ON A FUNCTIONAL CONNECTED TO THE LAPLACIAN IN A FAMILY OF PUNCTURED REGULAR POLYgons IN $\mathbb{R}^2$

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Let $p_1$ and $p_0$ be closed, regular, convex, concentric polygons having $n$ sides in $\mathbb{R}^2$ such that the circumradius of $p_0$ is strictly less than the inradius of $p_1$. We fix $p_1$ and vary $p_0$ by rotating it about its center. Let $\Omega$ be the interior of $p_1 \setminus p_0$. Let $u$ be the solution of the stationary problem $-\Delta u = 1$ in $\Omega$ vanishing on the boundary. We show that the associated Dirichlet energy functional $J(\Omega)$ attains its extremum values when the axes of symmetry of $p_0$ coincide with those of $p_1$.

Key words: Laplacian operator; boundary value problems for second order elliptic equations; variational methods for second order elliptic equations; second order elliptic equations; variational methods.

1. INTRODUCTION

Let $R > 0$ be an arbitrary constant. Fix $n \in \mathbb{N}$ with $n \geq 3$. Let $p_1$ be a regular polygon of $n$ sides in $\mathbb{R}^2$ circumscribed by the circle $C(0, R)$. Without loss of generality, let $(0, R) \in \mathbb{R}^2$ be one of the vertices of $p_1$. Let $r > 0$ be such that the circle $C(0, r)$ is contained in the interior of $p_1$. Throughout the paper the objects $R$, $r$ and $p_1$ are fixed. Let $p_0$ denote any regular polygon of $n$ sides circumscribed by the circle $C(0, r)$ and let $\mathcal{F}_n = \mathcal{F}_n(R, r)$ denote the family of domains $\Omega = (p_1 \setminus p_0)^c$, the interior of $p_1 \setminus p_0$ in $\mathbb{R}^2$.

We consider the following stationary problem on $\Omega$:

$$
\begin{align*}
-\Delta u &= 1 \text{ on } \Omega \text{ in the sense of distributions,} \\
u &= 0 \text{ on } \partial \Omega.
\end{align*}
$$

(1.0.1)

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Let \( u = y(\Omega) \) be the unique solution of problem (1.0.1). We study the extrema of the functional

\[
J(\Omega) = \int_{\Omega} \| \nabla y(\Omega) \|^2 \, dx
\]

on \( \mathcal{F}_n \) associated to (1.0.1).

Following [1], we introduce two special positions of \( p_0 \) in \( p_1 \): If a vertex of \( p_0 \) and a vertex of \( p_1 \) lie on the same half-axis of symmetry emanating from the origin \( 0 \) then we say that \( p_0 \) occupies 
\textit{on position} in \( p_1 \). If a vertex of \( p_1 \) lies on a half-axis of symmetry passing through the origin and the midpoint of one of the sides of \( p_0 \) then we say that \( p_0 \) occupies \textit{off position} in \( p_1 \). In the general position of \( p_0 \) neither of the two conditions is satisfied. Rotation by an integral multiple of \( 2\pi/n \) is a
symmetry of \( \Omega \) at all positions of \( p_1 \). However the on and off positions possess additional reflectional symmetry along the axes mentioned above.

\[\begin{align*}
\text{p}_0 \text{ in ‘on position’ for } n=3. \\
p_0 \text{ in ‘off position’ for } n=3.
\end{align*}\]

We state our main result.

\textbf{Theorem 1.1} — For \( \Omega \in \mathcal{F}_n \), the functional \( J(\Omega) \) is optimized exactly when the axes of symmetry of \( p_0 \) coincide with those of \( p_1 \). The minimizing configuration for \( J(\Omega) \) corresponds to case when \( p_0 \) occupies ‘on position’ in \( p_1 \). The maximizing configuration for \( J(\Omega) \) corresponds to the case when \( p_0 \) occupies ‘off position’ in \( p_1 \).

The above result has been proved in [5] for a family of domains \( \Omega = B_1 \setminus B_0 \) where \( B_0 \) is an
arbitrary closed ball contained in the interior \( B_1^c \) of the closed ball \( B_1 \) in Euclidean space \( \mathbb{R}^n \).

It is well known that for any bounded domain \( \Omega \) having smooth boundary the solution \( y(\Omega) \) of (1.0.1) belongs to \( C^\infty(\Omega) \subset H^2(\Omega) \). However the domains we consider are non-convex polygons for which the solution of (1.0.1) is not a smooth function on \( \Omega \) and instead lies in the Sobolev space \( H_0^{1+\delta}(\Omega) \), where \( \delta \in \left[ \frac{1}{2}, \frac{3}{5} \right] \) ([3]).
The Dirichlet energy functional $J(\Omega)$ associated to the stationary problem (1.0.1) is differentiable with respect to variation of domain when $\Omega$ has convex or $C^2$-boundary and expressions for the derivative are well-known ([8]) in such cases. Here we derive an expression for the derivative of $J(\Omega)$ when $\Omega \in \mathcal{F}_n$ in §3 and apply the standard technique of domain reflection pioneered by Serrin in [7] to prove our main result in §4.

2. Preliminaries

Let $\Omega$ be a bounded polygonal domain in $\mathbb{R}^d$.

