THE REPRESENTATION OF FRACTIONAL POWERS OF
COERCIVE DIFFERENTIAL OPERATORS

Jingren Qiang*, Quan Zheng* and Miao Li**1

*Department of Mathematics, Huazhnog University of Science and Technology,
Wuhan 430074, P. R. China,

**Department of Mathematics, Sichuan University, Chengdu,
Sichuan 610064, P. R. China

e-mails: qiangjingren@hotmail.com, qzheng@hust.edu.cn, mli@scu.edu.cn

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In this paper, under the definition of fractional powers by Straub, we will give the representa-
tion of fractional powers of coercive differential operators by using pseudo differential
operators.

Key words: Coercive differential operator; pseudo differential operator; fractional power;
Fourier multiplier.

1. INTRODUCTION

Fractional powers of linear operators in a Banach space has been a concern problem from the
50th of last century (cf. [7, 10, 11]). In 1990s, Straub [5] gave a generalization of fractional
powers of closed operators with polynomially bounded resolvents, which often appear in the
abstract Cauchy problems.

On the other hand, the representation of fractional powers of a elliptic operator under cer-
tain boundary conditions already had a lot of results (see, c.g., [3, 4, 9]). Because the co-
ercive differential operators under certain conditions have polynomially bounded resolvents,

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the generalized fractional power of such differential operators exists. So it is significance to characterize the fractional powers of the coercive differential operators.

Our paper is organized as follows. In Section 2 we give preliminaries on fractional powers of closed operators, coercive differential operators and pseudo differential operators. And then, in Section 3, we characterize the fractional powers of coercive differential operators by using pseudo differential operators.

In this paper, we always assume that $P(\xi) := \sum_{|\mu| \leq m} a_\mu \xi^\mu (\xi \in \mathbb{R}^n)$ is a complex-valued polynomial and the differential operator $P(D)$ defined in $L^p(\mathbb{R}^n)$ has the maximal domain in distributional sense, i.e.,

$$
\begin{align*}
D(P(D)) &= \{ f \in L^p(\mathbb{R}^n) : \mathcal{F}^{-1}(P\mathcal{F}f) \in L^p(\mathbb{R}^n) \} \\
D(P(D))f &= \mathcal{F}^{-1}(P\mathcal{F}f), \quad f \in D(P(D))
\end{align*}
$$

(1)

where $D = \frac{1}{i} \left( \frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_n} \right)$, $\mathcal{F}$ is the Fourier transform and $\mathcal{F}^{-1}$ is its inverse. Then $P(D)$ is closed and densely defined in $L^p(\mathbb{R}^n)$ ($1 \leq p < \infty$).

2. Preliminaries

We will denote by $\mathcal{S}$ the Schwartz space on $\mathbb{R}^n$, $X$ a Banach space, $\rho(A)$ the resolvent set of a linear operator $A$ in $X$ and $R(\lambda, A)$ the resolvent for $\lambda \in \rho(A)$. Moreover, set $1 \leq p < \infty$ and $n_p = n \left[ 1 - \frac{1}{p} - \frac{1}{2} \right]$.

**Definition 2.1** — Let $A$ be a closed linear operator in $X$. A subspace $D$ of $D(A)$ is said to be a core of $A$ if $D$ is dense in $D(A)$ with respect to the graph norm $\|x\|_A := \|x\| + \|Ax\|$.

Now we recall the generalization of fractional powers for the closed operators with polynomially bounded resolvents [2].

**Condition 2.2** — Let $A$ be a closed, densely defined, linear operator in $X$. We assume that there are constants $0 < \varphi < \frac{\pi}{2}$, $M > 0$, $\gamma \geq -1$ such that the closed sector

$$
\Sigma(\varphi) := \{ \lambda \in \mathbb{C} : |\arg \lambda| \leq \varphi \} \cup \{0\}
$$

is contained in $\rho(A)$ and $\|R(\lambda, A)\| \leq M(1 + |\lambda|)^\gamma$ for all $\lambda \in \Sigma(\varphi)$.
Definition 2.3 — Let $A$ satisfy condition 2.2. For $b \in \mathbb{R}$, set $< b > = \max\{0, [b] - \lfloor -\gamma \rfloor + 2\}$, where $[b]$ denotes the largest integer smaller than or equal to $b$. We define the fractional power $(-A)^b$ of the operator $-A$ by

$$(-A)^b = \begin{cases} 	ext{the usual power of } -A & \text{if } b \text{ is an integer} \\ \text{the closure of the operator } J^b & \text{otherwise} \end{cases}$$

where the operator $J^b$ on $D(A^{<b>})$ is given by

$$J^b_x = \frac{\sin(b - [b])\pi}{\pi} \int_0^\infty \lambda^{b-[b]-1} R(\lambda, A)(-A)^{[b]+1} x d\lambda.$$

