ON HERMITIAN GENERALIZED INVERSES AND POSITIVE SEMIDEFINITE GENERALIZED INVERSES

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The main aim of this paper is to investigate the Hermitian and positive semidefinite generalized inverses of a square matrix. First, we present some conditions for the existence of Hermitian and positive semidefinite generalized inverses. Further, expressions of these generalized inverses are given. Finally, we give two numerical examples to demonstrate our results.

Key words: Generalized inverse; Hermitian generalized inverse; positive semidefinite generalized inverse.

1. INTRODUCTION

Let $\mathbb{C}^{m \times n}$ denote the set of all $m \times n$ matrices over the complex field $\mathbb{C}$, $\mathbb{C}^{n \times n}_{\geq 0}$ denote the set of all $n \times n$ positive semidefinite matrices. For $A \in \mathbb{C}^{m \times n}$, its range space, null space, rank and conjugate transpose will be denoted by $R(A)$, $N(A)$, $r(A)$ and $A^*$ respectively. For positive semidefinite matrix $A \succeq 0$ (or positive definite matrix $A > 0$), its positive index of inertia is symbolled by $i_+(A)$.

For a matrix $A \in \mathbb{C}^{m \times n}$, the Moore-Penrose inverse $A^+$ is defined to be the unique solution of the four Penrose equations [1]

\begin{align*}
1) & \ AXA = A, & \ (2) & \ XAX = X, & \ (3) & \ (AX)^* = AX, & \ (4) & \ (XA)^* = XA.
\end{align*}
Let \( \emptyset \neq \eta \subseteq \{1, 2, 3, 4\} \). Then \( A_\eta \) denotes the set of all matrices \( X \) satisfy (i) for all \( i \in \eta \). Any matrix \( X \in A_\eta \) is called an \( \eta \)-inverse of \( A \). One usually denotes any \( \{1\} \)-inverse of \( A \) as \( A^- \). Any \( \{1, 2\} \)-inverse of \( A \) is denoted by \( A^{(1,2)} \), and any \( \{1, 3\} \)-inverse of \( A \) is denoted by \( A^{(1,3)} \). For convenience, we denote \( R_A = I - AA^\dagger \) and \( L_A = I - A^\dagger A \). Let \( A^{(i,j,\cdots,k)}_\eta \) and \( A^{(i,j,\cdots,k)} > \) be the Hermitian and positive semidefinite \( \{i, j, \cdots, k\} \)-inverse of \( A \).

For a matrix \( A \in \mathbb{C}^{n \times n} \), the group inverse, denoted by \( A^\# \), is the unique matrix \( X \) satisfying

\[
AXA = A, \quad XAX = X, \quad AX = XA.
\]

A simple result on the Moore-Penrose inverse of \( A \) is \( (A^*)^\dagger = (A^\dagger)^* \). So, for each Hermitian matrix, then \( A^\dagger = (A^\dagger)^* \), i.e., there exists Hermitian Moore-Penrose inverse. In [2], Tian presented a general expression for each Hermitian generalized inverse of a Hermitian matrix. However, Hermitian generalized inverses of a general square matrix do not necessarily exist, a natural consideration is to see when Hermitian generalized inverses exist for a square matrix. Some research work was done on this topic, for example, Khatri and Mitra [3] studied the Hermitian and nonnegative definite solutions of linear matrix equations, based on these results, the existence and expressions for Hermitian and nonnegative definite \( \{1\} \)-inverses and \( \{1, i\} \)-inverses \( (i = 3, 4) \) can be deduced. For Hermitian \( \{1, i\} \)-inverse, Liu and Yang [4] obtained the same results. Liu [10] established some rank conditions for Hermitian generalized inverses existing by extremal rank in his Ph.D. Thesis, but the expressions for these Hermitian generalized inverses were not presented.

In this article, we consider the Hermitian and positive semidefinite \( \{1, i\} \)-inverses, \( \{1, 2, i\} \)-inverses and \( \{1, 3, 4\} \)-inverses of a square matrix \( A, i = 3, 4 \), and establish some conditions for the existence as well as expressions for these generalized inverses.

We will need the following Lemmas.

**Lemma 1.1** [5] — Let \( A \in \mathbb{C}^{n \times n} \) be of rank \( r \). Then there exists unitary \( U \in \mathbb{C}^{n \times n} \) such that

\[
A = U \begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix} U^*,
\]

where \( \Sigma = \text{diag}(\sigma_1 I_{r_1}, \ldots, \sigma_t I_{r_t}) \) is the diagonal matrix of singular values of \( A \), \( \sigma_1 > \sigma_2 > \cdots > \sigma_t > 0 \), \( r_1 + r_2 + \cdots + r_t = r \), and \( K \in \mathbb{C}^{r \times r}, L \in \mathbb{C}^{r \times n-r} \) satisfy

\[
KK^* + LL^* = I_r.
\]
Furthermore, \( r(A^2) = r(A) \) if and only if \( K \) is nonsingular.