**Definition 2.1** — For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d$, $|\alpha| = \alpha_1 + \ldots + \alpha_d$. A function $v \in L^2(\Omega)$ is said to be an $\alpha$-th weak partial derivative of a function $u \in L^2(\Omega)$ if

$$\int_{\Omega} u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_{\Omega} v \varphi \, dx, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

The $\alpha$-th weak derivative of $u$ is denoted by $\partial^\alpha u$, if it exists.

We define the Sobolev space $H^s(\Omega)$ for $s \geq 0$:

(i) For $s = m \in \mathbb{Z}^+$,

$$H^m(\Omega) = \left\{ u \in L^2(\Omega) \; \bigg| \; \partial^\alpha u \in L^2(\Omega), \; \forall |\alpha| \leq m \right\}.$$

For $u, v \in H^m(\Omega)$, define

$$\langle u, v \rangle_{H^m} = \sum_{|\alpha| \leq m} \langle \partial^\alpha u, \partial^\alpha v \rangle_{L^2(\Omega)}.$$

(ii) For $s = m + \sigma$, where $m \in \mathbb{Z}^+$ and $\sigma \in [0, 1[$,

$$H^s(\Omega) = \left\{ u \in H^m(\Omega) \; \bigg| \; \sum_{|\alpha| = m} \int_{\Omega \times \Omega} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{\|x - y\|^{d+2\sigma}} \, dx \, dy < \infty \right\}.$$

For $u, v \in H^s(\Omega)$, define

$$\langle u, v \rangle_{H^s} = \langle u, v \rangle_{H^m} + \sum_{|\alpha| = m} \int_{\Omega \times \Omega} \frac{(\partial^\alpha u(x) - \partial^\alpha u(y)) (\partial^\alpha v(x) - \partial^\alpha v(y))}{\|x - y\|^{d+2\sigma}} \, dx \, dy.$$

$H^s(\Omega)$ is a Hilbert space with above inner product and the induced norm is called the Sobolev norm. The closure of $\mathcal{D}(\Omega)$ in $H^s(\Omega)$ is denoted by $H^s_0(\Omega)$. 
Definition 2.2 — Let \( \Omega \) be an open subset of \( \mathbb{R}^d \). We say \( \Omega \) has Lipschitz (polygonal) boundary if for every \( p \in \partial \Omega \), there exists a neighborhood \( U \) of \( p \) in \( \mathbb{R}^d \) such that \( \partial \Omega \cap U \) is the graph of a Lipschitz (continuous piecewise linear) function.

Proposition 2.3 (cf. Theorem 1.2.7, p. 4 of [3]) — Let \( \Omega \) be an open subset of \( \mathbb{R}^d \) having Lipschitz boundary. Then \( \mathcal{D}'(\bar{\Omega}) \) is dense in \( H^s(\Omega) \) for all \( s \geq 0 \).

Definition 2.4 — Let \( \mathcal{D}'(\Omega) \) denote the space of all distributions on \( \Omega \). For \( T \in \mathcal{D}'(\Omega) \), we define \( \Delta T \in \mathcal{D}'(\Omega) \) by

\[
\langle \Delta T; \varphi \rangle := \langle T; \Delta \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\Omega).
\]

Definition 2.5 — Let \( u \in L^1_{loc}(\Omega) \). Define \( T_u \in \mathcal{D}'(\Omega) \) by

\[
\langle T_u; \varphi \rangle := \int_{\Omega} u \varphi \, dx, \quad \forall \varphi \in \mathcal{D}(\Omega).
\]

For \( u, v \in L^1_{loc}(\Omega) \), we say \( \Delta u = v \) in the sense of distributions if \( \Delta T_u = T_v \).

For \( s \geq 0 \), the following definition now makes sense:

Definition 2.6 — \( E^s(\Delta, L^2(\Omega)) = \left\{ u \in H^s(\Omega) \mid \Delta u \in L^2(\Omega) \right\} \).

Thus we have an operator \( \Delta : E^s(\Delta, L^2(\Omega)) \to L^2(\Omega) \). For \( u, v \in E^s(\Delta, L^2(\Omega)) \), define

\[
\langle u, v \rangle_{E^s(\Delta, L^2(\Omega))} = \langle u, v \rangle_{H^s(\Omega)} + \langle \Delta u, \Delta v \rangle_{L^2(\Omega)}.
\]

\( E^s(\Delta, L^2(\Omega)) \) is a Hilbert space with this inner product. We denote by \( E^s_0(\Delta, L^2(\Omega)) \) (\( s \geq 1 \)) the Hilbert space \( E^s(\Delta, L^2(\Omega)) \cap H^1_0(\Omega) \) with the above inner product.

Proposition 2.7 (cf. Lemma 3.1, p. 261 of [2]) — Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^d \) having Lipschitz boundary. Then \( \mathcal{D}'(\bar{\Omega}) \) is dense in \( E^s(\Delta, L^2(\Omega)) \) for any \( 0 \leq s < 2 \) with \( s \neq \frac{3}{2}, \frac{1}{2} \).

For the remainder of this section \( \Omega \) denotes a bounded open set in \( \mathbb{R}^2 \) which has polygonal boundary and let \( \Gamma_1, \Gamma_2, \ldots, \Gamma_N \) be open line segments such that \( \partial \Omega = \bigcup_{j=1}^{N} \Gamma_j \). Let \( n_j \) denote the outward unit normal to \( \Omega \) on the side \( \Gamma_j \) (\( 1 \leq j \leq N \)).