Definition 2.4 — $P(D)$ is called a $r$-coercive differential operator if $P$ is $r$-coercive, i.e.,

$$|P(\xi)|^{-r} = O(|\xi|^{-r}) \text{ as } |\xi| \to \infty$$

where $r > 0$.

The following property of $r$-coercive differential operators will be used later.

Theorem 2.5 — Let $P$ be $r$-coercive for some $r \in (0, m]$, $Re P(\xi) \leq 0$ for all $\xi \in \mathbb{R}^n$, and $0 \in \rho(P(D))$. If $\gamma > n r m / r$, then the operator $P(D)$ satisfies Condition 2.2 in $L^p(\mathbb{R}^n)$.

Proof: Since $0 \in \rho(P(D))$, there exists constants $M_1, \delta > 0$ such that $\|R(\lambda, P(D))\| \leq M_1$ for $|\lambda| < \delta$. On the other hand, by the results in [12], there exists a strongly continuous family $T(t)$ ($t \geq 0$) in $L^p(\mathbb{R}^n)$ such that $\|T(t)\| \leq M_2(1 + t^\beta)$ ($t \geq 0$) for some constants $M_2 > 0$ and $\beta > \gamma$, $\{\lambda \in \mathbb{C} : Re \lambda > 0\} \subseteq \rho(P(D))$, and

$$R(\lambda, P(D)) f = \lambda^\gamma \int_0^\infty e^{-\lambda t} T(t) f dt$$

for all $f \in L^p(\mathbb{R}^n)$ and $Re \lambda > 0$. If $\lambda \in \Sigma(\pi/4)$ and $|\lambda| \geq \delta$, then

$$\|R(\lambda, P(D))\| \leq |\lambda|^{\gamma} \int_0^\infty e^{-Re\lambda t} \|T(t)\| dt$$

$$\leq 2M_2 |\lambda|^{\gamma} \left( \int_0^1 e^{-Re\lambda t} dt + \int_1^\infty e^{-Re\lambda t} t^\beta dt \right)$$

$$\leq M_3 \left( |\lambda|^\gamma \frac{|\lambda|^{\gamma}}{Re \lambda} + \frac{|\lambda|^\gamma}{(Re \lambda)^{\beta+1}} \right)$$

$$\leq M_3 (\sqrt{2}|\lambda|^{\gamma-1} + (\sqrt{2})^{\beta+1} \delta^{\gamma-\beta-1})$$

$$\leq M_4 (1 + |\lambda|)^{\gamma-1}.$$
It follows now that
\[ \|R(\lambda, P(D))\| \leq M(1 + |\lambda|)^{-1} \text{ for } \lambda \in \Sigma(\pi/4), \]
where \( M = \max\{M_1, M_4\}. \)

The assumption “0 ∈ ρ(P(D))” in Theorem 2.5 can be replaced by “ρ(P(D)) ≠ ∅ and
\( P(\xi) \neq 0 \) for all \( \xi \in \mathbb{R}^n \)” (cf. [8]).

In order to characterize the fractional powers of coercive differential operators, we need the
following definition of pseudo differential operators in \( L^p(\mathbb{R}^n) \).

**Definition 2.6** — Let \( Q \in C^\infty(\mathbb{R}^n, \mathbb{C}) \) is polynomially bounded. The pseudo differential
operator \( Op(Q) \) is defined by
\[
\begin{aligned}
D(Op(Q)) &= \{ f \in L^p(\mathbb{R}^n) : \mathcal{F}^{-1}(Q\mathcal{F}f) \in L^p(\mathbb{R}^n) \} \\
Op(Q)f &= \mathcal{F}^{-1}(Q\mathcal{F}f), \ \ f \in D(Op(Q)).
\end{aligned}
\]
(3)

We know that \( Op(Q) \) is a closed and densely defined linear operator in \( L^p(\mathbb{R}^n) \).