Without loss of generality, in the following content, we always assume \( A \) can be represented as
\[
A = \begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix}.
\] (1.1)

Hence,
\[
\begin{align*}
A^\dagger &= \begin{pmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{pmatrix}, \\
A^\# &= \begin{pmatrix} K^{-1}\Sigma^{-1} & K^{-1}\Sigma^{-1}K^{-1}L \\ 0 & 0 \end{pmatrix}, \text{if } A^\# \text{ exists,}
\end{align*}
\] (1.2)
(1.3)

where \( \Sigma, K \) and \( L \) are defined as in Lemma 1.1.

**Lemma 1.2 [11]** — Let \( A \in \mathbb{C}^{p \times q}, B \in \mathbb{C}^{q \times p} \) and \( C \in \mathbb{C}^{p \times p} \) be given matrices. Then the matrix equation \( AXB = C \) is consistent if and only if
\[
AA^\dagger CB^\dagger B = C.
\]

**Lemma 1.3 [4]** — Let \( A \in \mathbb{C}^{p \times q}, B \in \mathbb{C}^{q \times p} \) and \( C \in \mathbb{C}^{p \times p} \) be given matrices. If the equation \( AXB = C \) is consistent, then it has a Hermitian solution if and only if
\[
\begin{pmatrix}
-C & 0 & A \\
0 & C^* & B^* \\
B & A^* & 0
\end{pmatrix}
\]

\[\begin{pmatrix}
-X & -X^*
\end{pmatrix} = 2r \begin{pmatrix}
A^* & B
\end{pmatrix}.
\]

In this case, the general Hermitian solution can be expressed as
\[
X_h = \frac{X + X^*}{2},
\]

where
\[
X = A^\dagger CB^\dagger + P^\dagger E(A^\dagger)^* - P^\dagger B^*M^\dagger R_P E(A^\dagger)^* + M^\dagger R_P EQ^\dagger
\]

\[-P^\dagger B^*L_M V Q(A^\dagger)^* + L_M V R_B + L_A L_P U + L_A Z L_A + W R_Q R_B,
\]

and \( P = B^*L_A, Q = R_B A^*, M = R_P B^*, E = C^* - B^* A^\dagger CB^\dagger A^*, U, V, W \) and \( Z \) are arbitrary matrices with appropriate sizes.
2. HERMITIAN GENERALIZED INVERSES

In this section, our purpose is to investigate some conditions for Hermitian generalized inverses existing, and then establish several general expressions for these Hermitian generalized inverses.

Liu and Yang [4] provided two conditions for the existence of Hermitian \{1, i\}-inverse of \( A, i = 3, 4 \), and representations of these two Hermitian generalized inverses were presented. Here, we restudy Hermitian \{1, i\}-inverse, and establish some new equivalent results.

**Theorem 2.1** — Let \( A \in \mathbb{C}^{n \times n} \). Then the following statements are equivalent:

1. \( A_h^{(1,3)} \) exists;

2. \((A^*)^2 A\) is Hermitian;

3. \( AA^\dagger A^* A = A^2;\)

4. \((A^\dagger)^2 A\) is Hermitian;

5. \( A^2 A^\dagger\) is Hermitian;

6. \( \Sigma K\) is Hermitian.

In this case, \( A_h^{(1,3)} \) can be expressed as

\[
A_h^{(1,3)} = A^\dagger + (A^\dagger)^* - \frac{1}{2} A^\dagger (A + A^*) (A^\dagger)^* + L_A H L_A
\]

\[
= A^\dagger + (A^\dagger)^* - (A^\dagger)^2 A + L_A H L_A, \quad (2.1)
\]

where \( H \) is an arbitrary Hermitian matrix over complex field.

**Proof:** By [Theorem 3.2, 4], it follows that the statements (1), (2) and (3) are equivalent.

If \((A^*)^2 A\) is Hermitian, then \( A^\dagger (A^\dagger)^* (A^*)^2 A A^\dagger (A^\dagger)^* = (A^\dagger)^2 A\) is also Hermitian. Conversely, if \((A^\dagger)^2 A\) is Hermitian, so is \( A^* A (A^\dagger)^2 A A^* A = (A^*)^2 A\). Thus, statements (2) and (4) are equivalent. According to statement (3), it follows that

\[
A^2 A^\dagger = A A^\dagger A^* A A^\dagger = A A^\dagger A^* = (A^2 A^\dagger)^*,
\]

i.e., \( A^2 A^\dagger \) is Hermitian. By the above equality, (5) \( \Rightarrow \) (3) is obvious. Hence, statements (3) and (5) are equivalent too.
According to (1.1) and (1.2), $A^2 A^\dagger$ can be partitioned as

$$A^2 A^\dagger = \begin{pmatrix} \Sigma K & 0 \\ 0 & 0 \end{pmatrix},$$

this shows that (5) $\Leftrightarrow$ (6).

