MATRIX PRODUCT(MODULO-2) OF GRAPHS

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(Received 22 April 2013; after final revision 23 September 2013;
accepted 8 November 2013)

In continuation of the results obtained in [3] for the realization of the product of adjacency matrices under usual matrix multiplication, this article presents some interesting characterizations and properties of the graphs for which the product of adjacency matrices under modulo-2 is graphical.

Key words: Adjacency matrix; matrix product; realizability.

1. INTRODUCTION

Graphs considered in this paper are simple, undirected, and without self loops. Let $G$ be a graph on the set of vertices $\{v_1, v_2, \ldots, v_n\}$. Two vertices $v_i$ and $v_j$, $i \neq j$, are said to be adjacent to each other if there is an edge between them. An adjacency between the vertices $v_i$ and $v_j$ is denoted by $v_i \sim_G v_j$, and $v_i \not\sim_G v_j$ denotes that $v_i$ is not adjacent with $v_j$ in the graph $G$. The adjacency matrix of $G$ is a matrix $A(G) = (a_{ij}) \in M_n(\mathbb{R})$ in which $a_{ij} = 1$ if $v_i$ and $v_j$ are adjacent, and $a_{ij} = 0$ otherwise. Given two graphs $G$ and $H$ on the same set of vertices $\{v_1, v_2, \ldots, v_n\}$, $G \cup H$ represents the union of graphs $G$ and $H$ on the same set of vertices, where two vertices are adjacent in $G \cup H$ if they are adjacent in at least one of $G$ and $H$. Graphs $G$ and $H$ on the same set of vertices are said to be (mutually) edge disjoint if $u \sim_G v$ implies that $u \not\sim_H v$. Equivalently, $H$ is a sub graph of $\overline{G}$ and vice-versa.

For any terminology or notation, which are not defined but used in this paper, we refer to the books by West [1] and Buckley–Harary [2].

Definition 1.1 (Graphical Matrix; Akbari [4]) — A symmetric $(0, 1)$-matrix is said to be graphical if all its diagonal entries equal zero.
In fact, a graphical matrix is an adjacency matrix of some graph. If \( B \) is a graphical matrix such that \( B = A(G) \) for some graph \( G \), then we often say that \( G \) is the realization of graphical matrix \( B \).

**Definition 1.2 (Matrix Product; Manjunatha Prasad et al. [3])** — Consider any two graphs \( G \) and \( H \) on the same set of vertices. A graph \( \Gamma \) is said to be the matrix product of \( G \) and \( H \) if \( A(G)A(H) \) is graphical and \( \Gamma \) is the realization of \( A(G)A(H) \).

We shall extend the above definition of matrix product of graphs when the matrix multiplications is considered over the integers modulo-2.

**Definition 1.3 (Matrix Product (mod-2) of Graphs)** — The graph \( \Gamma \) is said to be a matrix product (mod-2) of graphs \( G \) and \( H \) if \( A(G)A(H) \) (mod-2) is graphical and \( \Gamma \) is the realization of \( A(G)A(H) \) (mod-2).

In this paper, we focus our attention on studying the case when the modulo-2 product of adjacency matrices is graphical.

If \( G \) is any graph and \( H \) is a totally disconnected graph (i.e., empty graph with no edges) on the same set of vertices as that of \( G \), then the product \( A(G)A(H) \) (mod-2) is the null matrix and hence it is graphical. So, the case where either \( G \) or \( H \) is a totally disconnected graph is a trivial case and we consider only the nontrivial cases, where both the graphs \( G \) and \( H \) are nonnull for further discussion.