Proposition 2.8 (Trace operator: cf. Theorem 1.4.2, p. 12 of [3]) — Fix \( s > \frac{1}{2} \). The map \( u \mapsto u|_{\Gamma_j} \) from \( \mathcal{D}'(\bar{\Omega}) \) extends as a bounded linear operator from \( H^s(\Omega) \) into \( H^{s-\frac{1}{2}}(\Gamma_j) \).

Remark 2.9: By Proposition 2.8, \( \forall u \in H^1_0(\Omega), u|_{\Gamma_j} = 0 \).

We need the Green’s Identities:

Proposition 2.10 (cf. Proposition 3.4, p. 264 of [2])
Let $s > \frac{1}{2}$. Let $v \in E^{1+s}(\Delta, L^2(\Omega))$ and $u \in H^1(\Omega)$ be arbitrary. Then

$$
\int_{\Omega} u \Delta v = - \int_{\Omega} \langle \nabla u, \nabla v \rangle \, dx + \sum_{j=1}^{N} \int_{\Gamma_j} u_{|\Gamma_j} \frac{\partial v}{\partial n_j} \, ds.
$$

**Proposition 2.11** (cf. Proposition 3.5, p. 264 of [2]) — Let $s > \frac{1}{2}$. Let $u \in E^{1+s}_0(\Delta, L^2(\Omega))$ and $v \in E^s(\Delta, L^2(\Omega))$ be arbitrary. Then

$$
\int_{\Omega} u \Delta v - v \Delta u \, dx = - \sum_{j=1}^{N} \int_{\Gamma_j} v_{|\Gamma_j} \frac{\partial u}{\partial n_j} \, ds.
$$

**Proposition 2.12** (cf. Proposition 3.6, p. 265 of [2]) — For every $f \in L^2(\Omega)$ there exists a unique solution $u \in H^1_0(\Omega)$ of the Dirichlet boundary value problem:

$$
\begin{align*}
\Delta u &= f \text{ on } \Omega \text{ in the sense of distributions,} \\
\quad &\quad u = 0 \text{ on } \partial \Omega.
\end{align*}
$$

(2.12.1)

Let $\Lambda = \frac{\pi}{\max j \omega_j}$, where $\omega_j$ denotes the interior angle at the $j$-th vertex of $\Omega$. Then $\Lambda > \frac{1}{2}$. It may be noted that for a non-convex polygonal domain, $\Lambda < 1$. We state the following regularity result.

**Proposition 2.13** (cf. Corollary 2.4.4 and Remark 2.4.6, pp. 58-59 [3]) — Let $s_0 = \min\{1, \Lambda\} > \frac{1}{2}$. If $u \in H^1_0(\Omega)$ such that $\Delta u \in L^2(\Omega)$, then $u \in H^{1+s}(\Omega), \forall s < s_0$.

By the above result $E^{1+s}_0(\Delta, L^2(\Omega)) = E^{1+t}_0(\Delta, L^2(\Omega)), \forall 0 \leq s, t \leq s_0$. We fix an arbitrary constant $\delta \in \left[\frac{1}{2}, \frac{3}{5}\right]$ for the remainder of this paper.

**Proposition 2.14** (cf. Proposition 4.1, p. 267 of [2]) — $\Delta : E^{1+\delta}_0(\Delta, L^2(\Omega)) \rightarrow L^2(\Omega)$ is an invertible bounded operator.

**Proposition 2.15** (The Maximum Principle, cf. Theorem 5, p. 61 of [6]) — Let $u(x_1, x_2, \ldots, x_d)$ satisfy the differential inequality

$$
L[u] = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial u}{\partial x_i} \geq 0
$$

in a domain $D \subset \mathbb{R}^d$ where $L$ is uniformly elliptic. Suppose the coefficients $a_{ij}$ and $b_i$ are uniformly bounded. If $u$ attains a maximum $M$ at a point of $D$, then $u \equiv M$ in $D$. 
Proposition 2.16 (The Hopf Maximum Principle, cf. Theorem 7, p. 65 [6]) — Let \( u \) satisfy the inequality
\[
L[u] \equiv \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial u}{\partial x_i} \geq 0
\]
in a domain \( D \subset \mathbb{R}^d \) where \( L \) is uniformly elliptic. Suppose that \( u \leq M \) in \( D \) and that \( u = M \) at a boundary point \( P \). Assume that \( P \) lies on the boundary of a ball \( K_1 \) in \( D \). If \( u \) is continuous in \( D \cup P \) and an outward directional derivative \( \partial u / \partial \nu \) exists at \( P \), then \( \partial u / \partial \nu > 0 \) at \( P \) unless \( u \equiv M \).

3. Material Derivative and Hadamard Formula

Recall that \( \mathcal{F}_n \) is the family of punctured sets \( \Omega = (p_1 \setminus p_0)^\circ \) where \( p_1 \) is a fixed closed, convex, regular polygon having \( n \) sides circumscribed on the circle \( C(0, R) \) and \( p_0 \) a similar polygon circumscribed on \( C(0, r) \) \((R > r > 0)\). Let \( B(0, s) \) denote the open disk of radius \( s > 0 \) centered at the origin \( 0 \) in \( \mathbb{R}^2 \).