By [Theorem 3.2, 4],

$$A_h^{(1,3)} = A^\dagger + (A^\dagger)^* - \frac{1}{2} A^\dagger (A + A^*) (A^\dagger)^* + L_A H L_A$$

$$= A^\dagger + (A^\dagger)^* - \frac{1}{2} A^\dagger A (A^\dagger)^* - \frac{1}{2} A^\dagger A^* (A^\dagger)^* + L_A H L_A$$

$$= A^\dagger + (A^\dagger)^* - \frac{1}{2} ((A^\dagger)^2 A)^* - \frac{1}{2} (A^\dagger)^2 A + L_A H L_A$$

$$= A^\dagger + (A^\dagger)^* - (A^\dagger)^2 A + L_A H L_A,$$

means that (2.1) follows. \qed

For Hermitian \{1, 4\}-inverse, we have the following result.

**Theorem 2.2** — Let $A \in \mathbb{C}^{n \times n}$. Then the following statements are equivalent:

1. $A_h^{(1,4)}$ exists;
2. $A^2 A^*$ is Hermitian;
3. $AA^* A^\dagger A = A^2$;
4. $A(A^\dagger)^2$ is Hermitian;
5. $A^\dagger A^2$ is Hermitian;
6. $K \Sigma$ is Hermitian.

In this case, $A_h^{(1,4)}$ can be expressed as

$$A_h^{(1,4)} = A^\dagger + (A^\dagger)^* - \frac{1}{2} (A^\dagger)^* (A + A^*) A^\dagger + R_A H R_A$$

$$= A^\dagger + (A^\dagger)^* - A(A^\dagger)^2 + R_A H R_A,$$

(2.2)

where $H$ is an arbitrary Hermitian matrix over complex field.

Next, we consider Hermitian \{1, 2, i\}-inverse, $i = 3, 4$. 
**Theorem 2.3** — Let $A \in \mathbb{C}^{n \times n}$. Then the following statements are equivalent:

1. $A_h^{(1,2,3)}$ exists;
2. $(A^*)^2A$ is Hermitian and $r(A) = r(A^2)$;
3. $AA^\dagger A^* A = A^2$ and $r(A) = r(A^2)$;
4. $(A^\dagger)^2A$ is Hermitian and $r(A) = r(A^2)$;
5. $A^2A^\dagger$ is Hermitian and $r(A) = r(A^2)$;
6. $\Sigma K$ is Hermitian and nonsingular.

If $A_h^{(1,2,3)}$ exists, then it is unique, and can be expressed as

$$A_h^{(1,2,3)} = A^\# A A^\dagger = A (A^* A^2)^\dagger A^* = A (A^2)^\dagger = (A^2 A^\dagger)^\dagger. \tag{2.3}$$

**Proof:** That the statements (2)-(5) are equivalent follows by Theorem 2.1 while the equivalence (5) and (6) follows by Lemma 1.1 and Theorem 2.1. We just need to prove (1) $\iff$ (5).

(1) $\Rightarrow$ (5): Suppose that $X$ is a Hermitian $\{1, 2, 3\}$-inverse of $A$, then $AX = AA^\dagger$, and $A^2A^\dagger$ is Hermitian. Thus

$$X = X A^* X = X^* A^* X = (AX)^* X = AA^\dagger X,$$

moreover,

$$AX = A^2 A^\dagger X = AA^\dagger,$$

which implies that $r(A) = r(A^2)$.

(5) $\Rightarrow$ (1): If (5) holds, then $A^\#$ exists. Denote $X = A^\# AA^\dagger$. It is easy to verify that $X$ is a $\{1, 2, 3\}$-inverse of $A$, so we only need to show that $X$ is Hermitian. According to the conditions in (5), we can deduce

$$A^2 A^\dagger = AA^\dagger A^*$$

$$\Rightarrow A^2 A^\dagger = AA^\dagger A^* (A^\#)^*$$

$$\Rightarrow A^2 A^\dagger = AA^\dagger A^* (A^\#)^*$$

$$\Rightarrow A^2 A^\dagger = AA^\dagger (A^\#)^*$$

$$\Rightarrow AA^\dagger = AA^\dagger (A^\#)^*$$

$$\Rightarrow A^\# AA^\dagger = AA^\dagger (A^\#)^*$$

$$\Rightarrow A^\# AA^\dagger = (A^\# AA^\dagger)^*.$$
Hence $X$ is a Hermitian $\{1, 2, 3\}$-inverse of $A$. Hence, the statements (1) and (5) are equivalent.

