \Delta(\beta) \)'s.

4. \( |R_H(f, g)| \leq |Aut(\tilde{N})| = [\pi_1(N) : H] \).

**Proof:** (1) Let \( \beta, \hat{\beta} \in Aut(\tilde{N}) \). Then,
\[
\Delta(\beta) = \Delta(\hat{\beta}) \Rightarrow (\tilde{f}, \beta \tilde{g}) \in \Delta(\hat{\beta})
\]
\[
\Rightarrow (\tilde{f}, \beta \tilde{g}) = (\mu \tilde{f}, \mu \hat{\beta} \tilde{g}), \mu \in Aut(\tilde{N})
\]
\[
\Rightarrow \tilde{f} = \mu \tilde{f} \text{ and } \beta \tilde{g} = \mu \hat{\beta} \tilde{g}
\]
\[
\Rightarrow 1_{\tilde{N}} = \mu, \text{ and hence } \beta \tilde{g} = \hat{\beta} \tilde{g}
\]
\[
\Rightarrow \beta = \hat{\beta} .
\]

The converse is trivial.
On the other hand, assume that $\Delta(\beta) \cap \Delta(\hat{\beta}) \neq \emptyset$. Then, there exists $\mu \in Aut(\tilde{N})$ such that $(\mu \tilde{f}, \mu \beta \tilde{g}) \in \Delta(\hat{\beta})$. Hence, there exists $\hat{\mu} \in Aut(\tilde{N})$ such that $(\mu \tilde{f}, \mu \beta \tilde{g}) = (\hat{\mu} \tilde{f}, \hat{\mu} \beta \tilde{g})$. Thus, $\mu \tilde{f} = \hat{\mu} \tilde{f}$, and $\mu \beta \tilde{g} = \hat{\mu} \beta \tilde{g}$. Hence, $\mu = \hat{\mu}$ and $\mu \beta = \hat{\mu} \beta$. So, $\beta = \hat{\beta}$.

It is obvious that if $\beta = \hat{\beta}$, then $\Delta(\beta) \cap \Delta(\hat{\beta}) \neq \emptyset$.

(2) Let $(\tilde{f}_1, \tilde{g}_1)$ be a lift of $(f, g)$. Then, there exist $\beta_1, \beta_2 \in Aut(\tilde{N})$ such that $\tilde{f}_1 = \beta_1 \tilde{f}$, and $\tilde{g}_1 = \beta_2 \tilde{g}$. Thus,

$$(\tilde{f}_1, \tilde{g}_1) = (\beta_1 \tilde{f}, \beta_2 \tilde{g}) = \beta_1 (\tilde{f}, \beta_1^{-1} \beta_2 \tilde{g}).$$

So, $(\tilde{f}_1, \tilde{g}_1) \in \Delta(\beta_1^{-1} \beta_2) \subseteq \bigcup_{\beta \in Aut(\tilde{N})} \Delta(\beta)$. Since $\bigcup_{\beta \in Aut(\tilde{N})} \Delta(\beta) \subseteq Lift(f, g)$, we get that $Lift(f, g) = \bigcup_{\beta \in Aut(\tilde{N})} \Delta(\beta)$. Moreover, by (1), the family $\left\{ \Delta(\beta) \mid \beta \in Aut(\tilde{N}) \right\}$ is a partition of $Lift(f, g)$.

(3) The proof follows from the definition of conjugacy of pairs of lifts and Definition 5.8.

(4) By (3), $|R_H(f, g)| \leq |\Delta| = \left| Aut(\tilde{N}) \right| = |\pi_1(N) : H|$. □

Remark 5.10 : Any other lift $(\tilde{f}_1, \tilde{g}_1)$ is conjugate to $(\tilde{f}, \beta \tilde{g})$ for some $\beta \in Aut(\tilde{N})$. If we define the action of $Aut(\tilde{M})$ on the set $\left\{ \Delta(\beta) \mid \beta \in Aut(\tilde{N}) \right\}$ from the right by

$$\Delta(\beta) \cdot \alpha = \left\{ \mu (\tilde{f}, \beta \tilde{g}) \alpha \mid \mu \in Aut(\tilde{N}) \right\},$$

then the union of the elements of each orbit, under this action, is a conjugacy class.

Definition 5.11 — A set $\Omega' \subseteq Lift(f, g)$ is said to be a set of Reidemeister representatives if each conjugacy class is represented exactly once in $\Omega'$.

Proposition 5.12 — Let $\Omega = \left\{ (\tilde{f}, \beta \tilde{g}) \mid \beta \in Aut(\tilde{N}) \right\}$, and let $\Omega' = \{(\tilde{f}_i, \tilde{g}_i) : i = 1, \ldots , r\}$, be a subset of $\Omega$, where $\tilde{g}_i = \beta_i \tilde{g}$ for some $\beta_i \in Aut(\tilde{N})$. Then, $\Omega'$ is the set of Reidemeister representatives, which appear in Equation 1.3 if and only if $\Omega'$ satisfies the following conditions:

1. Any two distinct pairs in $\Omega'$ are not conjugate.

2. If we add any $(\tilde{f}, \beta' \tilde{g}) \notin \Omega'$ from $\Omega$ to $\Omega'$, then $\Omega' \cup \{(\tilde{f}, \beta' \tilde{g})\}$ is not pairwise non conjugate; that is, $(\tilde{f}, \beta' \tilde{g})$ must be conjugate to some pair in $\Omega'$.

Proof : Apply Lemma 5.9. □

So then, Proposition 5.12 implies that we can make a suitable choice of $\beta \in Aut(\tilde{N})$, and use this choice to determine a set of Reidemeister representatives.
Notation 5.13 — From now on, we will assume that we have chosen an appropriate set of Reidemeister representatives. We use the notation $\Lambda \subseteq \text{Aut}(\tilde{N})$ to denote the corresponding choice of $\beta$’s.

The following theorem allows us to move one step closer to enumerate the $II$—Reidemeister representatives.

**Theorem 5.14** — We have

$$\text{Aut}(\tilde{N}) = \bigcup_{\beta \in \Lambda} \tilde{G}(\tilde{f}, \beta \cdot \tilde{g}).$$

(5.4)

where the union is a disjoint union.

**Proof:** It is enough to show that $\text{Aut}(\tilde{N}) \subseteq \bigcup_{\beta \in \Lambda} \tilde{G}(\tilde{f}, \beta \cdot \tilde{g})$. Let $\hat{\beta} \in \text{Aut}(\tilde{N})$. By the definition of $\Lambda$, $(\tilde{f}, \hat{\beta} \cdot \tilde{g})$ belongs to the Reidemeister class represented by $(\tilde{f}, \beta \cdot \tilde{g})$ for some $\beta \in \Lambda$. This implies that $\hat{\beta} = [\tilde{f} : \alpha] \cdot \beta \cdot [\tilde{g} : \alpha^{-1}]$ for some $\alpha \in \text{Aut}(\tilde{M})$. Thus, $\hat{\beta} \in \tilde{G}(\tilde{f}, \beta \cdot \tilde{g})$. Therefore, $\hat{\beta} \in \bigcup_{\beta \in \Lambda} \tilde{G}(\tilde{f}, \beta \cdot \tilde{g})$. The union is disjoint by Lemma 5.7 and the definition of $\Lambda$. 

**Corollary 5.15** — The following equation holds

$$|\text{Aut}(\tilde{N})| = \sum_{\beta \in \Lambda} |\tilde{G}(\tilde{f}, \beta \cdot \tilde{g})|. $$

We define next another set $G(\tilde{f}, \beta \cdot \tilde{g})$ which is related to $\tilde{G}(\tilde{f}, \beta \cdot \tilde{g})$. The new set is also related to $\Gamma(\tilde{f}, \beta \cdot \tilde{g})$ as we shall see. Moreover, under certain conditions it is a subgroup of $\text{Aut}(\tilde{N})$. These facts will be useful in computing $|\Lambda|$ in some special cases.

**Definition 5.16** — Let $\beta \in \text{Aut}(\tilde{N})$. The set $G(\tilde{f}, \beta \cdot \tilde{g})$ is defined by

$$G(\tilde{f}, \beta \cdot \tilde{g}) = \left\{ [\tilde{f} : \alpha] \cdot [\beta \cdot \tilde{g} : \alpha^{-1}] \mid \alpha \in \text{Aut}(\tilde{M}) \right\}$$

$$= \tilde{G}(\tilde{f}, \beta \cdot \tilde{g}) \cdot \beta^{-1}. $$

**Lemma 5.17** — Let $\beta \in \text{Aut}(\tilde{N})$. Then,

$$|G(\tilde{f}, \beta \cdot \tilde{g})| = |\tilde{G}(\tilde{f}, \beta \cdot \tilde{g})|. $$

**Proof:** It is easy to see that the function $\tilde{G}(\tilde{f}, \beta \cdot \tilde{g}) \rightarrow G(\tilde{f}, \beta \cdot \tilde{g})$ defined by $[\tilde{f} : \alpha] \cdot \beta \cdot [\tilde{g} : \alpha^{-1}] \mapsto [\tilde{f} : \alpha] \cdot \beta \cdot [\tilde{g} : \alpha^{-1}] \cdot \beta^{-1}$ is bijective. 

**Corollary 5.18** — We have the following equation

$$|\text{Aut}(\tilde{N})| = \sum_{\beta \in \Lambda} |G(\tilde{f}, \beta \cdot \tilde{g})|. $$

(5.5)
Proof: Apply Corollary 5.15 and Lemma 5.17.

Remark 5.19: The set $G(\tilde{f}, \beta \cdot \tilde{g})$ is not necessarily a subgroup of $\text{Aut}(\tilde{N})$ since it is not always closed under the multiplication, nor does the inverse of an element necessarily belong to $G(\tilde{f}, \beta \cdot \tilde{g})$. However, the identity $1_{\tilde{N}}$ always belongs to this set.

The main results of this section are the following theorem and corollary.

Theorem 5.20 — If $\Gamma(\tilde{f}, \beta \cdot \tilde{g})$ is a normal subgroup of $\text{Aut}(\tilde{M})$, for each $\beta \in \text{Aut}(\tilde{N})$, then

$$|\text{Aut}(\tilde{N})| = |\text{Aut}(\tilde{M})| \cdot \sum_{\beta \in \Lambda} \frac{1}{I(\tilde{f}, \beta \cdot \tilde{g})}. \quad (5.6)$$

Proof: Define $\varphi : \text{Aut}(\tilde{M}) \rightarrow G(\tilde{f}, \beta \cdot \tilde{g})$ by $\varphi(\alpha \cdot \Gamma(\tilde{f}, \beta \cdot \tilde{g})) = [\tilde{f} : \alpha] \cdot \beta \cdot [\tilde{g} : \alpha^{-1}] \cdot \beta^{-1}$.