Similarly, if \( \Gamma \) is the realization of modulo-2 product of \( G \) and \( H \), and \( v \) is an isolated vertex in \( G \) or in \( H \), then the vertex \( v \) remains to be an isolated vertex in \( \Gamma \). This can be verified easily through the matrix multiplication. If \( v \) is an isolated vertex in \( G \), then the column and rows corresponding to that isolated vertex in \( A(G) \) contain only zeros. Therefore, the problem of realization of the product \( A(G)A(H) \) (mod-2) reduces to the case of realization of \( A_1B_1 \) (mod-2), where \( A_1 \) and \( B_1 \) are the block matrices of \( A(G) \) and \( A(H) \), respectively, and corresponding to the set of vertices excluding the isolated vertex. Therefore, we consider the graphs \( G \) and \( H \) which have no isolated vertices, for all further discussion.

**Definition 1.4 (GH path; Manjunatha Prasad et al. [3])** — Given graphs \( G \) and \( H \) on the same set of vertices \( \{v_1, v_2, \ldots, v_n\} \), two vertices \( v_i \) and \( v_j \) \((i \neq j)\) are said to have a \( GH \) path if there exists a vertex \( v_k \), different from \( v_i \) and \( v_j \), such that \( v_i \sim_G v_k \) and \( v_k \sim_H v_j \) (See Figure 1).

Capital letter \( G \) (similarly, \( H \)) that occur beside any line/edge joining two vertices in the Figure 1 does not represents the label of that edge, but it represents a graph in which the corresponding end vertices are adjacent.

**Definition 1.5 (Parity regular graphs)** — A graph \( G \) is said to be a parity regular if all the vertices
are either of even degree or of odd degree.

Section 2 contains some results dealing with the conditions on graphs $G$ and $H$ for which $A(G)$ $A(H) \pmod{2}$ is graphical.

2. Main Results

2.1 Realization of $A(G)A(H)$

In Manjunatha Prasad et al. [3], the ordinary matrix product was considered and some properties of graphs $G$ and $H$ were studied for the realization of $A(G)A(H)$. A necessary and sufficient condition for such a realization was that $H$ must be a sub graph of complement of $G$ and for each ordered pair of distinct vertices, the numbers of $GH$ paths and $HG$ paths are the same and equal 1 or 0. When the matrix product is considered over modulo-2, this condition would not remain as a necessary condition. In fact, the class of graphs $H$, for which $A(G)A(H) \pmod{2}$ is graphical, is found to be a larger class. In the following Example 2.1, we have a graph $H$ such that $A(G)A(H) \pmod{2}$ is graphical but $H$ is not a sub graph of $\overline{G}$. Henceforth in this paper, all the matrix multiplications are considered over integers modulo-2 and writing $A(G)A(H)$ would mean $A(G)A(H) \pmod{2}$, unless indicated otherwise.

Example 2.1 : In the following graphs $G$ and $H$ (as shown in Figure 2) on seven vertices, $H$ is not a sub graph of $\overline{G}$. Note that the adjacency matrices of $G$ and $H$ are

$$
A(G) = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 
\end{bmatrix} \quad \text{and} \quad A(H) = \begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 
\end{bmatrix}
$$
respectively, and $A(G)A(H) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$. Clearly, $A(G)A(H)$ is graphical and $K_7$ is the graph realizing $A(G)A(H)$.

Figure 2: Graphs $G$ and $H$ for which $K_7$ is the realization of $A(G)A(H)$

In the following lemmas and theorem, we shall consider the graphs $G$ and $H$ on the same set of vertices $\{v_1, v_2, \ldots, v_n\}$.

**Lemma 2.1** — The diagonal entries of the matrix product $A(G)A(H)$ are zeros if and only if for each vertex $v_i$, the cardinality of the set of vertices $\{v_k : v_k \sim_G v_i, v_k \sim_H v_i\}$ is even.

**Proof**: Let $A(G) = (a_{ij}), A(H) = (b_{ij})$ and $A(G)A(H) = (c_{ij})$. Since the adjacency matrices are symmetric, we get that $b_{kj} = b_{jk}$ and

$$c_{ij} = \sum_k a_{ik}b_{kj} = \sum_k a_{ik}b_{jk} \pmod{2}. \tag{1}$$

In the above, taking $i = j$, we get that $c_{ii} = 0$ if and only if $a_{ik}b_{ik} \neq 0$ for even number of cases. The proof of lemma follows immediately by noting that $a_{ik}b_{ik} \neq 0$ is equivalent to say that the $i^{th}$ and $k^{th}$ vertices are adjacent in both the graphs.