Next, we prove the uniqueness. Assume that $Y$ is another Hermitian $\{1, 2, 3\}$-inverse of $A$. Then

$$A(A^\# AA^\dagger - Y) = 0, \quad (2.4)$$
$$A(A^\# AA^\dagger - Y)AA^\dagger = A^\# AA^\dagger - Y. \quad (2.5)$$

Note that, $(2.4)$ and $(2.5)$ mean that $R(A^\# AA^\dagger - Y) \subseteq N(A)$ and $R((A^\# AA^\dagger - Y)^*) = R(A^\# AA^\dagger - Y) \subseteq R(A)$, respectively. By $r(A) = r(A^2)$, it follows that $R(A) \cap N(A) = \{0\}$. Therefore, $A^\# AA^\dagger - Y = 0$, i.e., $A^\# AA^\dagger = Y$.

Since $r(A) = r(A^2)$, then $R(A^2) = R(A)$ and $R((A^2)^*) = R(A^*)$, it follows from [Lemma 2.4, 6] that $A^2(A^2)^\dagger = AA^\dagger$ and $(A^2)^\dagger A^2 = A^\dagger A$. By [Theorem 2.2, 7], we have

$$(A^\dagger A^2)^\dagger = (A^2)^\dagger (A^\dagger)^* \text{ and } (A^2 A^\dagger)^\dagger = A(A^2)^\dagger.$$

Therefore,

$$A(A^\dagger A^2)^\dagger A^* = A(A^2)^\dagger (A^\dagger)^* A^*$$
$$= A(A^2)^\dagger AA^\dagger$$
$$= A(A^2)^\dagger$$
$$= A^\# A^2(A^2)^\dagger$$
$$= A^\# AA^\dagger. \quad \square$$

For Hermitian $\{1, 2, 4\}$-inverse we have the following result.

**Theorem 2.4** — Let $A \in \mathbb{C}^{n \times n}$. Then the following statements are equivalent:

1. $A_h^{(1, 2, 4)}$ exists;
2. $A^2 A^*$ is Hermitian and $r(A) = r(A^2)$;
3. $A A^* A^\dagger A = A^2$ and $r(A) = r(A^2)$;
4. $(A A^\dagger)^2$ is Hermitian, and $r(A) = r(A^2)$;
(5) $A^\dagger A^2$ is Hermitian, and $r(A) = r(A^2)$;

(6) $K\Sigma$ is Hermitian and nonsingular.

If $A^{(1,2,4)}_h$ exists, then it is unique, and can be expressed as

$$A^{(1,2,4)}_h = A^\dagger AA^\# = A^*(A^2 A^\dagger)^\dagger A = (A^2)^\dagger A = (A^\dagger A^2)^\dagger.$$  

According to Theorem 2.1 and Theorem 2.2, we can derive the following result.

**Theorem 2.5** — Let $A \in \mathbb{C}^{n \times n}$. Then the following statements are equivalent:

(1) $A^{(1,3,4)}_h$ exists;

(2) $A^{(1,3)}_h$ and $A^{(1,4)}_h$ exist.

If $A^{(1,3,4)}_h$ exists, then it can be expressed as

$$A^{(1,3,4)}_h = A^\dagger + (A^\dagger)^* - A(A^\dagger)^2 - \frac{1}{2} \left( R_A (AR_A)^\dagger A(A^\dagger)^* (I + L_{AR_A}) R_A ight. + R_A (I + L_{AR_A}) A^\dagger A^*( (AR_A)^\dagger )^* R_A 
\left. + R_A (I + L_{AR_A}) A^\dagger A^*( (AR_A)^\dagger )^* R_A - R_A L_{AR_A} W L_{AR_A} R_A \right) \tag{2.6}$$

$$= A^\dagger + (A^\dagger)^* - (A^\dagger)^2 A - \frac{1}{2} \left( L_A (LA A)^\dagger A^*( I + R_{LA A} ) L_A + L_A (I + R_{LA A} ) (A^\dagger)^* A (LA A)^\dagger L_A - L_A R_{LA A} Z R_{LA A} L_A \right). \tag{2.7}$$

where $W$ and $Z$ are arbitrary Hermitian matrix over complex field.