Let $\alpha_1, \alpha_2 \in \text{Aut}(\tilde{M})$. Then

$$\alpha_1 \cdot \Gamma(\tilde{f}, \beta \cdot \tilde{g}) = \alpha_2 \cdot \Gamma(\tilde{f}, \beta \cdot \tilde{g})$$

$$\Leftrightarrow \alpha_1^{-1} \cdot \alpha_2 \in \Gamma(\tilde{f}, \beta \cdot \tilde{g})$$

$$\Leftrightarrow [\tilde{f} : \alpha_1^{-1} \cdot \alpha_2] = \beta \cdot [\tilde{g} : \alpha_1^{-1} \cdot \alpha_2] \cdot \beta^{-1}$$

$$\Leftrightarrow [\tilde{f} : \alpha_1^{-1}] \cdot [\tilde{f} : \alpha_2] = \beta \cdot [\tilde{g} : \alpha_1^{-1}] \cdot [\tilde{g} : \alpha_2] \cdot \beta^{-1}$$

$$\Leftrightarrow [\tilde{f} : \alpha_2] \cdot \beta \cdot [\tilde{g} : \alpha_2^{-1}] = [\tilde{f} : \alpha_1] \cdot \beta \cdot [\tilde{g} : \alpha_1^{-1}]$$

$$\Leftrightarrow \varphi(\alpha_1 \cdot \Gamma(\tilde{f}, \beta \cdot \tilde{g})) = \varphi(\alpha_2 \cdot \Gamma(\tilde{f}, \beta \cdot \tilde{g})).$$

Thus, $\varphi$ is a well-defined injection. Since it is obvious that $\varphi$ is onto, we get that $\varphi$ is bijective. Hence, we get that

$$|G(\tilde{f}, \beta \cdot \tilde{g})| = |\text{Aut}(\tilde{M}) : \Gamma(\tilde{f}, \beta \cdot \tilde{g})|.$$

Since $\Gamma(\tilde{f}, \beta \cdot \tilde{g})$ is normal, we further have

$$|G(\tilde{f}, \tilde{g})| = \frac{|\text{Aut}(\tilde{M})|}{|\Gamma(\tilde{f}, \beta \cdot \tilde{g})|} = \frac{|\text{Aut}(\tilde{M})|}{I(\tilde{f}, \beta \cdot \tilde{g})}.$$

By Corollary 5.18, we get

$$|\text{Aut}(\tilde{N})| = |\text{Aut}(\tilde{M})| \cdot \sum_{\beta \in \Lambda} \frac{1}{I(\tilde{f}, \beta \cdot \tilde{g})}. \quad \square$$

The next corollary gives conditions for counting for the $H$—Reidemeister representatives. It follows directly from Theorem 5.20.
Corollary 5.21 — Suppose $\Gamma(\tilde{f}, \beta \cdot \tilde{g})$ is a normal subgroup of $Aut(\tilde{M})$, for each $\beta \in Aut(\tilde{N})$, and $I$ is the same for all $H$-Nielsen classes. Then,

1. We have
   $$|\Lambda| = \frac{|Aut(\tilde{N})| \cdot I}{|Aut(\tilde{M})|}. \quad (5.7)$$

2. if $|Aut(\tilde{N})| = |Aut(\tilde{M})|$, then $|\Lambda| = I$.

3. if $|Aut(\tilde{N})|$ and $|Aut(\tilde{M})|$ are prime numbers and not equal, then $I = |Aut(\tilde{M})|$ and $|\Lambda| = |Aut(\tilde{N})|$.

Next we give sufficient conditions under which $\Gamma(\tilde{f}, \beta \cdot \tilde{g})$ is a normal subgroup of $Aut(\tilde{M})$, for each $\beta \in Aut(\tilde{N})$.

Proposition 5.22 — The following hold true

1. If $[G(\tilde{g}), G(\tilde{f})] = < 1 >$, then $G(\tilde{f}, \tilde{g})$ is a subgroup of $Aut(\tilde{N})$.

2. If $G(\tilde{g}) \subseteq Z(Aut(\tilde{N}))$, then $G(\tilde{f}, \beta \cdot \tilde{g}) = G(\tilde{f}, \tilde{g})$, for every $\beta \in Aut(\tilde{N})$. However, if $G(\tilde{f}) \subseteq Z(Aut(\tilde{N}))$, then $G(\tilde{f}, \beta \cdot \tilde{g})$ is a subgroup which is conjugate to $G(\tilde{f}, \tilde{g})$ by $\beta$, for every $\beta \in Aut(\tilde{N})$.

3. If $G(\tilde{g}) \subseteq Z(Aut(\tilde{N}))$ or $G(\tilde{f}) \subseteq Z(Aut(\tilde{N}))$, then $\Gamma(\tilde{f}, \beta \cdot \tilde{g}) = \Gamma(\tilde{f}, \tilde{g})$, for every $\beta \in Aut(\tilde{N})$, and $\Gamma(\tilde{f}, \tilde{g})$ is a normal subgroup of $Aut(\tilde{M})$.

Proof: (1) Let $\alpha, \alpha_1, \alpha_2 \in Aut(\tilde{M})$. Then,

$$\left( [\tilde{f} : \alpha_1] \cdot [\tilde{g} : \alpha_1^{-1}] \right) \cdot \left( [\tilde{f} : \alpha_2] \cdot [\tilde{g} : \alpha_2^{-1}] \right) = [\tilde{f} : \alpha_1] \cdot \left( [\tilde{g} : \alpha_1^{-1}] \cdot [\tilde{f} : \alpha_2] \right) \cdot [\tilde{g} : \alpha_2^{-1}]$$

$$= [\tilde{f} : \alpha_1] \cdot \left( [\tilde{f} : \alpha_2] \cdot [\tilde{g} : \alpha_1^{-1}] \right) \cdot [\tilde{g} : \alpha_2^{-1}]$$

$$= \left( [\tilde{f} : \alpha_1] \cdot [\tilde{f} : \alpha_2] \right) \cdot \left( [\tilde{g} : \alpha_1^{-1}] \cdot [\tilde{g} : \alpha_2^{-1}] \right)$$

$$= \left( [\tilde{f} : \alpha_1] \cdot [\tilde{f} : \alpha_2] \right) \cdot \left( [\tilde{g} : \alpha_2^{-1}] \cdot [\tilde{g} : \alpha_1^{-1}] \right)$$

$$= [\tilde{f} : \alpha_1 \cdot \alpha_2] \cdot [\tilde{g} : \alpha_1^{-1} \cdot \alpha_1^{-1}]$$

$$= [\tilde{f} : \alpha_1 \cdot \alpha_2] \cdot [\tilde{g} : (\alpha_1 \cdot \alpha_2)^{-1}] \in G(\tilde{f}, \tilde{g}).$$

On the other hand, it is easy to see that $[\tilde{f} : \alpha] \cdot [\tilde{g} : \alpha^{-1}]$ has $[\tilde{f} : \alpha^{-1}] \cdot [\tilde{g} : \alpha] \in G(\tilde{f}, \tilde{g})$ as an inverse.

(2) Apply Proposition 5.2 and Proposition 5.22.
(3) Assume \( G(\bar{g}) \subseteq Z(\text{Aut}(\bar{N})) \) or \( G(\bar{f}) \subseteq Z(\text{Aut}(\bar{N})) \). It is easy to see that \( \Gamma(\bar{f}, \beta \cdot \bar{g}) = \Gamma(\bar{f}, \bar{g}) \) for every \( \beta \in \text{Aut}(\bar{N}) \). Now, let \( \alpha \in \Gamma(\bar{f}, \bar{g}) \). Hence, \( [\bar{f} : \alpha] = [\bar{g} : \alpha] \in G(\bar{f}) \cap G(\bar{g}) \). Let \( \lambda \in \text{Aut}(\bar{M}) \). We need to show that \( \lambda \cdot \alpha \cdot \lambda^{-1} \in \Gamma(\bar{f}, \bar{g}) \). In fact,

\[
[\bar{f} : \lambda \cdot \alpha \cdot \lambda^{-1}] = [\bar{f} : \lambda] \cdot [\bar{f} : \alpha] \cdot [\bar{f} : \lambda^{-1}]
= [\bar{f} : \lambda] \cdot [\bar{f} : \lambda^{-1}] \cdot [\bar{f} : \alpha]
= [\bar{f} : \alpha]
= [\bar{g} : \alpha]
= [\bar{g} : \lambda] \cdot [\bar{g} : \lambda^{-1}] \cdot [\bar{g} : \alpha]
= [\bar{g} : \lambda] \cdot [\bar{g} : \alpha] \cdot [\bar{g} : \lambda^{-1}]
= [\bar{g} : \alpha \cdot \lambda^{-1}]
\]

Therefore, \( \lambda \cdot \alpha \cdot \lambda^{-1} \in \Gamma(\bar{f}, \bar{g}) \).

\[\square\]

6. **The Case Where \(|\text{Aut}(\bar{M})|\) and \(|\text{Aut}(\bar{N})|\) Are Prime Numbers**

In this section, unless otherwise stated, we study the case where \(|\text{Aut}(\bar{M})|\) and \(|\text{Aut}(\bar{N})|\) are prime numbers. We show how Corollary 5.21 explicitly determines, in this case, the \( H \)-Reidemeister representatives. Moreover, it allows us to refine Equation (1.3). In other words, this section generalizes [3], where we use the notion of \( \delta(f, g) \) rather than the notion of even and odd lifts introduced in [3]. Also, we generalize Theorem 2.5 of [3] by giving sufficient and necessary conditions for our refined equation to hold.

Let \( M \) and \( N \) be path connected, locally path connected topological spaces, \((\bar{M}, p)\) and \((\bar{N}, p)\) be regular coverings corresponding to the normal subgroups \( K \subseteq \pi_1(M) \) and \( H \subseteq \pi_1(N) \) of \( M \) and \( N \) respectively. We assume the coverings are finite, and unless otherwise stated that \(|\text{Aut}(\bar{M})|\) and \(|\text{Aut}(\bar{N})|\) are prime numbers. Let \((f, g) : M \longrightarrow N\) be a pair of maps for which there exists a pair of lifts \((\bar{f}, \bar{g}) : \bar{M} \longrightarrow \bar{N}\). Consider the commutative diagram (2.1).

**Lemma 6.1** — Let \( \alpha \in \text{Aut}(\bar{M}) \). If \( \delta(\bar{f}, \bar{g}; \alpha) = 1 \), then \( \delta(\bar{f}, \bar{g}; \sigma) = 1 \) for all \( \sigma \in \langle \alpha \rangle \), where \( \langle \alpha \rangle \) is the cyclic subgroup of \( \text{Aut}(\bar{M}) \) generated by \( \alpha \).

**Proof:** By Proposition 5.2, we have \( [\bar{f} : \alpha^k] = [\bar{f} : \alpha]^k \) for every \( \alpha \in \text{Aut}(\bar{M}) \) and every
integer $k$. Hence, 
\[
\delta(\tilde{f}, \tilde{g}; \alpha) = 1 \iff [\tilde{f} : \alpha] = [\tilde{g} : \alpha]
\]
\[
\iff [\tilde{f} : \alpha]^k = [\tilde{g} : \alpha]^k
\]
\[
\iff [\tilde{f} : \alpha^k] = [\tilde{g} : \alpha^k]
\]
\[
\iff \delta(\tilde{f}, \tilde{g}; \alpha^k) = 1. 
\]

**Proposition 6.2** — Let $\alpha \in \text{Aut}(\widetilde{M})$. Then, $\Box$

1. We have $\delta(\tilde{f}, \tilde{g}; \alpha) = 1$ if and only if $\delta(\tilde{f}, \tilde{g}; \sigma) = 1$ for every $\sigma \in \text{Aut}(\widetilde{M})$.

2. If $(\tilde{f}_0, \tilde{g}_0)$ is another lifting pair of $(f, g)$, then $[\tilde{f}_0 : \alpha] = [\tilde{f} : \alpha]$ and $\delta(\tilde{f}, \tilde{g}; \alpha) = 1$ if and only if $\delta(\tilde{f}_0, \tilde{g}_0; \alpha) = 1$.

**Proof:** (1) Since $\text{Aut}(\widetilde{M})$ has a prime order, we have $\text{Aut}(\widetilde{M}) = \langle \alpha \rangle = \langle \sigma \rangle$. By Lemma 6.1, part (1) holds.

(2) Assume $\delta(\tilde{f}, \tilde{g}; \alpha) = 1$ and $[\tilde{f} : \alpha] = [\tilde{g} : \alpha] = \beta$ for some $\beta \in \text{Aut}(\widetilde{N})$. Let $(\tilde{f}_0, \tilde{g}_0)$ be another lift of $(f, g)$. Since $\text{Aut}(\widetilde{N})$ has a prime order and by Remark 5.9 there exist integers $k$ and $l$ such that $\tilde{f}_0 = \beta^k \tilde{f}$ and $\tilde{g}_0 = \beta^l \tilde{g}$. Hence,
\[
\tilde{f}_0 \alpha = \beta^k \tilde{f} \alpha = \beta^k \beta \tilde{f} = \beta^{k+1} \tilde{f} = \beta \beta^k \tilde{f} = \beta \tilde{f}_0.
\]

Similarly, we get $\tilde{g}_0 \alpha = \beta \tilde{g}_0$. Thus, $[\tilde{f}_0 : \alpha] = [\tilde{g}_0 : \alpha] = \beta$. Therefore, $\delta(\tilde{f}_0, \tilde{g}_0; \alpha) = 1$. The converse can be proved in a similar way, since there are no restrictions on the lifting pairs involved. $\Box$

Proposition 6.2 emphasizes that when $|\text{Aut}(\widetilde{M})|$ and $|\text{Aut}(\widetilde{N})|$ are prime numbers, the value of $[\tilde{f} : \alpha]$ is independent of the selected lift $\tilde{f}$ of $f$, and the value of $\delta(\tilde{f}, \tilde{g}; \alpha)$ is independent of the chosen lift $(\tilde{f}, \tilde{g})$ of $(f, g)$ or $\alpha \in |\text{Aut}(\widetilde{M})|$. Equivalently, the values of $[\tilde{f} : \alpha]$ and $\delta(\tilde{f}, \tilde{g}; \alpha)$ depend only on $f$ and $g$. So, Proposition 6.2 allows us to generalize Definition 3.8 and Notation 5.1 as follows.