The in-radius of \( p_1 \) is defined as \( \sup \{ s > 0 \mid C(0, s) \subset p_1 \} \). We denote the in-radius of \( p_1 \) by \( \text{inrad}(p_1) \). Let \( r_1 = \frac{r + \text{inrad}(p_1)}{2} \). Let \( \varphi : \mathbb{R}^2 \to \mathbb{R} \) be a radial, non-negative \( C^\infty \)-function having compact support such that \( \varphi(x) = 1, \forall x \in B(0, r_1) \) and \( \varphi(x) = 0, \forall x \in \mathbb{R}^2 \setminus \overline{B}(0, \text{inrad}(p_1)) \).

The sides are numbered such that \( \Gamma_1, \ldots, \Gamma_n \) are contained in the inner polygonal boundary and \( \Gamma_{n+1}, \ldots, \Gamma_{2n} \) are contained in the outer polygonal boundary. Following the same convention, let \( Q_1, \ldots, Q_n \) denote the vertices on the inner boundary and let \( Q_{n+1}, \ldots, Q_{2n} \) denote the vertices on the outer boundary. Let \( \omega_j \) denote the interior angle at vertex \( Q_j \) for \( j = 1, \ldots, 2n \).

We denote by \( R_\theta \) a counter-clockwise rotation of \( \mathbb{R}^2 \) by an angle of \( \theta \). Consider the vector field \( V \) on \( \mathbb{R}^2 \) defined by
\[
V(x) = \varphi(x)R_{\pi/2}(x), \ (x \in \mathbb{R}^2).
\]

Throughout this section we fix \( \Omega \in \mathcal{F}_n \) arbitrarily and \( y \) denotes the unique solution \( y(\Omega) \) of (1.0.1).

Let \( \{ \psi_t \}_{t \in \mathbb{R}} \) be the 1-parameter group of diffeomorphisms of \( \mathbb{R}^2 \) associated with \( V \). Thus \( \psi_t \) is given explicitly by the formula
\[
\psi_t(x) = R_{t, \varphi(x)}(x) \ \forall \ x \in \mathbb{R}^2.
\]

Let \( \Omega_t \) denote \( \psi_t(\Omega) \). Observe that \( \mathcal{F}_n = \{ \Omega_t \mid t \in \mathbb{R} \} \). Let \( y_t \in H^1_0(\Omega_t) \) be the unique solution of (1.0.1) with \( \Omega = \Omega_t \) and let \( y^t(t \in \mathbb{R}) \) denote the composite \( y_t \circ \psi_{t|\Omega} \). It follows by Proposition
2.13 that \( y^t \in \mathcal{E}_0^{1+\delta}(\Delta, L^2(\Omega)) \) and by the Elliptic Regularity Theorem \( y^t \in C^\infty(\Omega) \). Following [4] we prove the following result.

**Proposition 3.1** — The map \( t \mapsto y^t \) is a \( C^1 \)-curve in \( \mathcal{E}_0^{1+\delta}(\Delta, L^2(\Omega)) \) from a neighborhood of zero in \( \mathbb{R} \).

**Proof:** For each \( t \in \mathbb{R} \) let \( \gamma_t(x) = \text{det}((d\psi_t)_x) \) where \( (d\psi_t)_x \) denotes the total derivative of \( \psi_t \) at \( x \in \mathbb{R}^2 \).

Let \( \varphi \in \mathcal{D}(\Omega) \) be arbitrary. Since \( \psi_t^{-1} \) is a diffeomorphism, \( \varphi \circ \psi_t^{-1} \in \mathcal{D}(\Omega_t) \). Therefore

\[
\int_{\Omega_t} \varphi \circ \psi_t^{-1} \, dx = \int_{\Omega_t} (-\Delta y_t)(\varphi \circ \psi_t^{-1}) \, dx \quad \text{(by (1.0.1))}
\]

\[
= \int_{\Omega_t} \langle \nabla y_t, \nabla (\varphi \circ \psi_t^{-1}) \rangle \, dx \quad \text{(by Proposition 2.10).}
\]

By the change of variable \( \psi_t : \Omega \to \Omega_t \) we obtain

\[
\int_{\Omega_t} \langle \nabla y_t, \nabla (\varphi \circ \psi_t^{-1}) \rangle \, dx = \int_{\Omega_t} \langle (\nabla y_t) \circ \psi_t, (\nabla (\varphi \circ \psi_t^{-1})) \circ \psi_t \rangle \gamma_t \, dx
\]

\[
= \int_{\Omega} \langle (d\psi_t)^* x (d\psi_t)^* \nabla (y_t \circ \psi_t), (d\psi_t)^* x (d\psi_t)^* \nabla (\varphi \circ \psi_t^{-1} \circ \psi_t) \rangle \gamma_t \, dx
\]

\[
= \int_{\Omega} \langle A_t \nabla y^t, \nabla \varphi \rangle \, dx .
\]

where \( A_t(x) = \gamma_t(x) ((d\psi_t)_x (d\psi_t)_x)^{-1} \) (the superscript * indicates the adjoint of the linear map).