**Proof:** (1) $\Rightarrow$ (2): This is obvious.

(2) $\Rightarrow$ (1): If $A^{(1,4)}_h$ exists, in order to prove the existence of $A^{(1,3,4)}_h$, we just need to show that there exists at least one Hermitian matrix $H$ such that $A^{(1,4)}_h$ given by (2.2) is a $\{1,3\}$-inverse of $A$, i.e.,

$$AA^{(1,4)}_h = AA^\dagger. \tag{2.8}$$

Substituting (2.2) in (2.8) gives

$$-AR_A H R_A = AR_A \tilde{H} R_A = A(A^\dagger)^* R_A. \tag{2.9}$$

That is to say, we only need to prove that the equation (2.9) is consistent and has a Hermitian solution $\tilde{H}$ (or $-H$). In view of Lemma 1.2, (2.9) is consistent if and only if

$$AR_A (AR_A)^\dagger A(A^\dagger)^* R_A = A(A^\dagger)^* R_A.$$
By (1.1) and (1.2), we have

\[
AR_A = \begin{pmatrix} 0 & \Sigma L \\ 0 & 0 \end{pmatrix}, \quad (AR_A)^\dagger = \begin{pmatrix} 0 & 0 \\ \Sigma L^\dagger & 0 \end{pmatrix}, \quad A(A^\dagger)^* R_A = \begin{pmatrix} 0 & K^*L \\ 0 & 0 \end{pmatrix}.
\]

Hence, we only need to verify that \(\Sigma L(\Sigma L)^\dagger K^*L = K^*L\), which is equivalent to \(\Sigma L(\Sigma L)^\dagger K^*LL^* = K^*L\). Since

\[
K^*LL^* = \Sigma \Sigma^{-1}K^*(I - KK^*)
= \Sigma(K\Sigma^{-1})^*(I - KK^*)
= \Sigma(K\Sigma^{-1} - KK^*)
= \Sigma[K\Sigma^{-1} - K(K^*\Sigma^{-1})K^*]
= \Sigma(K\Sigma^{-1} - KK^*\Sigma^{-1}K^*)
= \Sigma(K\Sigma^{-1} - KK^*K\Sigma^{-1})
= \Sigma LL^*K\Sigma^{-1},
\]

where the conditions \((\Sigma K)^* = \Sigma K\) and \((K\Sigma)^* = K\Sigma\) are used. According to the above analyses, the consistency of matrix equation (2.9) is evident.

On the other hand, the consistent equation (2.9) has a Hermitian solution if and only if

\[
r\begin{pmatrix} -A(A^\dagger)^* R_A & 0 & AR_A \\ 0 & R_A A^\dagger A^* & R_A \\ R_A & R_A A^* & 0 \end{pmatrix} = r\begin{pmatrix} 0 & A(A^\dagger)^* R_A A^* & AR_A \\ 0 & R_A A^\dagger A^* & R_A \\ R_A & R_A A^* & 0 \end{pmatrix}
= r\begin{pmatrix} 0 & 0 & AR_A \\ 0 & 0 & R_A \\ R_A & R_A A^* & 0 \end{pmatrix}
= 2r\begin{pmatrix} (AR_A)^* \\ R_A \end{pmatrix},
\]

where the condition \(A(A^\dagger)^* R_A A^* = (A(A^\dagger)^* R_A A^*)^* = AR_A A^\dagger A^*\) is used.

On account of Lemma 1.3, we know that (2.9) has a Hermitian solution, and which can be expressed as

\[
\tilde{H} = \frac{1}{2}(X + X^*),
\]

(2.10)
where,
\[
X = (AR_A)^\dagger A(A^\dagger)^* R_A + P^\dagger E((AR_A)^\dagger)^* - P^\dagger R_A M^\dagger R_P E((AR_A)^\dagger)^*
+ L_M V R_{R_A} + L_{AR_A} L_P U + L_{AR_A} Z L_{AR_A} + W R_{R_A},
\]
and \( P = R_A L_{AR_A}, \) \( M = R_P R_A, \) \( E = PA^\dagger A^*, U, V, W \) and \( Z \) are arbitrary matrices with appropriate sizes.

Moreover, by (1.1) and (1.2), we have
\[
M = (AR_A)^\dagger A R_A, \quad L_M = L_{AR_A}, \quad P^\dagger = P = P^2,
\]
and
\[
R_P E = 0, \quad L_{AR_A} L_P = 0.
\]

Hence,
\[
R_A X R_A = R_A (AR_A)^\dagger A(A^\dagger)^* R_A + R_A L_{AR_A} A^\dagger A^* ((AR_A)^\dagger)^* R_A + R_A L_{AR_A} Z L_{AR_A} R_A.
\] (2.11)

Together with (2.9), (2.10) and (2.11), the formula (2.6) is evident.