**Definition 2.7** — A function \( m \in L^\infty(\mathbb{R}^n) \) is called a Fourier multiplier on \( L^p(\mathbb{R}^n) \) if
\( \mathcal{F}^{-1}(m\mathcal{F}\varphi) \in L^p(\mathbb{R}^n) \) for all \( \varphi \in \mathcal{S} \) and if
\[
\|m\|_{\mathcal{M}_p} := \sup\{ \|\mathcal{F}^{-1}(m\mathcal{F}\varphi)\|_{L^p} : \varphi \in \mathcal{S}, \|\varphi\|_{L^p} \leq 1 \} < \infty.
\]

The space of all such \( m \) is denoted by \( \mathcal{M}_p(\mathbb{R}^n) \) with norm \( \| \cdot \|_{\mathcal{M}_p} \).

It is known that \( \mathcal{M}_p(\mathbb{R}^n) \) is a Banach space.

3. The Main Results

We start with two lemmas.

**Lemma 3.1** — If the hypotheses of Theorem 2.5 are fulfilled, then for all \( \gamma > n_p m/r, \)
\( (-P)^{-\gamma} \in \mathcal{M}_p(\mathbb{R}^n) \).

**Proof:** Since \( P \) is \( r \)-coercive, there exist constants \( \delta, c > 0 \) such that \( |P(\xi)| \geq \delta |\xi|^r \) for
\( |\xi| \geq c \). Let \( \mu \) is a multi-index with \( |\mu| \leq [n_p] + 1 \), then a direct calculation yields that
\[
|D^\mu(-P(\xi))^{-\gamma}| \leq M |\xi|^{(m-r-1)|\mu|-r\gamma}.
\]
for $|\xi| \geq c$. Also, by $0 \in \rho(P(D))$, one has that $1/P \in C^\infty(\mathbb{R}^n, \mathbb{C})$ (cf. [8]). It follows therefore from Theorem E in [1] that $(-P)^{-\gamma} \in \mathcal{M}_p(\mathbb{R}^n)$.

\textbf{Lemma 3.2} — If the hypotheses of Theorem 2.5 are fulfilled, then for $\gamma > n_p m/r$,

(a) $\mathcal{S}$ is the core of the operator $\text{Op}((-P)^\gamma)$ in $L^p(\mathbb{R}^n)$;

(b) $D((-P(D))^{-\gamma}) \subset D(\text{Op}((-P)^\gamma))$, where $\prec \gamma \succ = 2[\gamma] + 2$.

\textbf{Proof} : (a) It is obvious that $\text{Op}((-P)^\gamma) : \mathcal{S} \to \mathcal{S}$, in particular, $\mathcal{S} \subset D(\text{Op}((-P)^\gamma))$. Since $\mathcal{S}$ is a dense subset of the $L^p(\mathbb{R}^n)$, for every $f \in D(\text{Op}((-P)^\gamma))$ one can choose $g_n \in \mathcal{S}$ such that $\|g_n - \text{Op}((-P)^\gamma)f\|_{L^p} \to 0 \ (n \to \infty)$. By Lemma 3.1, we have $f_n = \mathcal{F}^{-1}((-P)^{-\gamma}\mathcal{F}g_n) \in \mathcal{S}$, then

\[
\|f_n - f\|_{L^p} = \|\mathcal{F}^{-1}((-P)^{-\gamma}\mathcal{F}(g_n - \mathcal{F}^{-1}((-P)^\gamma f)))\|_{L^p} \\
\leq \|(-P)^{-\gamma}\|_{\mathcal{M}_p}\|g_n - \text{Op}((-P)^\gamma)f\|_{L^p} \\
\to 0 \ (n \to \infty).
\]

Also,

\[
\|\text{Op}((-P)^\gamma)(f_n - f)\|_{L^p} = \|g_n - \text{Op}((-P)^\gamma)f\|_{L^p} \to 0 \ (n \to \infty).
\]

Thus $\mathcal{S}$ is a core of the operator $\text{Op}((-P)^\gamma)$ in $L^p(\mathbb{R}^n)$.