From (3.1.1) and (3.1.2) we obtain

\[
\int_{\Omega} \langle A_t \nabla y^t, \nabla \varphi \rangle \, dx = \int_{\Omega_t} \varphi \circ \psi_t^{-1} \, dx
\]

\[
= \int_{\Omega} \varphi \gamma_t \, dx .
\]

Let \( \Omega' \) be any relatively compact open subset of \( \Omega \) having \( C^\infty \) boundary and containing \( \text{supp}(\varphi) \). By the Divergence Theorem of Gauss applied to the smooth vector field \( X = \varphi(A_t \nabla y^t) \) defined on \( \Omega' \) we obtain

\[
\int_{\Omega'} \langle A_t \nabla y^t, \nabla \varphi \rangle \, dx + \int_{\Omega'} (\text{div}(A_t \nabla y^t)) \varphi \, dx = 0.
\]

Hence

\[
\int_{\Omega} \langle A_t \nabla y^t, \nabla \varphi \rangle \, dx = - \int_{\Omega} (\text{div}(A_t \nabla y^t)) \varphi \, dx .
\]

\[
\text{(3.1.4)}
\]
From (3.1.3) and (3.1.4) we get
\[
\int_{\Omega} \gamma_t \varphi \, dx = -\int_{\Omega} (\text{div}(A_t \nabla y^t)) \varphi \, dx, \quad \forall \varphi \in \mathcal{D}(\Omega).
\]  
(3.1.5)

It follows that
\[-\text{div}(A_t \nabla y^t) = \gamma_t \text{ on } \Omega.
\]  
(3.1.6)

Define \( \delta_{A_t} : E_{0}^{1+\delta}(\Delta, L^2(\Omega)) \to L^2(\Omega) \) by
\[
\delta_{A_t} u(x) = -\text{div}(A_t \nabla u)(x).
\]

Note that \( \delta_{A_t} u(x) = -\Delta u(x) \) for \( x \in \Omega \cap B(0, r_1) \) and \( x \in \Omega \setminus \overline{B}(0, \text{inrad}(p_1)) \). Thus each \( \delta_{A_t} : E_{0}^{1+\delta}(\Delta, L^2(\Omega)) \to L^2(\Omega) \) is a bounded operator.

Next define \( F : \mathbb{R} \times E_{0}^{1+\delta}(\Delta, L^2(\Omega)) \to L^2(\Omega) \) by
\[
F(t, u) = -\delta_{A_t}(u) + \gamma_t.
\]

Then \( F \) is a \( C^1 \)-map and by (3.1.6) \( F(t, y^t) = 0 \). Also \( (D_2 F)(0, y)(0, u) = \Delta u \) and hence by Proposition 2.14 \( (D_2 F)(0, y) : E_{0}^{1+\delta}(\Delta, L^2(\Omega)) \to L^2(\Omega) \) is an isomorphism. By the Implicit Function Theorem \( t \mapsto y^t \) is a \( C^1 \)-map in a neighborhood of zero in \( \mathbb{R} \).

Let \( \Omega' \) be any relatively compact open subset of \( \Omega \) with \( C^\infty \)-boundary.

**Lemma 3.2 (cf. [8])** — Let \( t \mapsto f(t) \) be a differentiable curve in \( L^2(\Omega) \), with \( f(0) \in H^1(\Omega) \). Let \( \{\psi_t\}_{t \in \mathbb{R}} \) be the one parameter group of diffeomorphisms of \( \mathbb{R}^2 \) associated with a compactly supported smooth vector field \( V \) defined on \( \mathbb{R}^2 \). Then for \( t \) sufficiently close to zero \( t \mapsto f(t) \circ \psi_t \) is a differentiable curve in \( L^2(\Omega') \) and at \( t = 0 \)
\[
\left. \frac{d(f(t) \circ \psi_t)}{dt} \right|_{t=0} = \left. \frac{df}{dt} \right|_{t=0} + \langle \nabla f(0), V \rangle_{\Omega'} \in L^2(\Omega').
\]

As a consequence of Lemma 3.2 we have the following two corollaries.

**Corollary 3.3** — The map \( t \mapsto y_{t|_{\Omega'}} \) is a \( C^1 \)-curve in \( L^2(\Omega') \) from a neighborhood of zero in \( \mathbb{R} \)
and
\[
\left. \frac{d}{dt} \right|_{t=0} (y_{t|_{\Omega'}}) = \left( \left. \frac{dy^t}{dt} \right|_{t=0} - \langle \nabla y^t, V \rangle_{\Omega'} \right)_{t|_{\Omega'}} \in L^2(\Omega').
\]

**PROOF:** Observe that \( y_t = y^t \circ \psi_t^{-1} = y^t \circ \psi_{-t} \). By Proposition 3.1 the map \( t \mapsto y^t \) is a \( C^1 \) curve in \( H^1(\Omega) \), so \( t \mapsto y_t \) satisfies the conditions of Lemma 3.2. The result now follows from the
observation that \( \{ \psi_{-t} \}_{t \in \mathbb{R}} \) is the one parameter group of diffeomorphisms of \( \mathbb{R}^2 \) associated with the vector field \( -V \).