Similarly, we can obtain the representation (2.7). The proof of this theorem is complete. □

### 3. Positive Semidefinite Generalized Inverses

In this section, conditions for the existence and general expressions for positive semidefinite generalized inverses are presented.

First, we consider positive semidefinite \( \{1, 3\} \)-inverse.

**Theorem 3.1** — Let \( A \in C^{n \times n} \). Then the following statements are equivalent:

1. \( A^{(1,3)} \) exists;
2. \( (A^*)^2 A \succeq 0 \) and \( r(A) = r(A^2) \);
3. \( (A^\dagger)^2 A \succeq 0 \) and \( r(A) = r(A^2) \);
4. \( A^2 A^\dagger \succeq 0 \) and \( r(A) = r(A^2) \);
(5) $A^\# AA^\dagger \succeq 0$;

(6) $\Sigma K > 0$.

In this case, $A_{\#}^{(1,3)}$ can be expressed as

$$A_{\#}^{(1,3)} = A(A^*A^2)^\dagger A^* + L_AWL_A$$  \hspace{1cm} (3.1)

$$= A(A^2)^\dagger + L_AWL_A$$  \hspace{1cm} (3.2)

$$= A^{\#} AA^\dagger + L_AWL_A$$  \hspace{1cm} (3.3)

$$= (A^2A^\dagger)^\dagger + L_AWL_A,$$  \hspace{1cm} (3.4)

where $W$ is an arbitrary positive semidefinite matrix over complex field.

**Proof:** Recall that, positive semidefinite generalized inverse $A_{\#}^{(1,3)}$ can be regarded as the positive semidefinite solution of consistent matrix equation $A^*AX = A^*$. Hence, the equivalence of (1) and (2), formula (3.1) are followed by [Theorem 2.2, 3] or [Lemma 5.1, 8].

(2) $\Leftrightarrow$ (3): Since $A^\dagger [(A^*)^2 A] A^\dagger (A^\dagger)^* = (A^\dagger)^2 A$ and $A^* A [(A^\dagger)^2 A] A^* A = (A^*)^2 A$.

Together with $i_+(P^*AP) \leq i_+(A)$, we have

$$i_+((A^*)^2 A) = i_+((A^\dagger)^2 A) \text{ and } r((A^*)^2 A) = r((A^\dagger)^2 A).$$

It is well known that $C \succeq 0$ is equivalent to $i_+(C) = r(C)$. Hence, statements (2) and (3) are equivalent. Using $A^*A^2 \succeq 0$, the equivalence of (2) and (4) can be verified by a similar method. The equivalence of (4), (5) and (6) can be deduced by Lemma 1.1, (1.1), (1.2) and (1.3).

Formulae (3.2), (3.3) and (3.4) are followed by Theorem 2.3.

For positive semidefinite $\{1,4\}$-inverse, we have the following result.

**Theorem 3.2** — Let $A \in \mathbb{C}^{n \times n}$. Then the following statements are equivalent:

(1) $A_{\#}^{(1,4)}$ exists;

(2) $A^2A^* \succeq 0$ and $r(A) = r(A^2)$;

(3) $A(A^\dagger)^2 \succeq 0$ and $r(A) = r(A^2)$;
(4) $A^\dagger A^2 \succeq 0$ and $r(A) = r(A^2)$;

(5) $A^\dagger AA^\# \succeq 0$;

(6) $K\Sigma > 0$.

In this case, $A^{(1,4)}_\succ$ can be expressed as

$$A^{(1,4)}_\succ = A^* (A^2 A^*)^\dagger A + R_A W R_A$$
$$= (A^2)^\dagger A + R_A W R_A$$
$$= A^\dagger AA^\# + R_A W R_A$$
$$= (A^\dagger A^2)^\dagger + R_A W R_A,$$

where $W$ is an arbitrary positive semidefinite matrix over complex field.

Similarly, we can derive the following results.

**Theorem 3.3** — Let $A \in \mathbb{C}^{n \times n}$. Then the following statements are equivalent:

1. $A^{(1,2,3)}_\succ$ exists;
2. $A^{(1,3)}_\succ$ exists.