**Lemma 2.2** — The $(i,j)^{th}$ entry of the matrix product $A(G)A(H)$ is either 0 or 1 depending on whether the number of $GH$ paths from $v_i$ to $v_j$ is even or odd, respectively.

**Proof**: Let $A(G) = (a_{ij}), A(H) = (b_{ij})$ and $A(G)A(H) = (c_{ij})$ as in the proof of Lemma 2.1. For $i \neq j$, note that $a_{ik}b_{jk} = 1$ if and only if there exists a $GH$ path $v_i \sim_G v_k \sim_H v_j$ from $v_i$ to $v_j$. Now from (1), the lemma follows. \(\square\)
Lemma 2.3 — The matrix product $A(G)A(H)$ is symmetric if and only if for each pair of distinct vertices $v_i$ and $v_j$, the numbers of $GH$ paths and $HG$ paths from $v_i$ to $v_j$ have the same parity (both are even or both are odd).

Proof: Note that each of $A(G)$ and $A(H)$ is symmetric, and therefore, their product $A(G)A(H)$ is symmetric if and only if $A(G)A(H) = A(H)A(G)$. Now the lemma follows from Lemma 2.2. □

Theorem 2.1 — The product $A(G)A(H)$ is graphical if and only if the following statements are true.

(i) For every $i$ ($1 \leq i \leq n$), there are even number of vertices $v_k$ such that $v_i \sim_G v_k$ and $v_k \sim_H v_i$.

(ii) For each pair of vertices $v_i$ and $v_j$ ($i \neq j$), the numbers of $GH$ paths and $HG$ paths from $v_i$ to $v_j$ have same parity.

2.2 Realization of $A(G)A(\overline{G})$

In [3], the authors have presented a characterization of nontrivial graphs $G$ for which the matrix product $A(G)A(\overline{G})$ with respect to the ordinary matrix multiplication is graphical, and the characterization is that $G$ is either 1-regular or $(n - 2)$ regular or $C_5$, a cycle graph on five vertices. In fact, whenever $G$ is different from $C_5$ and the product $A(G)A(\overline{G})$ under ordinary matrix multiplication is graphical, it has been observed in Theorem 10 of [3] that $A(G)A(\overline{G})$ is either $A(G)$ or $A(\overline{G})$. In the following Example 2.2, we consider the matrix multiplication under modulo-2 and present a graph $G$ which is neither amongst 1-regular, $(n - 2)$ regular or $C_5$ but $A(G)A(\overline{G})$ is graphical.

Example 2.2: Consider a 2-regular graph $G$ on six vertices and its complement $\overline{G}$, as shown in Figure 3. Note that

$$A(G)A(\overline{G}) = \begin{bmatrix}
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0
\end{bmatrix},$$

and the cocktail party graph shown in Figure 3 is the graph realizing $A(G)A(\overline{G})$.

The following theorem characterizes the class of all graphs $G$ for which $A(G)A(\overline{G})$ is graphical.
Figure 3: A 2-regular graph $G$ on 6 vertices for which $\Gamma$ is the graph realizing $A(G)A(\bar{G})$

**Theorem 2.2** — For any graph $G$ and its compliment $\bar{G}$ on the set of vertices $\{v_1, v_2, \ldots, v_n\}$, the following statements are equivalent:

(i) The matrix product $A(G)A(\bar{G})$ is graphical.

(ii) For every $i$ and $j$, $1 \leq i, j \leq n$, $\text{deg}_Gv_i - \text{deg}_Gv_j \equiv 0 \pmod{2}$.

(iii) The graph $G$ is parity regular.