**Definition 3.4** — The material derivative of \( y = y(\Omega) \) in the direction of \( V \) is defined by

\[
\dot{y}(\Omega, V) := \left. \frac{dy}{dt} \right|_{t=0} \in E_0^{1+\delta}(\Delta, L^2(\Omega)).
\]

**Definition 3.5** — The shape derivative \( y' \) in the direction of \( V \) is defined by

\[
y' = \dot{y} - \langle \nabla y, V \rangle.
\]

**Corollary 3.6** —

\[
y'_{\Omega_t} = \left. \frac{d}{dt} \right|_{t=0} (y_{\Omega_t}).
\]

Consider the domain functional \( J(\Omega_t) \) \( t \in \mathbb{R} \) defined in §1. By Proposition 2.10 we have

\[
J(\Omega_t) = \int_{\Omega_t} y_t \, dx.
\]

**Definition 3.7** — The Eulerian derivative \( dJ(\Omega, V) \) of \( J(\Omega_t) \) at \( t = 0 \) is defined as

\[
dJ(\Omega, V) := \lim_{t \to 0} \frac{J(\Omega_t) - J(\Omega)}{t}.
\]

For the sake of completeness we include a proof of the following Proposition which can be found in [8].

**Proposition 3.8** — The functional \( J(\Omega_t) \) is differentiable at \( t = 0 \) and

\[
dJ(\Omega, V) = \int_{\Omega} y' \, dx.
\]

**Proof:** Let \( \omega \) denote the volume form on \( \mathbb{R}^2 \). Let \( L_V \omega \) denote the Lie derivative of \( \omega \) with respect to \( V \) and \( i_V \omega \) denote the interior multiplication of \( \omega \) with respect to \( V \). Then

\[
\frac{d}{dt} (\psi_t^* \omega) \bigg|_{t=0} = L_V \omega - (di_V + i_V d)\omega = d(\iota_V \omega) = \text{div}(V) \omega
\]
and
\[
dJ(\Omega, V) = \lim_{t \to 0} \frac{J(\Omega_t) - J(\Omega)}{t} = \lim_{t \to 0} \int_{\Omega} \left( \frac{y' \gamma_t \omega - y \omega}{t} \right)\qquad \text{(where } \gamma_t = det(d\psi_t))
\[
= \int_{\Omega} \left( \frac{d}{dt} \left( y' \gamma_t \omega \right) \right) |_{t=0} \quad \text{(by Proposition 3.1)}
\[
= \int_{\Omega} \left( \dot{y} + y \text{div}(V) \right) \omega
\[
= \int_{\Omega} \left( y' + \langle \nabla y, V \rangle + y \text{div}(V) \right) \omega
\[
= \int_{\Omega} y' \omega + \int_{\Omega} d(y i_V \omega)
\[
= \int_{\Omega} y' \omega.
\]

As in §2 let \( \Gamma_1, \ldots, \Gamma_{2n} \) be the open segments of the boundary \( \partial \Omega = \bigcup_{j=1}^{2n} \Gamma_j \) and let \( n_j \) denote the outward unit normal to \( \Omega \) on the side \( \Gamma_j \). The sides \( \Gamma_j \) are numbered such that \( \Gamma_1, \ldots, \Gamma_n \) are contained in the inner polygonal boundary and \( \Gamma_{n+1}, \ldots, \Gamma_{2n} \) are contained in the outer polygonal boundary.

**Proposition 3.9** — The shape derivative \( y' \in H^\delta(\Omega) \) is a weak solution of the following Dirichlet Boundary Value Problem:

\[
\Delta v = 0 \text{ on } \Omega
\]
\[
v|_{\Gamma_j} = -\frac{\partial v}{\partial n_j} \langle V, n_j \rangle \in H^{\delta-1/2}(\Gamma_j), \forall j = 1, \ldots, 2n.
\]

(3.9.1)

Thus \( y' \in E^\delta(\Delta, L^2(\Omega)). \)

**Proof:** Fix \( \varphi \in \mathcal{D}(\Omega) \) arbitrarily. Choose a relatively compact open subset \( \Omega' \subset \Omega \) having \( C^\infty \) boundary such that \( \text{supp}(\varphi) \subset \Omega' \). Choose \( a > 0 \) such that \( \Omega'' \subset \Omega_t, \forall |t| < a \). Then for each \( |t| < a, \varphi \in \mathcal{D}(\Omega_t) \) and we have

\[
\int_{\Omega_t} \varphi \, dx = -\int_{\Omega_t} (\Delta y_t) \varphi \, dx \quad \text{(by (1.0.1))}
\]
\[
= -\int_{\Omega_t} y_t (\Delta \varphi) \, dx \quad \text{(by Definition 2.5)}.
\]

Hence

\[
\int_{\Omega'} \varphi \, dx = \int_{\Omega'} y_t (\Delta \varphi) \, dx.
\]

(3.9.2)
By Corollary 3.6 we can differentiate both sides of (3.9.2) with respect to $t$ (at $t = 0$) to obtain

$$\int_{\Omega'} y'(\Delta \varphi) \, dx = 0.$$  

Since $\varphi \in \mathcal{D}(\Omega)$ is arbitrary, $\Delta y' = 0$ in the sense of distributions. 