If $A^{(1,2,3)}_\succ$ exists, then it is unique, and can be expressed as

$$A^{(1,2,3)}_\succ = A^\# AA^\dagger = A(A^* A^2)^\dagger A^* = A(A^2)^\dagger = (A^2 A^\dagger)^\dagger.$$

**Proof:** (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1): If $A^{(1,3)}_\succ$ exists, then $A^\# AA^\dagger \succeq 0$ and $r(A) = r(A^2)$. It is easy to verify that $A^\# AA^\dagger$ is a positive semidefinite $\{1, 2, 3\}$-inverse of $A$. The uniqueness can be proved by a similar approach with Theorem 2.3. And the other formulae are given by Theorem 3.1. $\Box$

**Theorem 3.4** — Let $A \in \mathbb{C}^{n \times n}$. Then the following statements are equivalent:

1. $A^{(1,2,4)}_\succ$ exists;
2. $A^{(1,4)}_\succ$ exists.
If $A_{\geq}^{(1,2,4)}$ exists, then it is unique, and can be expressed as

$$A_{\geq}^{(1,2,4)} = A^\dagger A A^\# = A^* (A^2 A^*)^\dagger A = (A^2)^\dagger A = (A^\dagger A^2)^\dagger.$$  

**Theorem 3.5** — Let $A \in \mathbb{C}^{n \times n}$. Then the following statements are equivalent:

1. $A_{\geq}^{(1,3,4)}$ exists;
2. $M = \begin{pmatrix} (A^2)^* A & A^* A A^* \\ A A^* A & A^2 A^* \end{pmatrix} \succeq 0$ and $r(M) = r \left( \begin{array}{cc} A & A^* \end{array} \right)$;
3. $A_{\geq}^{(1,3)}$, $A_{\geq}^{(1,4)}$ exist, and $A^2 A^* \succeq A (A A^\#)^* A A^*$, $(A^2)^* A \succeq A^* A A^\# A^* A$.

If $A_{\geq}^{(1,3,4)}$ exists, then it can be expressed as

$$A_{\geq}^{(1,3,4)} = A^\dagger A A^\# - R_A (A A^\#)^* (A R_A (A A^\#)^*)_A A^\# R_A$$
$$+ R_A L A W R_A L A R_A$$
$$A^\# A A^\dagger - L A A A^\# (A A^\#)^* L A A^\# (A A^\#)^* L A$$
$$+ L A R L A W R L A A L A,$$

where $W$ is an arbitrary positive semidefinite matrix over complex field.

**Proof:** (1) $\iff$ (2): As is well known, there exists a $A_{\geq}^{(1,3,4)}$ if and only if the consistent matrix equations $A^* A X = A^*$ and $X A A^* = A^*$ have a common positive semidefinite solution. By [Theorem 2.3, 3], the equivalence of statements (1) and (2) is evident.

(2) $\Rightarrow$ (3): If (2) holds, then $A_{\geq}^{(1,3)}$ and $A_{\geq}^{(1,4)}$ exist, so $(A^2)^* A \succeq 0$, $A^2 A^* \succeq 0$ and $r(A) = r(A^2)$. The following equalities can be verified,

$$M = \begin{pmatrix} I & 0 \\ A A^* A ((A^2)^* A)^\dagger & I \end{pmatrix} \begin{pmatrix} (A^2)^* A & 0 \\ 0 & A^2 A^* - A (A A^\#)^* A A^\# \end{pmatrix}$$
$$\times \begin{pmatrix} I & 0 \\ A A^* A ((A^2)^* A)^\dagger & I \end{pmatrix}^*$$
$$= \begin{pmatrix} I & A^* A A^\# (A^2 A^*)^\dagger \\ 0 & I \end{pmatrix} \begin{pmatrix} (A^2)^* A - A^* A A^\# A A^* & 0 \\ 0 & A^2 A^* \end{pmatrix}$$
$$\times \begin{pmatrix} I & A^* A A^\# (A^2 A^*)^\dagger \\ 0 & I \end{pmatrix}^*.$$
These two equalities imply that \( A^2A^* \geq A(AA^#)^*AA^* \) and \( (A^2)^*A \geq A^*AA^#A^*A \).

(3) \( \Rightarrow \) (2): If (3) holds, according to the above analyses, we have \( M \geq 0 \).

On the other hand, since

\[
\begin{pmatrix}
(A^2)^*A & A^*AA^* \\
AA^*A & A^2A^*
\end{pmatrix}
= \begin{pmatrix}
(A^2)^* & A^*A \\
AA^* & A^2
\end{pmatrix}
\begin{pmatrix}
A & 0 \\
0 & A^*
\end{pmatrix}
\]

\[
\begin{pmatrix}
(A^2)^*A & A^*AA^* \\
AA^* & A^2
\end{pmatrix}
= \begin{pmatrix}
(A^2)^*A & A^*AA^* \\
AA^*A & A^2A^*
\end{pmatrix}
\begin{pmatrix}
A^\dagger & 0 \\
0 & (A^*)^\dagger
\end{pmatrix}.
\]

Hence,

\[
r(M) = r \left( \begin{pmatrix}
(A^2)^* & A^*A \\
AA^* & A^2
\end{pmatrix} \right)
= r \left( \begin{pmatrix}
A^*A & (A^2)^* \\
A^2 & AA^*
\end{pmatrix} \right)
= r \left\{ \left( \begin{pmatrix}
A & A^*
\end{pmatrix} \right)^* \left( \begin{pmatrix}
A & A^*
\end{pmatrix} \right) \right\}
= r \left( \begin{pmatrix}
A & A^*
\end{pmatrix} \right).
\]

Now, the equivalence of statements (1), (2) and (3) has been proved.