(b) If $f \in D((-P(D))^{-\gamma})$, then

\[
\text{Op}((-P)^\gamma)f = \mathcal{F}^{-1}((-P)^\gamma\mathcal{F}f) = \mathcal{F}^{-1}((-P)^{-k}\mathcal{F}((-P(D))^{-\gamma-k}f))
\]

where $k = \prec \gamma \succ - \gamma > n_p m/r$. Hence, by Lemma 3.1, we have that $(-P)^{-k} \in \mathcal{M}_p(\mathbb{R}^n)$, which implies that $\text{Op}((-P)^\gamma)f \in L^p(\mathbb{R}^n)$, i.e., $f \in D(\text{Op}((-P)^\gamma))$, as desired.

\textbf{Theorem 3.3} — If the hypotheses of Theorem 2.5 are fulfilled, then for $\gamma > n_p m/r$, $(-P(D))^\gamma = \text{Op}((-P)^\gamma)$ in $L^p(\mathbb{R}^n)$.

\textbf{Proof} : If $\gamma$ is an integer, the assertion is obvious. If $\gamma$ is not an integer, we claim that $J^\gamma f = \text{Op}((-P)^\gamma)f$ for $f \in D((-P(D))^{-\gamma})$, where

\[
\prec \gamma \succ = \max\{0, [\gamma] - [1 - \gamma] + 2\} = 2[\gamma] + 2.
\]
In fact, for \( f \in D((-P(D))^\gamma) \), we obtain by Definition 2.3 and Lemma 3.2(b) that

\[
J^\gamma f = \frac{\sin(b\pi)}{\pi} \int_0^\infty \lambda^{b-1} R(\lambda, P(D))(-P(D))^{[\gamma]+1} f d\lambda \\
= \frac{\sin(b\pi)}{\pi} \int_0^\infty \lambda^{b-1} \mathcal{F}^{-1}\left(\frac{1}{\lambda - P}(-P)^{[\gamma]+1} \mathcal{F} f\right) d\lambda \\
= \frac{\sin(b\pi)}{\pi} \mathcal{F}^{-1}\left(\int_0^\infty \lambda^{b-1} \frac{1}{\lambda - P}(-P)^{-(b-1)} d\lambda \mathcal{F} \left(\text{Op}((-P)^\gamma) f\right)\right)
\]

where \( b = \gamma - [\gamma] > 0 \). Since \( \frac{z^{b-1}}{z+1} \) is an analytic function on \( \{ z \in \mathbb{C} : |\arg z| < \pi \} \), by Cauchy integral formula one gets

\[
\int_0^\infty \lambda^{b-1} \frac{1}{\lambda - z_0}(-P)^{-(b-1)} d\lambda = \int_{\Gamma} \frac{z^{b-1}}{z+1} dz = \int_0^\infty \frac{r^{b-1}}{r+1} dr = \frac{\pi}{\sin(b\pi)}
\]

for all \( z_0 \notin [0, \infty) \), where \( \Gamma = \{ z \in \mathbb{C} : |\arg(z + z_0)| = \pi \} \). Also, it follows from our assumptions that \( P(\xi) \notin [0, \infty) \) for all \( \xi \in \mathbb{R}^n \), and therefore \( J^\gamma f = \text{Op}((-P)^\gamma) f \).

Next, since \( \text{Op}((-P)^\gamma) \) is a closed operator in \( L^p(\mathbb{R}^n) \),

\[
(-P(D))^\gamma = J^\gamma \subset \text{Op}((-P)^\gamma).
\]

On the other hand, if \( f \in D(\text{Op}((-P)^\gamma)) \), by Lemma 3.2(a), there exists \( f_n \in \mathcal{S} \) such that

\[
\| f_n - f \|_{L^p} + \| \text{Op}((-P)^\gamma)(f_n - f) \|_{L^p} \to 0 \ (n \to \infty).
\]

Also, it is obvious that \( \mathcal{S} \subset \mathcal{D}((-\mathcal{D}(\mathcal{D}))^\gamma) \). Then

\[
(-P(D))^\gamma f_n \to \text{Op}((-P)^\gamma) f \ (n \to \infty)
\]

in \( L^p(\mathbb{R}^n) \). This implies that \( f \in D((-P(D))^\gamma) \), and therefore \( (-P(D))^\gamma = \text{Op}((-P)^\gamma) \) in \( L^p(\mathbb{R}^n) \).

\[\square\]

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