Since $y' \in E^{k}(\Delta, L^{2}(\Omega)), y'_{|\Gamma_j} \in H^{\delta-1/2, \forall j = 1, \ldots, 2n}$ by Proposition 2.8. Further

$$y'_{|\Gamma_j} = \hat{y}_{|\Gamma_j} - \langle \nabla y, V \rangle_{|\Gamma_j}$$

$$= - \langle \nabla y, V \rangle_{|\Gamma_j} \quad (\because \hat{y} \in E^{1,\delta}_{0}(\Delta, L^{2}(\Omega)) \subset H^{1}_{0}(\Omega) \text{ and by Remark 2.9})$$

$$= - \frac{\partial y}{\partial n} \langle V, n_{j} \rangle \quad (\because y \in H^{1}_{0}(\Omega)). \quad \square$$

Now we derive what is commonly known as the ‘Hadamard formula’.

**Theorem 3.10** — Define $j(t) = J(\Omega_{t})$. Then $j(t)$ is differentiable at $t = 0$ and its derivative is given by

$$j'(0) = \sum_{j=1}^{n} \int_{\Gamma_{j}} \left( \frac{\partial y}{\partial n_{j}} \right)^{2} \langle V, n \rangle \, ds.$$ 

**PROOF** : By Proposition 3.8, $j'(0) = \int_{\Omega} y' \, dx$. Since $-\Delta y = 1$ on $\Omega$ by (1.0.1) and $\Delta y' = 0$ on $\Omega$ by (3.9.1) we have

$$\int_{\Omega} y' \, dx = \int_{\Omega} ((\Delta y') y - (\Delta y)y') \, dx$$

$$= - \sum_{j=1}^{2n} \int_{\Gamma_{j}} y'_{|\Gamma_{j}} \frac{\partial y}{\partial n_{j}} \, ds \quad (\text{by Proposition 2.11})$$

$$= \sum_{j=1}^{n} \int_{\Gamma_{j}} \left( \frac{\partial y}{\partial n_{j}} \right)^{2} \langle V, n_{j} \rangle \, ds \quad (\text{by Proposition 3.9}) \quad \square$$

4. PROOF OF THE MAIN THEOREM

Recall $F_{n}, p_{1}, R$ and $r$ from §1. Let $p_{2}$ be a regular polygon having $n$ sides which has $(0, r) \in \mathbb{R}^{2}$ as one of its vertices, circumscribed on the circle $C(0, r)$. Let $\Omega_{0} = (p_{1} \setminus p_{2})^{o}$ be fixed throughout this section.

Let $V$ be the vector field defined in §3 and $\{\psi_{t}\}$ the 1-parameter group of diffeomorphisms of $\mathbb{R}^{2}$ associated with $V$. Let $\Omega_{t} = \psi_{t}(\Omega_{0})$. Then $F_{n} = \{\Omega_{t} \mid t \in \mathbb{R}\}$. Since $R_{2\pi/n}(p_{2}) = p_{2}$, $\psi_{2\pi/n}(\Omega) = \Omega, \forall \Omega \in F_{n}$. Therefore $F_{n} = \left\{ \Omega_{t} \mid t \in \left[ -\frac{\pi}{n}, \frac{\pi}{n} \right] \right\}$.
Recall $J(\Omega_t) = \int_{\Omega_t} y_t \, dx$ from §3 where $y_t$ is the unique solution of (1.0.1) with $\Omega = \Omega_t$. Define $j(t) = J(\Omega_t)$, for $t \in \mathbb{R}$.

We use the following notations:

1. For $p, q \in \mathbb{R}^2$, $[p, q] = \{ p + t(q - p) \mid 0 \leq t \leq 1\}$ denotes the closed line segment joining $p$ & $q$. Half-open or open segments are represented similarly.

2. $\Gamma_{1,t}, \ldots, \Gamma_{n,t}$ denote the open segments of the inner polygonal boundary of $\Omega_t$ as in §3 numbered counter-clockwise and $n_{j,t}$ denotes the outward unit normal to side $\Gamma_{j,t}$. The side $\Gamma_{1,0}$ of $\Omega_0$ meets the side $\Gamma_{n,0}$ at vertex $(0, r) \in \mathbb{R}^2$ and $\Gamma_{j,t} = R_t(\Gamma_{j,0})$ for $j = 1, \ldots, n$. $\Gamma_{n+1,}, \ldots, \Gamma_{2n}$ denote the open segments of the outer polygonal boundary of $\Omega$ which is the same for every $\Omega \in \mathcal{F}_n$.

Since $\psi_{s+t} = \psi_s \circ \psi_t$, $j(t)$ is differentiable as a function of $t$ by Theorem 3.10 applied to $\Omega = \Omega_t$ and

$$j'(t) = \sum_{j=1}^{n} \int_{\Gamma_{j,t}} \left( \frac{\partial y_t}{\partial n_{j,t}} \right)^2 \langle V, n_{j,t} \rangle \, ds.$$ 

Since $R_{2\pi/n}$ is a symmetry of $\Omega_t$

$$j'(t) = n \int_{\Gamma_{1,t}} \left( \frac{\partial y_t}{\partial n_{1,t}} \right)^2 \langle V, n_{1,t} \rangle \, ds. \quad (4.0.1)$$

Let $m_t$ be the midpoint of $\Gamma_{1,t} = [b_t, c_t]$, where $b_t$ and $c_t$ denote the left and right endpoints of $\Gamma_{1,t}$ respectively. See Figure 1. Let $\sigma_t$ denote the reflection map of $\mathbb{R}^2$ about the line passing through $m_t$ and the origin $0$.