**Proof:** Note that (ii) $\iff$ (iii) follows from the definition of parity regular graphs. Now, we shall prove (i) $\iff$ (ii). Let $(A(G))_{ij} = a_{ij}$. From the definitions of the complement of a graph and $GH$ path, $H = \bar{G}$ implies that

$$\text{deg}_Gv_i = \text{Number of walks of length 2 from } v_i \text{ to } v_j \text{ in } G$$

$$+ \text{Number of } G\bar{G} \text{ paths from } v_i \text{ to } v_j + a_{ij},$$

and similarly,

$$\text{deg}_Gv_j = \text{Number of walks of length 2 from } v_j \text{ to } v_i \text{ in } G$$

$$+ \text{Number of } G\bar{G} \text{ paths from } v_j \text{ to } v_i + a_{ji},$$

for every distinct pair of vertices $v_i$ and $v_j$. Since a $G\bar{G}$ path from $v_j$ to $v_i$ is a $G\bar{G}$ path from $v_i$ to $v_j$, now from Theorem 2.1 and comparing the right hand sides of (2) and (3), we get that $A(G)A(\bar{G})$ is graphical if and only if $\text{deg}_Gv_i \equiv \text{deg}_Gv_j \pmod{2}$.

**Remark 2.1:** It is also possible for one to prove (i) $\iff$ (ii), by taking $A(\bar{G}) = J - A(G) - I$ in the matrix products $A(G)A(\bar{G})$ and $A(\bar{G})A(G)$, where $J$ is the $n \times n$ matrix with all 1s and $I$ is the $n \times n$ identity matrix.

**Remark 2.2:** For any two graphs $G$ and $H$ such that $A(G)A(H)$ is graphical under ordinary matrix multiplication, it has been noted in [3] that any of $G$ and $H$ is connected implies that the other is regular. When we consider matrix multiplication under modulo-2, we observe that the connected
graphs $G$ and $\overline{G}$ as shown in the Figure 4 deviate from the said property, where both the graphs are just parity regular and neither of them are regular. Note that

\[
A(G)A(\overline{G}) = \begin{bmatrix}
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0
\end{bmatrix}
\]

realizing the graph $\Gamma$ as shown in Figure 4.

We shall now characterize all the graphs $G$ for which $A(G)A(\overline{G}) = A(G)$, when we consider the matrix multiplication with reference to modulo-2. This is in line with Theorems 9 and 10 of [3], which provide a characterization for such matrices when we consider regular matrix multiplication.

**Theorem 2.3** — Consider a graph $G$ and its complement $\overline{G}$ defined on the set of vertices $\{v_1, v_2, \ldots, v_n\}$. Then $A(G)A(\overline{G}) = A(G)$ if and only if $(A(G))^2$ is either a null matrix or the matrix $J$ with all entries equals to 1.

**Proof:** Let $A(G) = (a_{ij})$. From Lemma 2.2, taking $H = \overline{G}$, we get that $A(G)A(\overline{G}) = A(G)$ implies the number of $G\overline{G}$ paths from $v_i$ to $v_j$ is $a_{ij}$. Now substituting in (2), we get that

\[
deg_G v_i \equiv \text{number of walks of length 2 from } v_i \text{ to } v_j \text{ in } G \pmod{2} \text{ for } i \neq j. \tag{4}
\]

From Theorem 2.2, we have that $G$ is a parity regular and therefore $\deg_G v_i - \deg_G v_j \equiv 0 \pmod{2}$.

So, from (4) we get that $(A(G))^2$ is either 0 or $J$.

Conversely, suppose that $(A(G))^2$ is either 0 or $J$. If $(A(G))^2 = 0$ we get that the degree of all the vertices in $G$ are even and $(A(G))^2 = J$ would mean that degree of all the vertices are odd. By
taking \( A(\overline{G}) = J - A(G) - I = J + A(G) + I \), as we know that the minus (-) is the same as the plus (+) under modulo-2), we get

\[
A(G)A(\overline{G}) = A(G)(J + A(G) + I) = A(G)J + (A(G))^2 + A(G).
\]

In either of cases that \((A(G))^2\) is 0 or \(J\), we get that the right hand side of the above reduces to \(A(G)\). \(\Box\)

The following corollary follows from Theorem 2.3, which also characterizes the graphs \(G\) with property \(A(G)A(\overline{G}) = A(G)\) in terms of characteristics of \(\overline{G}\).