**Definition 6.3** — Define $[f : \alpha]$ by $[f : \alpha] := [\tilde{f} : \alpha]$, and $\delta(f, g)$ by $\delta(f, g) := \delta(\tilde{f}, \tilde{g}; \alpha)$, where $\tilde{f}$ and $\tilde{g}$ are any lifts of $f$ and $g$ respectively, and $\alpha \in \text{Aut}(\widetilde{M})$.

The next proposition shows the way to find the $H$–Reidemeister representatives for the case where $|\text{Aut}(\widetilde{M})|$ and $|\text{Aut}(\widetilde{N})|$ are prime numbers.
Proposition 6.4 — Let \( \beta \in Aut(\tilde{N}) - \{1_{\tilde{N}}\} \). Then, there exist exactly \(|Aut(\tilde{N})|^{\delta(f, g)} \) \(H\)-Reidemeister classes, each of which can be represented by a pair of lifts of \( f \) and \( g \) of the form \((\tilde{f}, \beta^i \tilde{g})\) where \( 0 \leq i \leq |Aut(\tilde{N})|^{\delta(f, g)} - 1 \).

PROOF : By Remark 5.9, the action of \( Aut(\tilde{M}) \) on the sets \( \Delta(\beta^i) \), where \( 0 \leq i \leq |Aut(\tilde{N})| - 1 \), places them in their conjugacy classes. The number of these classes depends on the value of \( \delta(f, g) \).

So, we differentiate between two cases.

In the first case, we assume that \( \delta(f, g) = 1 \). Let \( \alpha \in Aut(\tilde{M}) \), \( 0 \leq i \leq |Aut(\tilde{N})| - 1 \), and \((\tilde{f}, \beta^i \tilde{g}) \in \Delta(\beta^i) \). Then,

\[
(\tilde{f}, \beta^i \tilde{g}) \cdot \alpha = (\tilde{f} \cdot \alpha, \beta^i \tilde{g} \cdot \alpha) = (\beta(\tilde{f} \alpha), \beta^i \tilde{g} \alpha) = (\beta[\tilde{f} : \alpha] \tilde{f}, \beta \beta^i[\tilde{g} : \alpha] \tilde{g}) = (\beta[\tilde{f} : \alpha] \tilde{f}, \beta \beta^i[\tilde{f} : \alpha] \tilde{g}) = (\tilde{f} \cdot \alpha, \beta^i \tilde{g}) = \Delta(\beta^i).
\]

That is, the action of \( Aut(\tilde{M}) \) on \( \Delta(\beta^i) \) carries it back on to itself, i.e., the elements of \( \Delta(\beta^i) \) are conjugate only to themselves for each respective \( i \). Hence, in this case, we have \(|Aut(\tilde{N})| \) conjugacy classes (namely \( \Delta(\beta^i) \), where \( 0 \leq i \leq |Aut(\tilde{N})| - 1 \)). That is, the number of \( H \)-Reidemeister classes is \(|Aut(\tilde{N})| \), and each \( H \)-Reidemeister class has \((\tilde{f}, \beta^i \tilde{g})\) as a representative for some \( i \).

In the second case, let us assume \( \delta(f, g) = 0 \). Let \( \alpha \in Aut(\tilde{M}) \), \( 0 \leq i \leq |Aut(\tilde{N})| - 1 \), and \((\tilde{f}, \beta^i \tilde{g}) \in \Delta(\beta^i) \). Suppose \([\tilde{g} : \alpha] = [\tilde{f} : \alpha]^t \) where \( t > 1 \). Then,

\[
(\tilde{f}, \beta^i \tilde{g}) \cdot \alpha = (\tilde{f} \cdot \alpha, \beta \beta^i \tilde{g} \cdot \alpha) = (\beta(\tilde{f} \alpha), \beta \beta^i \tilde{g} \alpha) = (\beta[\tilde{f} : \alpha] \tilde{f}, \beta \beta^i[\tilde{g} : \alpha] \tilde{g}) = (\beta[\tilde{f} : \alpha] \tilde{f}, \beta[\tilde{f} : \alpha]^t \tilde{g}) = (\beta[\tilde{f} : \alpha] \tilde{f}, \beta \beta^i \tilde{g}) = \beta[\tilde{f} : \alpha] \tilde{f}, \beta \beta^i \tilde{g} \in \Delta([\tilde{f} : \alpha]^{t-1} \beta^i).
\]
That is, the action of $\alpha$ maps $\Delta(\beta^i)$ bijectively onto $\Delta([f : \alpha]^{i-1}\beta^i)$, i.e., the elements of $\Delta(\beta^i)$ and $\Delta([f : \alpha]^{i-1}\beta^i)$ are conjugate to each other. Since $[f : \alpha]^{i-1}$ is fixed and $\text{Aut}(\tilde{N})$ is cyclic, if $i$ runs over the set $\{0, 1, \ldots, |\text{Aut}(\tilde{N})| - 1\}$, then each of the elements of $\bigcup_{i=0}^{\lfloor \frac{|\text{Aut}(\tilde{N})| - 1}{2} \rfloor} \Delta(\beta^i)$ is conjugate to the others. Thus, in this case we have only one conjugacy class (namely $\bigcup_{i=0}^{\lfloor \frac{|\text{Aut}(\tilde{N})| - 1}{2} \rfloor} \Delta(\beta^i)$), and hence one $H$-Reidemeister class, which of course can be represented by $(\tilde{f}, \tilde{g})$. $\square$

Next, we present many characterizations for which a pair of maps $(f, g)$ satisfies the condition $\delta(f, g) = 1$. Afterward, we collect our results in Corollary 6.10.

For each $x \in \Phi(f, g)$, we have the following diagram:

\[
\begin{array}{cccccc}
C(f\# , g\#)_x & \xrightarrow{i} & \pi_1(M, x) & \xrightarrow{g\# f^{-1}_\#} & \pi_1(N, f(x)) & \xrightarrow{j} & \pi_1(N, f(x)) / H(f(x)) \\
\downarrow \Theta_M & & (1) & & (2) & & (3) & \downarrow 1 \\
\Theta_M (C(f\# , g\#)_x) & \xrightarrow{i} & H_1(M) & \xrightarrow{g\# - f\#} & H_1(N) & \xrightarrow{j} & \pi_1(N, f(x)) / H(f(x)) \\
& & & \Theta (\tilde{g}\# - \tilde{f}\#) & & & \equiv \text{Aut}(\tilde{N}) \\
\end{array}
\]

(6.1)

where

- $i$ and $i$ are the inclusion homomorphisms on the corresponding groups.
- $\Theta_M$ and $\Theta_N$ are the abelianizations on the corresponding groups.
- 1 in diagram 6.1 denotes the identity.
- The function $g\# f^{-1}_\#$ is defined by $g\# f^{-1}_\#(a) = g\#(a) f\#(a)^{-1}$ for every $a \in \pi_1(M, x)$.

The next lemma does not require that $|\text{Aut}(\tilde{M})|$ and $|\text{Aut}(\tilde{N})|$ be prime.

Lemma 6.5 — If $\text{Aut}(\tilde{N})$ is abelian, then Diagram 6.1 is commutative with $\text{Ker}(\tilde{j}) = \Theta_N(H(f(x)))$ and $\Theta_M (C(f\# , g\#)_x) \subseteq \text{Ker}(\tilde{g}\# - \tilde{f}\#)$.

Proof: Let $F_M$ and $F_N$ be the commutator subgroups of $\pi_1(M, x)$ and $\pi_1(N, f(x))$ respectively.

- Commutativity of box (1): it is obvious that $i$ is well-defined. Let $a \in C(f\# , g\#)_x$. Then,

\[
i \circ \Theta_M(a) = i(\Theta_M(a)) = \Theta_M(a) = \Theta_M(i(a)) = \Theta_M \circ i(a).
\]
• Commutativity of box (2): first, $\bar{f}_{\#}$ is defined such that the diagram

$$
\begin{array}{ccc}
\pi_1(M, x) & \xrightarrow{f_{\#}} & \pi_1(N, f(x)) \\
\Theta_M \downarrow & & \downarrow \Theta_N \\
H_1(M) & \xrightarrow{\bar{f}_{\#}} & H_1(N)
\end{array}
$$

(6.2)

is commutative. What we need to show is that $\bar{f}_{\#}$ is well defined, which is true since $f_{\#}(F_M) \subseteq F_N$. The same is true for $g_{\#}$. Therefore, the homomorphism $g_{\#} - \bar{f}_{\#}$ is well defined. Now, let $a \in \pi_1(M, x)$. Then,

$$
\Theta_N \circ g_{\#} \bar{f}_{\#}^{-1}(a) = \Theta_N (g_{\#}(a) f_{\#}(a)^{-1})
$$

$$
= \Theta_N (g_{\#}(a)) + \Theta_N (f_{\#}(a)^{-1})
$$

$$
= \Theta_N (g_{\#}(a)) - \Theta_N (f_{\#}(a))
$$

$$
= \bar{g}_{\#} (\Theta_M(a)) - \bar{f}_{\#} (\Theta_M(a))
$$

$$
= \bar{g}_{\#} - \bar{f}_{\#} (\Theta_M(a))
$$

$$
= \bar{g}_{\#} - \bar{f}_{\#} \circ \Theta_M(a).
$$

• Commutativity of box (3): we have $\bar{j}$ is defined such that box (3) commutes. To show it is well-defined, it is sufficient to notice that since $\pi_1(N, f(x)) / H(f(x))$ is abelian, $F_N \subseteq H(f(x))$.

• Let $b \in \pi_1(N, f(x))$. Then,

$$
\Theta_N(b) \in Ker(\bar{j}) \iff \bar{j}(\Theta_N(b)) = 0
$$

$$
\iff \bar{j}(b) = 0
$$

$$
\iff b \in H(f(x))
$$

$$
\iff \Theta_N(b) \in \Theta_N(H(f(x))).
$$

• Let $a \in C(f_{\#}, g_{\#})$. Then,

$$
f_{\#}(a) = g_{\#}(a) \Rightarrow \Theta_N(f_{\#}(a)) = \Theta_N(f_{\#}(a))
$$

$$
\Rightarrow \bar{f}_{\#}(\Theta_M(a)) = \bar{g}_{\#}(\Theta_M(a))
$$

$$
\Rightarrow \bar{g}_{\#}(\Theta_M(a)) - \bar{f}_{\#}(\Theta_M(a)) = 0
$$

$$
\Rightarrow \Theta_M(a) \in Ker(\bar{g}_{\#} - \bar{f}_{\#}).
$$
Therefore,
\[ \Theta_M (C(f\#, g\#) \in S) \subseteq \text{Ker}(\tilde{g}\# - \tilde{f}\#). \]

The first characterization, for which a pair \((f, g)\) satisfies that \(\delta(f, g) = 1\), is geometric. The condition characterizes the fact \(\delta(f, g) = 1\) through the action of \(\text{Aut}(\tilde{M})\) on the coincidence set of every pair of lifts \((\tilde{f}, \tilde{g}) \in \text{Lift}(f, g)\).

**Proposition 6.6** — Assume \(|\text{Aut}(\tilde{M})|\) and \(|\text{Aut}(\tilde{N})|\) are prime numbers. The following are equivalent:

1. \(\delta(f, g) = 1\).

2. For every \((\tilde{f}, \tilde{g}) \in \text{Lift}(f, g)\), \(\bar{x} \in \Phi(\tilde{f}, \tilde{g})\), and \(\alpha \in \text{Aut}(\tilde{M})\), we have \(\alpha(\bar{x}) \in \Phi(\tilde{f}, \tilde{g})\).