![Figure 1](image-url)
Since \( n_{1,t} \) is constant along \( \Gamma_{1,t} \) and \( n_{1,t} = \frac{m_t}{\|m_t\|} \), \( \sigma_t \) fixes \( n_{1,t} \). Therefore \( x + \sigma_t(x) \) is a scalar multiple of \( n_{1,t} \) for each \( x \in \mathbb{R}^2 \).

Let \( \Gamma = \{m_t, \sigma_t\} \). Then \( \Gamma_{1,t} = \Gamma \cup \sigma_t(\Gamma) \). For any \( x \in \Gamma \),

\[
\langle V(x) + V(\sigma_t(x)), n_{1,t} \rangle = \langle R_{\pi/2}(x) + R_{\pi/2}(\sigma_t(x)), n_{1,t} \rangle \\
= \langle R_{\pi/2}(x + \sigma_t(x)), n_{1,t} \rangle \\
= 0 \quad (\cdot, x + \sigma_t(x) \in \mathbb{R}n_{1,t}).
\]

Hence

\[
\langle V(x), n_{1,t} \rangle = -\langle V(\sigma_t(x)), n_{1,t} \rangle, \quad \forall x \in \Gamma. \quad (4.0.2)
\]

Therefore by (4.0.2)

\[
j'(t) = n \left( \int_\Gamma \left( \frac{\partial y_t}{\partial n_{1,t}} \right)^2 \langle V, n_{1,t} \rangle \ ds - \int_\Gamma \left( \frac{\partial(y_t \circ \sigma_t)}{\partial n_{1,t}} \right)^2 \langle V, n_{1,t} \rangle \ ds \right). \quad (4.0.3)
\]

Note that \( \sigma_t \) is a symmetry of \( \Omega_t \) for \( t = 0, \pm \frac{\pi}{n} \) and hence \( y_t \circ \sigma_t = y_t, \forall t = 0, \pm \frac{\pi}{n} \). Therefore by (4.0.1) and (4.0.3), \( j'(t) \) vanishes at \( t = 0, \pm \frac{\pi}{n} \). Since \( \sigma_0(\Omega_t) = \Omega_t, j(t) \) is an even function of \( t \).

Hence the result follows if we show that \( j'(t) > 0, \forall t \in \left]0, \frac{\pi}{n}\right[ \).

So let \( t \in \left]0, \frac{\pi}{n}\right[ \). We identify \( z = (a, b) \in \mathbb{R}^2 \) with \( a + ib \in \mathbb{C} \). So \( n_{1,t} = \frac{m_t}{\|m_t\|} = -e^{i(t+\pi/2+\pi/n)} \). For \( z \in \Gamma, V(z) = iz \) and \( \text{Arg}(e^{-it}z) \in \left[\frac{\pi}{2}, \frac{\pi}{2} + \frac{\pi}{n}\right] \). It follows that

\[
\text{Re} (V(z)n_{1,t}) = -\text{Re} \left( iz e^{-i(t+\pi/2+\pi/n)} \right) = -\text{Re} \left( e^{-i\pi/n} e^{-it}z \right) \\
= -\cos \left( \text{Arg}(e^{-it}z) - \frac{\pi}{n} \right) < 0.
\]

Hence

\[
\langle V(x), n_{1,t} \rangle < 0, \forall x \in \Gamma. \quad (4.0.4)
\]

Let \( x^* \) denote \( \sigma_t(x) \) for \( x \in \mathbb{R}^2 \). Using (4.0.4) to prove that \( j'(t) > 0 \) it remains to show that

\[
\left| \frac{\partial y_t}{\partial n_{1,t}}(x^*) \right| > \left| \frac{\partial y_t}{\partial n_{1,t}}(x) \right|, \quad \forall x \in \Gamma.
\]

By (1.0.1), \( \Delta(-y_t) = 1 > 0 \) on \( \Omega_t \) and \( -y_t(x) = 0, \forall x \in \partial \Omega_t \). Hence by the Maximum Principle (Proposition 2.15), \( y_t(x) > 0, \forall x \in \Omega_t \). Then Proposition 2.16 applies and we have

\[
\frac{\partial y_t}{\partial n_{1,t}}(x) < 0, \quad \forall x \in \Gamma. \quad (4.0.5)
\]
Let \( p, q \in \Gamma_{n+1} \) be the points where the lines extending \([0, m_t]\) and \([0, c_t]\) meet \( \Gamma_{n+1} \). Consider the quadrilateral \( G \) having the vertices \( c_t, m_t, p, q \). Then \( \sigma_t(G) \subset \Omega_t \). We define \( w(x) = y_t(x) - y_t(x^*) \) \( \forall x \in G \). Then

\[
\Delta w = 0, \quad \text{on } G,
\]
\[
w < 0, \quad \text{on } [p, q],
\]
\[
w = 0, \quad \text{on } \Gamma.
\]

Hence by the Hopf Maximum Principle \( \frac{\partial w}{\partial n_{1,t}}(x) > 0, \forall x \in \Gamma \). By (4.0.5), \( \left| \frac{\partial y_t}{\partial n_{1,t}}(x^*) \right| > \left| \frac{\partial y_t}{\partial n_{1,t}}(x) \right|, \forall x \in \Gamma \).

\[\square\]

REFERENCES