**Corollary 2.1** — Consider a graph \(G\) such that \(A(G) = A\) and \(A(\overline{G}) = B\). Then the following statements are equivalent:

(i) \(AB = A\).

(ii) \(B^2 = I\) or \(J - I\).

(iii) \(\overline{G}\) is a graph with one of the following properties.

(a) Degree of each vertex is odd and the number of paths of length 2 between every pair of vertices is even.

(b) Degree of each vertex is even and the number of paths of length 2 between every pair of vertices is odd.

**Proof:** (i) \(\Rightarrow\) (ii). From Theorem 2.3, we get that \(AB = A\) implies \(A^2\) is either a null matrix or \(J\). By taking \(A = J - B - I\), we get that \(A^2 = J^2 + B^2 + I\). Further, note that \(J^2\) is either a null matrix or \(J\) itself, depending on the number of vertices on which the graph defined is even or odd. In either case, \(A^2 = 0\) or \(J\) implies \(B^2 = I\) or \(J - I\).

(ii) \(\Rightarrow\) (i). Noting that \(J^2 = 0\) or \(J\) itself, by proper substitution for \(B^2\), we get \(A^2 = J^2 + B^2 + I\) is either a null matrix or \(J\). So, we obtain (i) from Theorem 2.3.

As (iii) is graph theoretical interpretation of (ii), (ii) \(\Leftrightarrow\) (iii) is trivial. \(\Box\)

The following corollary characterizes the graphs \(G\) for which \(A(G)A(\overline{G}) = A(\overline{G})\) and the proof is immediate from Corollary 2.1.

**Corollary 2.2** — Consider a graph \(G\) such that \(A(G) = A\) and \(A(\overline{G}) = B\). Then the following statements are equivalent:
(i) $AB = B$.

(ii) $A^2 = I$ or $J - I$.

The idempotency of an adjacency matrix $A(G)$ with reference to regular matrix multiplication is impossible unless $G$ is a totally disconnected graph or equivalently, $A(G)$ is a null matrix. Whenever we consider matrix multiplication under modulo-2, as in the present case, from Theorem 2.1 it is clear that $(A(G))^2$ is graphical if and only if the degree of all the vertices are even. In the following theorem, we shall verify when does $(A(G))^2 = A(G)$.

**Theorem 2.4** — Let $G$ be a graph with adjacency matrix $A(G)$. Then the following statements are equivalent:

(i) $(A(G))^2 = A(G)$ i.e., $A(G)$ is idempotent.

(ii) $A(G)A(\overline{G}) = 0$ and the degree of every vertex in $G$ is even.

(iii) The number of $G\overline{G}$ paths of length 2 between every pair of vertices is even and the degree of every vertex in $G$ is even.

**Proof:** (i) $\Rightarrow$ (ii). $(A(G))^2$ is graphical implies that the diagonal entries of $(A(G))^2$ are zeros and in turn, we get that degree of each vertex in $G$ is zero modulo-2. In other words, degree of each vertex is even. So, we get that $A(G)J = 0$ and therefore $A(G)A(\overline{G}) = A(G)J - A(G) - I = -(A(G))^2 - A(G) = 0$ whenever $(A(G))^2 = A(G)$.

(ii) $\Rightarrow$ (iii) follows from Lemma 2.3 and (iii) $\Rightarrow$ (i) follows from Theorem 2.2. $\square$

The graph $G$ as shown in Figure 5 is such that $(A(G))^2 = A(G)$ and the degree of every vertex of $G$ is even. Further, $A(G)A(\overline{G}) = 0$.

![Graph G](image)

Figure 5: A graph $G$ for which $A(G)A(\overline{G}) = A(G)$
REFERENCES