3. There exist \((\tilde{f}, \tilde{g}) \in \text{Lift}(f, g)\), \(\bar{x} \in \Phi(\tilde{f}, \tilde{g})\), and \(\alpha \in \text{Aut}(\tilde{M})\) such that \(\alpha(\bar{x}) \in \Phi(\tilde{f}, \tilde{g})\).

**Proof**:

\((1) \Rightarrow (2)\) : Assume \(\delta(f, g) = 1\). Let \((\tilde{f}, \tilde{g}) \in \text{Lift}(f, g)\) such that \(\Phi(\tilde{f}, \tilde{g}) \neq \emptyset\), \(\bar{x} \in \Phi(\tilde{f}, \tilde{g})\), and \(\alpha \in \text{Aut}(\tilde{M})\). Then,
\[
\tilde{f}(\alpha(\bar{x})) = \tilde{f} \alpha(\bar{x}) = [\tilde{f} : \alpha] \tilde{f}(\bar{x}) = [\tilde{g} : \alpha] \tilde{g}(\bar{x}) = \tilde{g} \alpha(\bar{x}) = \tilde{g}(\alpha(\bar{x})).
\]

That is, \(\alpha(\bar{x}) \in \Phi(\tilde{f}, \tilde{g})\).

\((2) \Rightarrow (3)\) : Trivial.

\((3) \Rightarrow (1)\) : Suppose there exist \((\tilde{f}, \tilde{g}) \in \text{Lift}(f, g)\), \(\bar{x} \in \Phi(\tilde{f}, \tilde{g})\), and \(\alpha \in \text{Aut}(\tilde{M})\) such that \(\alpha(\bar{x}) \in \Phi(\tilde{f}, \tilde{g})\). Then,
\[
[\tilde{f} : \alpha] \tilde{f}(\bar{x}) = \tilde{f} \alpha(\bar{x}) = \tilde{f}(\alpha(\bar{x})) = \tilde{g}(\alpha(\bar{x})) = \tilde{g} \alpha(\bar{x}) = [\tilde{g} : \alpha] \tilde{g}(\bar{x}) = [\tilde{g} : \alpha] \tilde{f}(\bar{x}).
\]

Thus, \([\tilde{f} : \alpha] = [\tilde{g} : \alpha]\) and hence \(\delta(f, g) = 1\). \(\square\)

The second characterization, for which a pair \((f, g)\) satisfies that \(\delta(f, g) = 1\), is algebraic. It characterizes the fact \(\delta(f, g) = 1\) through relations of the fundamental groups of the considered spaces.

**Proposition 6.7** — Assume \(|\text{Aut}(\tilde{M})|\) and \(|\text{Aut}(\tilde{N})|\) are prime numbers. The following are equivalent:

1. \(\delta(f, g) = 1\).
2. For every \((\tilde{f}, \tilde{g}) \in \text{Lift}(f, g)\) and \(\tilde{x} \in \Phi(\tilde{f}, \tilde{g})\), we have
\[
g_\# f_\#^{-1} (\pi_1(M, p(\tilde{x}))) \subseteq H(\tilde{f}(\tilde{x})) .
\]

3. There exist a lift \((\tilde{f}, \tilde{g}) \in \text{Lift}(f, g)\) and \(\tilde{x} \in \Phi(\tilde{f}, \tilde{g})\) such that
\[
g_\# f_\#^{-1} (\pi_1(M, p(\tilde{x}))) \subseteq H(\tilde{f}(\tilde{x})) .
\]

**Proof:** \(\bullet(1) \Rightarrow (2)\) : Assume that \(\delta(f, g) = 1\). Let \((\tilde{f}, \tilde{g}) \in \text{Lift}(f, g)\) and \(\tilde{x} \in \Phi(\tilde{f}, \tilde{g})\). Put \(p(\tilde{x}) = x\). Let \(a \in \pi_1(M, x)\) and \(\tilde{a}\) be the lift of \(a\) at \(\tilde{x}\). Since the covering is regular, there exists \(\alpha \in \text{Aut}(\tilde{M})\) such that \(a(1) = \alpha(\tilde{x})\). By Proposition 6.6, \(\tilde{f}(\alpha(\tilde{x})) = \tilde{g}(\alpha(\tilde{x}))\). Hence,
\[
\tilde{g}(\tilde{a}) \tilde{f}(\tilde{a})^{-1} \in \pi_1(\tilde{N}, \tilde{f}(\tilde{x})).
\]

Therefore,
\[
g_\# f_\#^{-1}(a) = g(a)f(a^{-1})
\]
\[
= g(p(\tilde{a}))f(p(\tilde{a}^{-1}))
\]
\[
= p(\tilde{g}(\tilde{a}))p(\tilde{f}(\tilde{a}^{-1}))
\]
\[
= p(\tilde{g}(\tilde{a})\tilde{f}(\tilde{a}^{-1})) \in H(\tilde{f}(\tilde{x})).
\]

That is,
\[
g_\# f_\#^{-1} (\pi_1(M, p(\tilde{x}))) \subseteq H(\tilde{f}(\tilde{x})) .
\]

\(\bullet(2) \Rightarrow (3)\) : Trivial.

\(\bullet(3) \Rightarrow (1)\) : Assume
\[
g_\# f_\#^{-1} (\pi_1(M, p(\tilde{x}))) \subseteq H(\tilde{f}(\tilde{x})) .
\]

for some lift \((\tilde{f}, \tilde{g})\) of \((f, g)\) and \(\tilde{x} \in \Phi(\tilde{f}, \tilde{g})\). Let \(\alpha \in \text{Aut}(\tilde{M})\) and \(\tilde{a} : \tilde{x} \rightarrow \alpha(\tilde{x})\) be a path in \(\tilde{M}\).

We have \(p(\tilde{a})\) is a loop in \(M\) at \(p(\tilde{x})\). Thus, there exists \(b \in \pi_1(\tilde{N}, \tilde{f}(\tilde{x}))\) such that \(g_\# f_\#^{-1}(p(\tilde{a})) = p(b)\). Hence,
\[
g(p(\tilde{a}))f(p(\tilde{a}^{-1})) = p(b)
\]
\[
\Rightarrow g(p(\tilde{a})) = p(b)f(p(\tilde{a}))
\]
\[
\Rightarrow p(\tilde{g}(\tilde{a})) = p(b)p(\tilde{f}(\tilde{a}))
\]
\[
\Rightarrow p(\tilde{g}(\tilde{a})) = p(b\tilde{f}(\tilde{a})).
\]
However, \( \tilde{g}(\tilde{a}) \) and \( b \tilde{f}(\tilde{a}) \) are lifts to the same path and having the same initial point \( \tilde{f}(\tilde{x}) \). Thus, they are homotopic relative endpoints and have the same end point, i.e., \( \tilde{f}(\alpha(\tilde{x})) = \tilde{g}(\alpha(\tilde{x})) \) or \( \alpha(\tilde{x}) \in \Phi(\tilde{f}, \tilde{g}) \). By Proposition 6.6, we get \( \delta(f, g) = 1 \).

The third characterization, for which a pair \( (f, g) \) satisfies \( \delta(f, g) = 1 \), is also algebraic. It characterizes the fact \( \delta(f, g) = 1 \) through a sequence of homological groups and homomorphisms of the involved spaces.

**Proposition 6.8** — Assume \( |\text{Aut}(\tilde{M})| \) and \( |\text{Aut}(\tilde{N})| \) are prime numbers. The following are equivalent:

1. \( \delta(f, g) = 1 \).

2. The sequence

\[
\Theta_M(C(f_#, g_#)) \xrightarrow{\tilde{f}_#} H_1(M) \xrightarrow{\tilde{g}_# - \tilde{f}_#} H_1(N) \xrightarrow{\tilde{j}} \text{Aut}(\tilde{N})
\]  

(6.3)

is a chain complex.

**PROOF:** \( \bullet (1) \to (2) \) : Suppose \( \delta(f, g) = 1 \). Let \( x \in \Phi(f, g) \). By Lemma 6.5, \( \Theta_M(C(f_#, g_#)_x) \subseteq \text{Ker}(\tilde{g}_# - \tilde{f}_#) \). Moreover, let \( \Theta_N(b) \in (\tilde{g}_# - \tilde{f}_#)(H_1(M)) \). Then,

\[
\Theta_N(b) = (\tilde{g}_# - \tilde{f}_#)(\Theta_M(a)) \text{ for same } a \in \pi_1(M, x).
\]

\[
= \tilde{g}_#(\Theta_M(a)) - \tilde{f}_#(\Theta_M(a))
\]

\[
= \Theta_N(g_#(a)) - \Theta_N(f_#(a))
\]

\[
= \Theta_N(g_#(a)f_#(a)^{-1})
\]

\[
= \Theta_N(g_# f_#^{-1}(a)).
\]

Since \( g_# f_#^{-1}(\pi_1(M, x)) \in H(f(x)) \), we get that \( \Theta_N(b) \in \Theta_N(H(f(x))) = \text{Ker}\tilde{j} \). Therefore, the sequence 6.3 is a chain complex.

\( \bullet (2) \to (1) \) : Suppose the sequence 6.3 is a chain complex for some (or for every) \( x \in \Phi(f, g) \). Hence, \( (\tilde{g}_# - \tilde{f}_#)(H_1(M)) \subseteq \Theta_N(H(f(x))) \). Let \( b = g_# f_#^{-1}(a) \) and \( a \in \pi_1(M, x) \) Then,

\[
\Theta_N(b) = \Theta_N(g_# f_#^{-1}(a)) = (\tilde{g}_# - \tilde{f}_#)(\Theta_M(a)).
\]

Thus, \( \Theta_N(b) \in \Theta_N(H(f(x))) \). Since \( F_N \subseteq H(f(x)) \), where \( F_N \) is the commutator subgroup of \( \pi_1(N) \), we have \( b \in H(f(x)) \). That is, \( g_# f_#^{-1}(\pi_1(M, x)) \subseteq H(f(x)) \). It follows from Proposition 6.7 that \( \delta(f, g) = 1 \). \( \Box \)
Remark 6.9: In Proposition 6.8, we did not mention the coincidence point of \( f \) and \( g \) at which the sequence 6.3 is applied because the proposition is true whatever the coincidence point of \( f \) and \( g \) is.

Now, we summarize the previous characterizations in the following corollary.

Corollary 6.10 — Assume \( |\text{Aut}(\tilde{M})| \) and \( |\text{Aut}(\tilde{N})| \) are prime numbers. The following are equivalent:

1. \( \delta(f, g) = 1 \).
2. For every \((\tilde{f}, \tilde{g}) \in \text{Lift}(f, g)\), \( \bar{x} \in \Phi(\tilde{f}, \tilde{g}) \), and \( \alpha \in \text{Aut}(\tilde{M}) \), we have \( \alpha(\bar{x}) \in \Phi(\tilde{f}, \tilde{g}) \).
3. There exist \((\tilde{f}, \tilde{g}) \in \text{Lift}(f, g)\), \( \bar{x} \in \Phi(\tilde{f}, \tilde{g}) \), and \( \alpha \in \text{Aut}(\tilde{M}) \) such that \( \alpha(\bar{x}) \in \Phi(\tilde{f}, \tilde{g}) \).
4. There exist a lift \((\tilde{f}, \tilde{g}) \in \text{Lift}(f, g)\) and \( \bar{x} \in \Phi(\tilde{f}, \tilde{g}) \) such that
   \[
   g \# f^{-1} \theta_1(M, p(\bar{x})) \subseteq H(\tilde{f}(\bar{x})).
   \]
5. For every \((\tilde{f}, \tilde{g}) \in \text{Lift}(f, g)\) and \( \bar{x} \in \Phi(\tilde{f}, \tilde{g}) \), we have
   \[
   g \# f^{-1} \theta_1(M, p(\bar{x})) \subseteq H(\tilde{f}(\bar{x})).
   \]
6. The sequence
   \[
   \Theta_M (C(f \#, g \#)) \xrightarrow{\partial} H_1(M) \xrightarrow{\theta_1 \#} H_1(N) \xrightarrow{\partial} \text{Aut}(\tilde{N})
   \]
   is a chain complex for every \( x \in \Phi(f, g) \).
7. The sequence
   \[
   \Theta_M (C(f \#, g \#)) \xrightarrow{\partial} H_1(M) \xrightarrow{\theta_1 \#} H_1(N) \xrightarrow{\partial} \text{Aut}(\tilde{N})
   \]
   is a chain complex for some \( x \in \Phi(f, g) \).

PROOF: Apply Propositions 6.6, 6.7, and 6.8. \( \square \)

The following corollary generalizes part (3) of Corollary 5.21.

Corollary 6.11 — For every nonempty Nielsen class \( A \) of \( f \) and \( g \), we have \( I_A = |\text{Aut}(\tilde{M})|^{\delta(f,g)} \).

PROOF: Suppose \( \delta(f,g) = 1 \) and let \( A \) be a nonempty Nielsen class of \( f \) and \( g \). Let \( x \in A \) and \((\tilde{f}, \tilde{g})\) be a lift of \((f, g)\) such that \( A \subseteq p \Phi(\tilde{f}, \tilde{g}) \). By (2), Corollary 6.10, if \( \bar{x} \in p^{-1}(x) \cap \Phi(\tilde{f}, \tilde{g}) \), then
\( \alpha(\bar{x}) \in \Phi(\bar{f}, \bar{g}) \) for every \( \alpha \in \text{Aut}(\tilde{M}) \). That is, \( p^{-1}(x) \subseteq \Phi(\bar{f}, \bar{g}) \). Therefore, \( I_A = |p^{-1}(x)| = |\text{Aut}(\tilde{M})| \).

Suppose now \( \delta(f, g) = 0 \). Then, \( \Phi(f, g) = p \Phi(\bar{f}, \bar{g}) \). Let \( x \in \Phi(f, g) \) and \( \bar{x} \in p^{-1}(x) \cap \Phi(\bar{f}, \bar{g}) \).

By Proposition 6.6, \( \alpha(\bar{x}) \) does not belong to \( \Phi(\bar{f}, \bar{g}) \). Thus, \( |p^{-1}(x) \cap \Phi(\bar{f}, \bar{g})| = 1 \). This means that for any nonempty Nielsen class \( A \) of \( f \) and \( g \), we have \( I_A = 1 \). \( \Box \)

We now prepare for the main results in this section, Theorems 6.16, 6.18, and 6.19. For each \( x \in \Phi(f, g) \), Consider the diagram:

\[
\begin{array}{ccc}
C(f\#_x, g\#_x) & \xrightarrow{i} & \pi_1(M, x) \\
\downarrow{\Theta_M} & & \downarrow{\Theta_M} \\
\Theta_M(C(f\#_x, g\#_x)) & \xrightarrow{j} & \frac{\pi_1(M, x)}{K(x)} \\
\end{array}
\]

where the homomorphisms are as in 6.1.

**Lemma 6.12** — Let \( x \) be a coincidence point of \( f \) and \( g \). Then, diagram 6.4 commutes, and \( \text{ker} j = \Theta_M(K(x)) \).

**PROOF:** The proof is quite similar to Lemma 6.5. \( \Box \)

**Lemma 6.13** — Let \( x \) be a coincidence point of \( f \) and \( g \). In diagram 6.4, the first horizontal sequence is a chain complex if and only if the second horizontal sequence is a chain complex.

**PROOF:** Assume \( C(f\#_x, g\#_x) \subseteq \text{ker} j = K(x) \). By Lemma 6.12, we have

\( \Theta_M(C(f\#_x, g\#_x)) \subseteq \Theta_M(K(x)) = \text{ker} j \).

Conversely, suppose \( \Theta_M(C(f\#_x, g\#_x)) \subseteq \text{ker} j = \Theta_M(K(x)) \). Since \( F_M \subseteq K(x) \), we get that \( C(f\#_x, g\#_x) \subseteq K(x) \). \( \Box \)

**Lemma 6.14** — Let \( \bar{x}, \bar{y} \in \Phi(\bar{f}, \bar{g}) \) be in the same Nielsen class, and let \( x = p(\bar{x}) \) and \( y = p(\bar{y}) \).

Then,

\( C(f\#_x, g\#_x) \subseteq K(x) \) iff \( C(f\#_y, g\#_y) \subseteq K(y) \).

**PROOF:** It is sufficient to show that if \( C(f\#_x, g\#_x) \subseteq K(x) \) then \( C(f\#_y, g\#_y) \subseteq K(y) \). Let \( \tilde{\omega} : \bar{x} \longrightarrow \bar{y} \) be a path that establishes the Nielsen relation between \( \bar{x} \) and \( \bar{y} \). Put \( \omega = p(\tilde{\omega}) \). We have
the commutative diagram

\[
\begin{array}{cccc}
\pi_1(\tilde{M}, \tilde{x}) & \xrightarrow{\tilde{\omega}_\#} & \pi_1(\tilde{M}, \tilde{y}) \\
\downarrow p_\# & & \downarrow p_\# \\
\pi_1(M, x) & \xrightarrow{\omega_\#} & \pi_1(M, y) \\
f_\# \downarrow g_\# & & f_\# \downarrow g_\#
\end{array}
\]

\[
\pi_1(N, f(x)) \xrightarrow{f(\omega)_\#} \pi_1(N, f(y)).
\]

Notice that \(f(\omega)_\# = g(\omega)_\#\). Let \(a \in \pi_1(M, y)\). Then,

\[
a \in C(f_\#, g_\#)_y \Rightarrow \omega^{-1}_\#(a) \in C(f_\#, g_\#)_x \subseteq K(x)
\]

\[
\Rightarrow \omega^{-1}_\#(a) = p_\#(\tilde{d}); \tilde{d} \in \pi_1(\tilde{M}, \tilde{x})
\]

\[
\Rightarrow a = \omega_\# p_\#(\tilde{d}) = p_\#(\omega_\#(\tilde{d}))
\]

\[
\Rightarrow a \in p_\# (\pi_1(\tilde{M}, \tilde{y})) = K(y).
\]

Let \(\alpha \in Aut(\tilde{M})\) and \(\tilde{x} \in \Phi(\tilde{f}, \tilde{g})\). We define the set \(\alpha \cdot [\tilde{x}]\) by \(\{\alpha \cdot \tilde{y} \mid \tilde{y} \in [\tilde{x}]\}\). The following proposition is a generalization of a part of Theorem 2.5 of [3].

**Proposition 6.15** — Assume \(\delta(f, g) = 1\). Let \(x \in \Phi(f, g)\) and \(\tilde{x} \in p^{-1}(x) \cap \Phi(\tilde{f}, \tilde{g})\). Then, the family \(\{\alpha \cdot [\tilde{x}] \mid \alpha \in Aut(\tilde{M})\}\) is pairwise disjoint if and only if \(C(f_\#, g_\#)_x \subseteq K(x)\).

**Proof:** Assume the family \(\{\alpha \cdot [\tilde{x}] \mid \alpha \in Aut(\tilde{M})\}\) is pairwise disjoint. Let \(a \in C(f_\#, g_\#)_x\) and \(\tilde{a} : \tilde{x} \rightarrow \tilde{y}\) be the lift of \(a\) at \(\tilde{x}\). Hence, \(\tilde{y} \in p^{-1}(x)\). So, there exists \(\alpha \in Aut(\tilde{M})\) such that \(\tilde{y} = \alpha(\tilde{x})\). Moreover,

\[
f(a) = g(a) \Rightarrow f(p(\tilde{a})) = g(p(\tilde{a}))
\]

\[
\Rightarrow p(\tilde{f}(\tilde{a})) = p(\tilde{g}(\tilde{a}))
\]

Since \(\tilde{x}, \tilde{y} \in \Phi(\tilde{f}, \tilde{g})\), we have \(\tilde{f}(\tilde{a}) \sim_0 \tilde{g}(\tilde{a})\), i.e., \([\tilde{y}] = [\tilde{x}]\). However, \(\tilde{y} = \alpha(\tilde{x})\). By the assumptions, we get \(\alpha = 1_{\tilde{M}}\) and \(\tilde{y} = \tilde{x}\). Thus, \(a = p(\tilde{a}) \in p_\#(\pi_1(\tilde{M}, \tilde{x})) = K(x)\). Consequently, \(C(f_\#, g_\#)_x \subseteq K(x)\).

For the converse, suppose \(C(f_\#, g_\#)_x \subseteq K(x)\). Assume, for contrary, that there exists \(\alpha \in Aut(\tilde{M})\) such that \(\alpha \neq 1_{\tilde{M}}\) and \([\tilde{x}] \cap \alpha \cdot [\tilde{x}] \neq \emptyset\). Thus, there exists an open path \(\tilde{a} : \tilde{x} \rightarrow \alpha(\tilde{x})\) (that is, the endpoints of the path are different) in \(\tilde{M}\) such that \(\tilde{f}(\tilde{a}) \sim_0 \tilde{g}(\tilde{a})\). So, we get \(p(\tilde{f}(\tilde{a})) \sim_0 p(\tilde{g}(\tilde{a}))\) or \(f(p(\tilde{a})) \sim_0 g(p(\tilde{a}))\). That is, \(p(\tilde{a}) \in C(f_\#, g_\#)_x \subseteq K(x)\). Hence, there exists
\( \tilde{d} \in \pi_1(\tilde{M}, \tilde{x}) \) such that \( p(\tilde{a}) = p(\tilde{d}) \). The last statement implies that \( \alpha(\tilde{x}) = \tilde{a}(1) = \tilde{d}(1) = \tilde{x} \) and hence \( \alpha = 1_{\tilde{M}} \) which is a contradiction. Therefore, \( [\tilde{x}] \cap \alpha \cdot [\tilde{x}] = \phi \) for every \( \alpha \in \text{Aut}(\tilde{M}) - \{1_{\tilde{M}}\} \), and this in turn yields the information that the family \( \{ \alpha \cdot [\tilde{x}] \mid \alpha \in \text{Aut}(\tilde{M}) \} \) is pairwise disjoint. \( \square \)

The next two theorems generalize Theorem 2.5 of [3], and refine Equation (1.3). Their proofs have partially the same flow as that of Theorem 2.5 of [3].

**Theorem 6.16** — Assume \( \Phi(f, g) \) is finite and \( \delta(f, g) = 1 \). Let \( \beta \in \text{Aut}(\tilde{N}) - \{1_{\tilde{N}}\} \). Then,

\[
N(f, g) = \frac{1}{|\text{Aut}(\tilde{M})|} \cdot \sum_{i=0}^{[\text{Aut}(\tilde{N})]-1} N(\tilde{f}, \beta^i \tilde{g})
\]

if and only if \( \tilde{A} \cap (\alpha \cdot \tilde{A}) = \emptyset \) for every \( \alpha \in \text{Aut}(\tilde{M}) - \{1_{\tilde{M}}\} \), \( \beta^i \in \text{Aut}(\tilde{N}) \) for all \( i \), and for all \( \tilde{A} \in \Phi(\tilde{f}, \beta^i \tilde{g}) \) for which \( p(\tilde{A}) \) is an essential Nielsen class of \( f \) and \( g \).

**Proof:** Set \( |\text{Aut}(\tilde{M})| = P \) and \( |\text{Aut}(\tilde{N})| = Q \). Assume \( \tilde{A} \cap (\alpha \cdot \tilde{A}) = \emptyset \) for every \( \alpha \in \text{Aut}(\tilde{M}) \), \( \beta^i \in \text{Aut}(\tilde{N}) \), and \( \tilde{A} \in \Phi(\tilde{f}, \beta^i \tilde{g}) \) for which \( p(\tilde{A}) \) is an essential Nielsen class of \( f \) and \( g \). Let \( A = \{x_0, \ldots, x_s\} \) be an essential Nielsen class of \( f \) and \( g \). Then, there exists \( 0 \leq i \leq Q - 1 \) such that \( A \subseteq p(\Phi(\tilde{f}, \beta^i \tilde{g})) \). Let \( x_0 \in A \) and \( \tilde{x}_0 \in p^{-1}(x_0) \cap \Phi(\tilde{f}, \beta^i \tilde{g}) \). Let \( \omega_j : x_0 \rightarrow x_j \) be a path in \( M \) which establishes the Nielsen relation between \( x_0 \) and \( x_j \) for \( j = 0, \ldots, s \). Let \( \tilde{\omega}_j \) be the lift of \( \omega_j \) at \( \tilde{x}_0 \). Since the homotopy between \( f(\omega_j) \) and \( g(\omega_j) \) lifts to a homotopy between \( \tilde{f}(\tilde{\omega}_j) \) and \( \beta^i \tilde{g}(\tilde{\omega}_j) \), we get that the points \( \tilde{x}_0, \tilde{x}_1(1), \ldots, \tilde{x}_s(1) \) lie in the same Nielsen class of \( \tilde{f} \) and \( \beta^i \tilde{g} \). On the other hand, since \( \delta(f, g) = 1 \), \( \alpha \cdot [\tilde{x}_0] \) is also a Nielsen class of \( \tilde{f} \) and \( \beta^i \tilde{g} \) for each \( \alpha \in \text{Aut}(\tilde{M}) \); so by the assumptions, the family \( \{ [\tilde{x}_0], \alpha \cdot [\tilde{x}_0], \ldots, \alpha^{P-1} \cdot [\tilde{x}_0] \} \) is mutually disjoint. We show that the union of this family is \( p^{-1}(A) \). Obviously, \( \bigcup_{k=0}^{P-1} \alpha^k \cdot [\tilde{x}_0] \subseteq p^{-1}(A) \). Let \( \tilde{x} \in p^{-1}(A) \). Let \( \sigma : x_0 \rightarrow p(\tilde{x}) \) be a path that establishes the Nielsen relation between \( x_0 \) and \( p(\tilde{x}) \), and \( \tilde{\sigma} \) be its lift in \( \tilde{M} \) at \( \tilde{x}_0 \). Since \( \tilde{\sigma}(1) \in p^{-1}(p(\tilde{x})) \), there exists \( 0 \leq k \leq P - 1 \) such that \( \tilde{\sigma}(1) = \alpha^k(\tilde{x}) \). Thus, \( \alpha^k(\tilde{x}) \in [\tilde{x}_0] \) which implies \( \tilde{x} \in \alpha^{P-k} \cdot [\tilde{x}_0] \). Therefore, \( p^{-1}(A) \subseteq \bigcup_{k=0}^{P-1} \alpha^k \cdot [\tilde{x}_0] \).

Consequently, we have \( S_A = P \). That is, the number \( S \) is fixed for all Nielsen classes of \( f \) and \( g \) and equal to \( P \). Since Corollary 6.11 implies that \( I \) is also equal to \( P \) for all Nielsen classes of \( f \) and \( g \), we get that \( J_A = 1 \) for all Nielsen classes of \( f \) and \( g \). Hence by Theorem 4.4, and Lemma 6.4 we get

\[
N(f, g) = \frac{1}{P} \sum_{i=0}^{Q-1} N(\tilde{f}, \beta^i \tilde{g})
\]
or

\[ N(f, g) = \frac{1}{P} \cdot \sum_{i=0}^{Q-1} N(\tilde{f}, \beta^i \tilde{g}) \cdot \]

For the converse, assume

\[ N(f, g) = \frac{1}{P} \cdot \sum_{i=0}^{Q-1} N(\tilde{f}, \beta^i \tilde{g}) \cdot \]

Let \([x]\) be a nonempty Nielsen class of \(f\) and \(g\). Pick \(\tilde{x} \in p^{-1}(x) \cap \Phi(\tilde{f}, \beta^i \tilde{g})\) for some suitable \(i\). As in the above argument we have \(p^{-1}([x]) = \bigcup_{j=0}^{P-1} \alpha^j \cdot [\tilde{x}]\). We claim that this union is disjoint. Let \(\Psi([x]) = \{[\tilde{x}], \alpha \cdot [\tilde{x}], \ldots, \alpha^{P-1} \cdot [\tilde{x}]\}\) and \(\Psi = \{\Psi([x]) \mid [x] \in \Phi_E(f, g)\}\), where \(\Phi_E(f, g)\) is the set of all essential Nielsen classes of \(f\) and \(g\). Define the function

\[ \tilde{\Phi}_E(f, g) \longrightarrow \Psi \quad : \quad [x] \mapsto \Psi([x]) \cdot \]

We show that this function is a well-defined bijection.

- Let \([x], [y] \in \tilde{\Phi}_E(f, g)\) such that \([x] = [y]\). Let \(\omega : x \rightarrow y\) be a path that establishes the Nielsen relation and \(\tilde{\omega} : \tilde{x} \rightarrow \tilde{y}\) be its lift at \(\tilde{x}\), where \(\tilde{y} \in p^{-1}(y)\). Then, \([\tilde{x}] = [\tilde{y}]\) and hence \(\Psi([x]) = \Psi([y])\). Therefore, the function is well-defined.

- Suppose \(\Psi([x]) = \Psi([y])\) and let \(\tilde{x} \in p^{-1}(x) \cap \Phi(\tilde{f}, \beta^i \tilde{g})\) and \(\tilde{y} \in p^{-1}(y) \cap \Phi(\tilde{f}, \beta^{i_2} \tilde{g})\). Thus, \(i_1 = i_2\). Furthermore, there exists \(j\) with \(0 \leq j \leq P - 1\) such that \([\tilde{y}] = \alpha^j \cdot [\tilde{x}]\). Thus, \([y] = p([\tilde{y}]) = p(\alpha^j \cdot [\tilde{x}]) = p([\tilde{x}]) = [x]\). This implies that the function is one to one.

- Surjectivity is obvious.

Since the function is bijective we get that \(|\Psi| = N(f, g)\). Now, Let \(r\) denote the number of essential classes \([x] \in \Phi(f, g)\) such that \([\tilde{x}] = \alpha^j \cdot [\tilde{x}]\) for some \(j\) with \(0 \leq j \leq P - 1\) (and hence for all \(j\) with \(0 \leq j \leq P - 1\)). In other words, \(r\) is the number of essential classes \([x]\) such that \(|\Psi([x])| = 1\). So, there exist \(N(f, g) - r\) elements in \(\Psi\) each of which has a cardinality of \(P\). Hence,

\[ N(f, g) - r = \frac{1}{P} \cdot \left( \sum_{i=0}^{Q-1} N(\tilde{f}, \beta^i \tilde{g}) \right) - r \]

\[ = \frac{1}{P} \cdot \left( \sum_{i=0}^{Q-1} N(\tilde{f}, \beta^i \tilde{g}) \right) - \frac{r}{P} \]

\[ = N(f, g) - \frac{r}{P} \quad \text{(by the assumptions)} \]
which yields that \( r = 0 \). That is, \([\bar{x}] \cap \alpha \cdot [\bar{x}] = \emptyset\) for every \( \alpha \in Aut(\tilde{M}) \), \( \beta^i \in Aut(\tilde{N}) \), and \([\bar{x}] \in \Phi(\tilde{f}, \beta^i \tilde{g})\) such that \( p([\bar{x}])\) is an essential Nielsen class of \( f \) and \( g \).

\( \square \)

Notice that, although Theorem 6.16 gives a necessary and sufficient condition for Equation 6.5 to hold, it has a drawback. It uses the set of essential classes of \( f \) and \( g \) the very thing we are supposed to count. The following corollary helps us to get around this.

**Corollary 6.17** — Assume \( \delta(f, g) = 1 \). Let \( \beta \in Aut(\tilde{N}) - \{1_{\tilde{N}}\} \). If \( \tilde{A} \cap \left( \alpha \cdot \tilde{A} \right) = \emptyset \) for every \( \alpha \in Aut(\tilde{M}) - \{1_{\tilde{M}}\} \), \( \beta^i \in Aut(\tilde{N}) \) for all \( i \), and for all \( \tilde{A} \in \Phi(\tilde{f}, \beta^i \tilde{g}) \), then

\[
N(f, g) = \frac{1}{|Aut(\tilde{M})|} \cdot \sum_{i=0}^{|Aut(\tilde{N})|-1} N(\tilde{f}, \beta^i \tilde{g})
\]

**PROOF:** Apply Theorem 6.16.

\( \square \)

**Theorem 6.18** — If \( \delta(f, g) = 0 \), then \( N(f, g) = N(\tilde{f}, \tilde{g}) \).

**PROOF:** Assume that \( \delta(f, g) = 0 \). By Corollary 6.11, the number \( I = 1 \). Since \( J \cdot S = I \), we get that \( J = S = 1 \) for every nonempty Nielsen class of \( f \) and \( g \). Since \( J = 1 \) is the same for all classes, Theorem 4.4 implies that

\[
N(f, g) = \sum_{i=0}^{|Aut(\tilde{N})|} \frac{N(\tilde{f}, \beta^i \tilde{g})}{S(\tilde{f}, \beta^i \tilde{g})} = \frac{N(\tilde{f}, \tilde{g})}{1} = N(\tilde{f}, \tilde{g})
\]

We sum up Corollary 6.17 and Theorem 6.18 in the following theorem.

**Theorem 6.19** — Let \( M \) and \( N \) be connected closed manifolds of the same dimension, and let \((\tilde{M}, p)\) and \((\tilde{N}, p)\) be regular coverings corresponding to the normal subgroups \( K \subseteq \pi_1(M) \) and \( H \subseteq \pi_1(N) \) of \( M \) and \( N \) respectively. Assume the coverings are finite and that \( |Aut(\tilde{M})| \) and \( |Aut(\tilde{N})| \) are prime numbers. Let \((f, g) : M \rightarrow N\) be a pair of maps for which there exists a pair of lifts \((\tilde{f}, \tilde{g}) : \tilde{M} \rightarrow \tilde{N}\). If

i. \( \delta(f, g) = 0 \), or

ii. \( \delta(f, g) = 1 \) with \( C(f\# , g\#)_p(x) \subseteq K(p(\bar{x})) \) for every nonempty Nielsen class \([\bar{x}]\) of \((\tilde{f}, \beta^i \tilde{g})\) with \( 0 \leq i \leq |Aut(\tilde{N})| - 1 \),

then

\[
N(f, g) = \frac{1}{|Aut(\tilde{M})|^{\delta(f, g)} \cdot \sum_{i=0}^{|Aut(\tilde{N})|^{\delta(f, g)}-1} N(\tilde{f}, \beta^i \tilde{g})}
\]  \hspace{1cm} (6.6)
7. Examples

The examples in this section have two goals. The first one is to illustrate our method of finding the \( H \)-Reidemeister representatives. The other one is to show how to use Equations \( 1.3 \) and \( 1.4 \) to compute the Nielsen number \( N(f, g) \). The examples given in this section are all new in the literature. Although we could calculate the Nielsen number in some of the following examples by easier methods, we use our method to demonstrate its advantages.

**Example 7.1**: Define the maps \( f, g : L(5, 1) \rightarrow L(5, 1) \) by \( f [\rho_1 \, e^{i \theta_1}, \rho_2 \, e^{i \theta_2}] = [\rho_1 \, e^{4i \theta_1}, \rho_2 \, e^{4i \theta_2}] \) and \( g[z_1, z_2] = [z_2, z_1] \), respectively. The manifold \( L(5, 1) \) is the Lens space (for the definition of a Lens space see Example 2.43 of [5]). Let \( p = p' : S^3 \rightarrow L(5, 1) \) be the quotient map that defines the Lens space. Both maps are well-defined, differ from the identity map, and admit lifts \( \tilde{f}, \tilde{g} : S^3 \rightarrow S^3 \) defined in the natural way. We have the commutative diagram

\[
\begin{array}{ccc}
S^3 & \xrightarrow{\tilde{f}} & S^3 \\
\downarrow p & & \downarrow p \\
L(5, 1) & \xrightarrow{\tilde{f}} & L(5, 1)
\end{array}
\] (7.1)

- Notice that the covering \( (S^3, p) \) is universal. Also, \( Aut(S^3) \cong \mathbb{Z}_6 \). We have \( \delta(f, g) = 0 \), since \( g(\omega(z_1, z_2)) = \omega \tilde{g}(z_1, z_2) \) and \( \tilde{f}(\omega(z_1, z_2)) = \omega^4 \tilde{f}(z_1, z_2) \), or equivalently since \( [f : \omega] = \omega^4 \neq \omega = [g : \omega] \). So, we have one \( H \)-Reidemeister representative, namely \( (\tilde{f}, \tilde{g}) \). Thus, by Theorem 6.18, we have \( N(f, g) = N(\tilde{f}, \tilde{g}) \).

- Now, \( L(\tilde{f}, \tilde{g}) = -15 \neq 0 \). So, \( N(f, g) = N(\tilde{f}, \tilde{g}) = 1 \).

- Notice that \( \Phi(\tilde{f}, \tilde{g}) = \left\{ \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right\} \). Thus, \( |\Phi(f, g)| = 1 = N(f, g) \).

The previous example showed that Equation (6.19) sometimes give the best estimation of the minimum number of coincidence points in the homotopy classes of the considered maps. The next example shows that our method might not be good enough to estimate the number of coincidence points of the maps under consideration.

**Example 7.2**: We use the same spaces given in Example 7.1. Define the maps \( f \) and \( g \) by \( f(z_1, z_2) = [\overline{z_1}, \overline{z_2}] \) and \( g(z_1, z_2) = [z_2, z_1] \), respectively. Both \( f \) and \( g \) are well-defined maps which admit lifts \( \tilde{f} \) and \( \tilde{g} \) defined in the natural way.

- We have \( \delta(f, g) = 0 \), since \( g(\omega(z_1, z_2)) = \omega \tilde{g}(z_1, z_2) \) and \( \tilde{f}(\omega(z_1, z_2)) = \overline{\omega} \tilde{f}(z_1, z_2) \). Similar
to Example 7.1, by Theorem 6.18, we have \( N(f, g) = N(\tilde{f}, \tilde{g}) \). Since \( L(\tilde{f}, \tilde{g}) = 0 \), \( N(f, g) = N(\tilde{f}, \tilde{g}) = 0 \).

- Notice that \( \Phi(\tilde{f}, \tilde{g}) = \{ (z, \tilde{z}) \mid |z| = \frac{1}{\sqrt{2}} \} \). Thus, \( |\Phi(f, g)| = \infty \).

In the next example, we can determine the \( H \)–Reidemeister representatives, but we cannot apply Equation (6.19). However, we apply Equation (1.3) instead.

**Example 7.3:** As in Example 7.1, define the maps \( f, g : L(5, 1) \rightarrow L(5, 1) \) by \( f[\rho e^{i\theta}, z] = [\rho e^{-i\theta}, \tilde{z}] \) and \( g[z_1, z_2] = [z_2, z_1] \). Both maps are well-defined, differ from the identity map, and admit lifts \( \tilde{f} \) and \( \tilde{g} \) defined in the natural way.

- We have \( |Aut(S^3)| = 5 \). Since \( \tilde{f} \) and \( \tilde{g} \) are equivariant under the action of \( Aut(S^3) \cong \mathbb{Z}_5 \) on \( S^3 \), \( \delta(f, g) = 1 \). This means that we have five \( H \)–Reidemeister representatives given by \( (\tilde{f}, \omega^t \cdot \tilde{g}) \), where \( t \) satisfies that \( 0 \leq t \leq 4 \). However, we cannot apply Equation (6.19) or Theorem 6.16 to compute \( N(f, g) \) in this example, because the Nielsen classes \( \Phi(\tilde{f}, \tilde{g}) \) and \( \omega \cdot \Phi(\tilde{f}, \tilde{g}) \) are not disjoint. So, in this case we apply equation (1.3) that is used for the general case.

- Since the covering is universal, we have \( S(\tilde{f}, \omega^t \cdot \tilde{g}) = 1 \) for every \( t \) such that \( 0 \leq t \leq 4 \). This implies also that \( J \) is the same for all Nielsen classes. Therefore, we have

\[
N(f, g) = \sum_{t=1}^{5} N(\tilde{f}, \omega^t \cdot \tilde{g}) = \sum_{t=1}^{5} N(\tilde{f}, \omega^t \cdot \tilde{g}) .
\]

Now, \( \omega^t \cdot \tilde{g} \) is homotopic to \( \tilde{g} \) for all \( t \), we have \( L(\tilde{f}, \omega^t \cdot \tilde{g}) \neq 0 \) for all \( 0 \leq t \leq 4 \). Thus, we obtain that

\[
N(f, g) = 5 \times N(\tilde{f}, \tilde{g}) = 5 \times 1 = 5 .
\]

- It can be shown that

\[
\Phi(\tilde{f}, \omega^t \cdot \tilde{g}) = \left\{ \left( \frac{\omega^k}{\sqrt{2}}, \frac{\omega^{k+t}}{\sqrt{2}} e^{\frac{4\pi t}{25}} \right) \mid k = 0, \ldots, 4 \right\}
\]

for all \( 0 \leq t \leq 4 \). Hence, \( |\Phi(\tilde{f}, \omega^t \cdot \tilde{g})| = 5 \) and \( |\Phi(f, g)| = 5 \times 5 = 25 \).

Next, we give examples which study the general case, where the covering spaces possess covering transformation groups of non-prime cardinality. Of course, more work is expected to find out the \( H \)–Reidemeister representatives, and also to evaluate the Nielsen number. We show that counting the \( H \)–Reidemeister representatives sometimes is useful in determining the representatives, even for
cases where either \(|\text{Aut}(\widetilde{M})|\) or \(|\text{Aut}(\widetilde{N})|\) is not prime. The following lemma is useful in finding the \(H–\text{Reidemeister representatives for general cases. We shall use it in the next examples.}

**Lemma 7.4** — Assume the hypotheses given in the beginning of Section 5. Let \(\beta \in \text{Aut}(\widetilde{N})\). Then, \((\widetilde{f}, \beta \widetilde{g})\) and \((\widetilde{f}, \beta \widetilde{g})\) belong to the same \(H–\text{Reidemeister class if and only if there exist} \alpha \in \text{Aut}(\widetilde{M})\) and \(\hat{\beta} \in \text{Aut}(\widetilde{N})\) such that \((\widetilde{f}, \widetilde{g})\alpha = \hat{\beta}(\widetilde{f}, \beta \widetilde{g})\). Moreover, fixing \(\alpha\), such a \(\hat{\beta}\) is unique.

**Proof:** The proof is quite direct from definition of conjugate lifts. \(\square\)

**Example 7.5:** Let \(f_1, g_1 : L(5, 1) \to L(5, 1)\) and \(f_2, g_2 : S^1 \to S^1\) be maps defined by \(f_1[ r_1 e^{i\theta_1}, r_2 e^{i\theta_2}] = [ r_1 e^{i\theta_0}, r_2 e^{i\theta_2}], g_1 [z_1, z_2] = [z_1, z_2], f_2(e^{i\phi}) = e^{i\delta \phi}, \) and \(g_2(e^{i\phi}) = e^{i3\phi}.\)

Let \(p, p' : S^1 \to S^1\) be the covering maps defined by \(p(z) = z^2\) and \(p'(z) = z^3\), respectively, and \(q : S^3 \to L(5, 1)\) be the quotient map that defines the lens space. Define \(f, g : L(5, 1) \times S^1 \to L(5, 1) \times S^1\) by \(f = f_1 \times f_2\) and \(g = g_1 \times g_2\). We have that \(q \times p, q \times p' : S^3 \times S^1 \to L(5, 1) \times S^1\) are covering maps. Both \(f\) and \(g\) admit lifts \(\widetilde{f} = \widetilde{f}_1 \times \widetilde{f}_2\) and \(\widetilde{g} = \widetilde{g}_1 \times \widetilde{g}_2\) where \(\widetilde{f}_1(r_1 e^{i\theta_0}, r_2 e^{i\theta_2}) = (r_1 e^{i\theta_0}, r_2 e^{i\theta_2}), \widetilde{f}_2(z) = z^4, \widetilde{g}_1 = 1_{S^3},\) and \(\widetilde{g}_2(z) = z^2\), for \(z \in S^1\). Consider the commutative diagram

\[
\begin{array}{ccc}
S^3 \times S^1 & \to & S^3 \times S^1 \\
q \times p \downarrow & & \downarrow q \times p' \\
L(5, 1) \times S^1 & \to & L(5, 1) \times S^1 \\
\end{array}
\]

Notice that the space \(L(5, 1) \times S^1\) is a orientable connected closed smooth manifold. The covering are regular since the fundamental groups of \(L(5, 1), S^1\) and \(L(5, 1) \times S^1\) are abelian. Our goal is to compute \(N(f, g)\).

- **We have** \(\text{Aut}(S^3 \times S^1, q \times p) \cong \text{Aut}(S^3, q) \oplus \text{Aut}(S^1, p) \cong \mathbb{Z}_5 \oplus \mathbb{Z}_2 \cong \mathbb{Z}_{10}.\) Similarly, \(\text{Aut}(S^3 \times S^1, q \times p') \cong \text{Aut}(S^3, q) \oplus \text{Aut}(S^1, p') \cong \mathbb{Z}_5 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_{15}.\)

Let \(\omega, \lambda, \) and \(\mu\) be the 5th, the 3rd, and the square primitive roots of unity respectively. Then, we can write that \(\text{Aut}(S^3 \times S^1, q \times p) = \langle (\omega, \mu) \rangle\) and \(\text{Aut}(S^3 \times S^1, q \times p') = \langle (\omega, \lambda) \rangle\). We choose the \(H–\text{Reidemeister representatives among the members of the set} U = \{(\widetilde{f}, (\omega^l, \lambda^t) \, \widetilde{g}) : 0 \leq l \leq 4 \quad \text{and} \quad 0 \leq t \leq 2\}.\) In fact, we have two methods to do that.

- **Method 1:** Let us compute the number \(|\Lambda|\) of the \(H–\text{Reidemeister representatives. Using some group theory tools leads to the following equation}

\[
I(\widetilde{f}, \widetilde{g}) = I(\widetilde{f}_1, \widetilde{g}_1) \times I(\widetilde{f}_2, \widetilde{g}_2) = 5 \times 2 = 10.
\]

Thus, by Corollary 5.21, we obtain that \(|\Lambda| = \frac{15 \times 10}{10} = 15.\) That is, we have fifteen representatives which are all the members of the set \(U.\)
- **Method 2**: Let us directly find them. This implies using Lemma 7.4. Let $0 \leq k, l \leq 4$ and $0 \leq t \leq 2$. Since $\tilde{f}_2 \mu = \tilde{f}_2$ and $\tilde{g}_2 \mu = \tilde{g}_2$, we have

\[
\left( \tilde{f}, \left( \omega^t, \lambda^t \right) \tilde{g} \right) \left( \omega^k, \mu \right) = \left( \tilde{f}_1 \times \tilde{f}_2 \left( \omega^k, \mu \right), \left( \omega^l, \lambda^t \right) \tilde{g}_1 \times \tilde{g}_2 \left( \omega^k, \mu \right) \right)
\]
\[
= \left( \tilde{f}_1 \omega^k \times \tilde{f}_2 \mu, \omega^l \tilde{g}_1 \omega^k \times \lambda^t \tilde{g}_2 \mu \right)
\]
\[
= \left( \tilde{f}_1 \omega^k \times \tilde{f}_2, \omega^l \tilde{g}_1 \omega^k \times \lambda^t \tilde{g}_2 \right)
\]
\[
= \left( \tilde{f}, \left( \omega^l, \lambda^t \right) \tilde{g} \right) \left( \omega^k, 1_{S^1} \right)
\]
\[
= \left( \omega^k, 1_{S^1} \right) \left( \tilde{f}_1 \times \tilde{f}_2, \left( \omega^l, \lambda^t \right) \tilde{g}_1 \times \tilde{g}_2 \right)
\]
\[
= \left( \omega^k, 1_{S^1} \right) \left( \tilde{f}, \left( \omega^l, \lambda^t \right) \tilde{g} \right).
\]

So, we must choose 15 $H$–Reidemeister representatives, namely $\left( \tilde{f}, \left( \omega^l, \lambda^t \right) \tilde{g} \right)$ for $0 \leq l \leq 4$ and $0 \leq t \leq 2$.

Next, we compute $N(f, g)$. Since the fundamental group of $L(5, 1) \times S^1$ is abelian, the number $J$ only depends on the $H$–Nielsen class. Thus, by Theorem 4.4

\[
N(f, g) = \sum_{l=0}^{4} \sum_{t=0}^{2} \frac{N(\tilde{f}, \left( \omega^l, \lambda^t \right) \tilde{g})}{S(\tilde{f}, \left( \omega^l, \lambda^t \right) \tilde{g})}.
\]

Since $\left( \tilde{f}, \left( \omega^l, \lambda^t \right) \tilde{g} \right)$ is homotopic to $\left( \tilde{f}, \tilde{g} \right)$ and the fundamental group of $L(5, 1) \times S^1$ is abelian, it follows that

\[
N(f, g) = 15 \times \frac{N(\tilde{f}, \tilde{g})}{S(\tilde{f}, \tilde{g})}.
\]

(7.2)

- To compute $S(\tilde{f}, \tilde{g})$, notice that

\[
J(\tilde{f}, \tilde{g}) = J(\tilde{f}_1, \tilde{g}_1) \times J(\tilde{f}_2, \tilde{g}_2) = 5 \times 1 = 5.
\]

Hence, by Proposition 3.19, we have

\[
S(\tilde{f}, \tilde{g}) = \frac{10}{5} = 2.
\]

(7.3)

The computation of $N(\tilde{f}, \tilde{g})$ in this example depends on the Lefschetz number of $\left( \tilde{f}, \tilde{g} \right)$. It can be shown (in fact, it is not an easy calculation) that $L(\tilde{f}, \tilde{g}) = 10 \neq 0$. It follows that $L \left( \tilde{f}, \left( \omega^l, \lambda^t \right) \tilde{g} \right) \neq$
0 for all $t$ and $l$. Hence, $N(\tilde{f}, \tilde{g}) = |\text{Coker}(\tilde{g}_\# - \tilde{f}_\#)|$. We have $\pi_1(S^3 \times S^1, ((1,0),1)) \cong \pi_1(S^1,1)$. If $a_1$ is the generator of $\pi_1(S^1,1)$, by abuse of language we can write

$$(\tilde{g}_\# - \tilde{f}_\#)(a_1) = (\tilde{g}_2 - \tilde{f}_2)(a_1) = 2a_1 - 4a_1 = -2a_1.$$

That is $\text{Im}(\tilde{g}_\# - \tilde{f}_\#) = 2\mathbb{Z}$. Therefore, $|\text{Coker}(\tilde{g}_\# - \tilde{f}_\#)| = \left|\frac{\mathbb{Z}}{2\mathbb{Z}}\right| = 2$ and $N(\tilde{f}, \tilde{g}) = 2$.

- Finally, by Equations 7.2 and 7.3

$$N(f, g) = 15 \times \frac{2}{2} = 15.$$

**Example 7.6**: Let $f, g : S^1 \to S^1$ be maps defined by $f(z) = z^3$ and $g(z) = z^6$ for every $z \in S^1$. Let $p, \tilde{p} : S^1 \to S^1$ be the covering maps defined by $p(z) = z^4$ and $\tilde{p}(z) = z^6$. The maps $f$ and $g$ admit lifts $\tilde{f}$ and $\tilde{g}$ on $S^1$ defined by $\tilde{f}(z) = z^2$ and $\tilde{g}(z) = z^4$ respectively, where $z \in S^1$. Let $\omega$ be the 4th primitive root of unity and $\mu$ be the 6th primitive root of unity. Then, $\text{Aut}(S^1, p) = \langle \omega \rangle$ and $\text{Aut}(S^1, \tilde{p}) = \langle \mu \rangle$.

Now, let us choose the $H$–Reidemeister representatives among $(\tilde{f}, \tilde{g}), (\tilde{f}, \mu \tilde{g}), \ldots, (\tilde{f}, \mu^5 \tilde{g})$ (some of them may lie in the same $H$–Reidemeister class). We have $I(\tilde{f}, \tilde{g}) = \left|\Gamma(\tilde{f}, \tilde{g})\right|$, where $\Gamma(\tilde{f}, \tilde{g}) = \left\{ \alpha \in \text{Aut}(S^1, p) | \delta(\tilde{f}, \tilde{g}; \alpha) = 1 \right\}$. Notice that $[\tilde{f} : \omega] = \mu^3 \neq 1 = [\tilde{g} : \omega]$ and $[\tilde{f} : \omega^2] = 1 = [\tilde{g} : \omega^2]$. Since $\omega \notin \Gamma(\tilde{f}, \tilde{g})$, we have that $\omega^3 = \omega^{-1} \notin \Gamma(\tilde{f}, \tilde{g})$ either (recall that $\Gamma(\tilde{f}, \tilde{g})$ is a subgroup of $\text{Aut}(S^1, p)$). Thus, we get that $I(\tilde{f}, \tilde{g}) = 2$. By Corollary 5.21, we have $|\Lambda| = \frac{6 \times 2}{4} = 3$. That is, we have only three representatives. So, we use Lemma 7.4 to search for them.

- We start with the pair $(\tilde{f}, \tilde{g})$: applying the action of $\text{Aut}(S^1, p)$ from the right on this pair leads us to the following:

  - Assume $(\tilde{f}, \tilde{g}) \omega = \mu^k (\tilde{f}, \mu^j \tilde{g})$ for some $j$ and $k$ with $0 \leq k, j \leq 5$. Then, for every $z \in S^1$ we have

    $$(\tilde{f}, \tilde{g}) \omega (z) = \mu^k (\tilde{f}, \mu^j \tilde{g}) (z)$$

    $\Rightarrow (\omega^2 z^2, \omega^4 z^4) = (\mu^k z^2, \mu^{k+j} z^4)$

    $\Rightarrow k = \omega^2$ and $k + j = 0 \pmod{6}$

    $\Rightarrow k = j = 3$.

    The uniqueness in Lemma 7.4 guarantees that such $k$ and $j$ are unique.

  - Similarly, we have $(\tilde{f}, \tilde{g}) \omega^2 = (\tilde{f}, \tilde{g})$ and $(\tilde{f}, \tilde{g}) \omega^3 = (\tilde{f}, \tilde{g}) \omega = \mu^3 (\tilde{f}, \mu^3 \tilde{g})$ as shown before. Thus, we deduce that $(\tilde{f}, \tilde{g})$ and $(\tilde{f}, \mu^3 \tilde{g})$ belong to the same $H$–Reidemeister class.
A similar argument applied to \((\tilde{f}, \mu \tilde{g})\) and \((\tilde{f}, \mu^2 \tilde{g})\) leads us to deduce that \((\tilde{f}, \mu \tilde{g})\) and \((\tilde{f}, \mu^4 \tilde{g})\) belong to the same \(H\)-Reidemeister class and also that \((\tilde{f}, \mu^2 \tilde{g})\) and \((\tilde{f}, \mu^5 \tilde{g})\) belong to the same \(H\)-Reidemeister class, and there are no other equivalences. Therefore, we choose \((\tilde{f}, \tilde{g})\), \((\tilde{f}, \mu \tilde{g})\), and \((\tilde{f}, \mu^2 \tilde{g})\) as our Reidemeister representatives.

Next, we evaluate \(N(f, g)\). Since \(S^1\) is a Jiang space, the number \(J\) is fixed for all \(H\)-Nielsen classes. By Theorem 4.4

\[
N(f, g) = \sum \frac{N(\tilde{f}, \tilde{g})}{S(\tilde{f}, \tilde{g})},
\]

where the sum runs over all \(H\)-Reidemeister representatives. Since \(\pi_1(S^1, 1)\) is abelian, we get that

\[
N(f, g) = \sum_{i=0}^{i=2} \frac{N(\tilde{f}, \mu^i \tilde{g})}{S(\tilde{f}, \mu^i \tilde{g})} = 3 \times \frac{N(\tilde{f}, \tilde{g})}{S(\tilde{f}, \tilde{g})}.
\]

Now, we want to compute both \(S(\tilde{f}, \tilde{g})\) and \(N(\tilde{f}, \tilde{g})\). Since \(S^1\) is a Jiang space and \(deg(\mu^k \tilde{g}) = deg(\tilde{g})\) for all \(k \leq 5\), then \(\mu^k \tilde{g}\) is homotopic to \(\tilde{g}\) and hence \(L(\tilde{f}, \mu^k \tilde{g}) = L(\tilde{f}, \tilde{g}) = 4 - 2 = 2 \neq 0\) for all \(k \leq 5\). Thus, we get that \(N(\tilde{f}, \tilde{g}) = |deg(\tilde{g}) - deg(\tilde{f})| = |4 - 2| = 2\). On the other hand, let \(z = 1 \in \Phi(\tilde{f}, \tilde{g})\), then \(p(1) = 1 \in \Phi(f, g)\). We know that \(J(\tilde{f}, \tilde{g}) = |\{C(f_\#, g_\#)_{x=1}\}|\), but as in the previous example \(C(f_\#, g_\#)_{x=1} = Ker(g_\# - f_\#) = 0\) since \(g_\# - f_\#\) is a monomorphism. Hence, \(J(\tilde{f}, \tilde{g}) = 1\). Finally,

\[
S(\tilde{f}, \tilde{g}) = \frac{I(\tilde{f}, \tilde{g})}{J(\tilde{f}, \tilde{g})} = \frac{2}{1} = 2,
\]

and hence \(N(f, g) = \frac{3 \times 2}{2} = 3\). This, of course, is the same result obtained if we used the well known computation methods applied to the Jiang spaces.

Squares in plane have a smooth structure. However, not every atlas makes a square smooth (intuitively, squares have corners and so they are non-smooth). In fact, we should not worry about smoothness of squares because our method consists on orientability not smooth structures. The next example illustrates the application of Theorem 4.4 on maps on squares. Of course other methods might be easier than ours, however we want to represent how our method works.

**Example 7.7:** Let \(M\) be a square in the \(xy\)-plane, lying entirely inside the unit circle \(S^1\) and centered at the origin. Let \(h: S^1 \to M\) be the homeomorphism defined by \(z \mapsto w\), where \(w\) is the intersection point of the square and the line segment joining the origin and the point \(z\). Notice that \(Arg(z) = Arg(w) := \theta\), where \(Arg(z)\) represents the principal argument of the complex number \(z\). Let \(f, g: M \to M\) be maps defined by \(f(r(\theta)e^{i\theta}) = r((-\theta))e^{-i\theta}\) and \(g(r(\theta)e^{i\theta}) = r(2\theta)e^{i2\theta}\),
respectively, where \( r(\theta) = \frac{h(e^{i\theta})}{e^{i\theta}} \) the length of the complex number \( w = h(e^{i\theta}) \). We shall compute \( N(f, g) \). Consider the commutative diagram

\[
\begin{array}{ccc}
S^1 & \xrightarrow{\tilde{f}, \tilde{g}} & S^1 \\
\downarrow h & & \downarrow h \\
M & \xrightarrow{\tilde{f}, \tilde{g}} & M
\end{array}
\]  

(7.4)

The covering \((S^1, h)\) is a 1-fold regular covering for \(M\). There is only one Reidemeister representative \((\tilde{f}, \tilde{g})\) for the pair \((f, g)\), where \(\tilde{f}\) and \(\tilde{g}\) are defined by \(\tilde{f}(e^{i\theta}) = e^{-i\theta}\) and \(\tilde{g}(e^{i\theta}) = e^{i2\theta}\). Since \(|\text{Aut}(S^1, h)| = 1\), we have \(I(\tilde{f}, \tilde{g}) = J(\tilde{f}, \tilde{g}) = S(\tilde{f}, \tilde{g}) = 1\). Since \(S^1\) is a Jiang space, it can be shown, similarly to the previous example, that \(N(\tilde{f}, \tilde{g}) = 3\). Theorem 4.4 implies that

\[N(f, g) = \frac{N(\tilde{f}, \tilde{g})}{S(\tilde{f}, \tilde{g})} = \frac{3}{1} = 3.\]

**Remark 7.8:** The covering map (i.e. the homeomorphism) in the previous example is a 1-fold covering map. To obtain an \(n\)-fold covering map for the same square, compose the homeomorphism with the self-map on \(S^1\) defined by \(z \mapsto z^n\), where \(S^1\) is considered as a subset of the complex numbers.

**REFERENCES**