By \( (A^2)^*A \geq A^*AA^#A^*A \), it follows that \( (A^\dagger)^*(A^2)^*AA^\dagger \geq (A^\dagger)^*A^*AA^#A^*AA^\dagger \), i.e., \( A^2A^\dagger \geq AA^#A^* = A(AA^#)^* \), so \(-AR_A(AA^#)^* \geq 0 \). Let the right hand side of (3.5) be \( X \), it is clear that \( X \geq 0 \). Furthermore,

\[
r \left( \begin{pmatrix}
AR_A(AA^#)^* & AA^#R_A
\end{pmatrix} \right)
= r \left( \begin{pmatrix}
AA^#(AR_A)^* & AA^#R_A
\end{pmatrix} \right)
= r \left( \begin{pmatrix}
AA^#R_A & AA^#R_A
\end{pmatrix} \right)
= r(AR_A)
= r(AR_A)^\dagger
= r(AR_A(AA^#)^*)
= r(AR_A(AA^#)^*),
\]

means that

\[
R(AR_A(AA^#)^*) = R(AA^#R_A).
\]
Hence, it is easy to verify that $X$ is a $\{1, 3, 4\}$-inverse of $A$. Therefore, $X$ is a positive semidefinite $\{1, 3, 4\}$-inverse of $A$. Also, (3.6) can be verified similarly.

Remark: $A^1 AA^\#$ in (3.5) can be replaced by $A^*(A^2 A^*)^\dagger A$ or $(A^2)^\dagger A$ or $(A^\dagger A^2)^\dagger$, and $A^\# AA^\dagger$ in (3.6) can also be replaced by $A(A^* A^2)^\dagger A^*$ or $A(A^2)^\dagger$ or $(A^2 A^\dagger)^\dagger$.

4. Examples

In this section, we present two numerical examples to demonstrate an application of Theorem 2.3 and Theorem 3.3.

Example 4.1: Consider the matrix

$$A = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 \\
0 & 2 & 1 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix}.$$ 

It is easy to verify that $A$ satisfies $A^* A^2$ is Hermitian, and $r(A) = r(A^2)$. According to Theorem 2.3, the existence of $A^{(1,2,3)}_h$ is obvious, and

$$A^{(1,2,3)}_h = (A^2 A^\dagger)^\dagger = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -\frac{1}{3} & \frac{2}{3} & 0 \\
0 & \frac{2}{3} & -\frac{1}{3} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.$$ 

Example 4.2: Consider the matrix

$$A = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 \\
0 & 1 & 2 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix}.$$ 

It is easy to verify that $A$ satisfies $A^* A^2 \succeq 0$ and $r(A) = r(A^2)$. According to Theorem
3.3, the existence of $A_{\geq}^{(1,2,3)}$ are obvious, and

$$A_{\geq}^{(1,2,3)} = (A^2 A^\dagger)^\dagger = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{2}{3} & -\frac{1}{3} & 0 \\
0 & -\frac{1}{3} & \frac{2}{3} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.$$ 

5. CONCLUSIONS

In this paper, we have established some conditions for the existence and representations for Hermitian and positive semidefinite $\{1, i\}$-inverses, $\{1, 2, i\}$-inverses and $\{1, 3, 4\}$-inverses, $i = 3, 4$.

Using Lemma 1.3, we see that $A_{\leq}$ exists if and only if

$$r(A - A^*) = 2r \begin{pmatrix} A & A^* \end{pmatrix} - 2r(A),$$

and expression of $A_{\leq}$ can also be obtained by Lemma 1.3. Together with [Theorem 2.1, 9], we get some equivalent conditions for the existence of $A_{\leq}$ as follows

$$r(A - A^*) = r(AA^\dagger - A^\dagger A) = 2r(A - A^2 A^\dagger) = 2r(A - A^\dagger A^2).$$

Since $A_{\leq}$ and $A_{\geq}$ can be regarded as the Hermitian and positive semidefinite solutions of consistent matrix equation $A^*AXA^* = A^*AA^*$. Hence, by Theorem 2.4 and Theorem 2.5 in [3], conditions for the existence and the representations for $A_{\leq}$ and $A_{\geq}$ can be deduced.

However, these representations for $A_{\leq}$ and $A_{\geq}$ are very complicated obtained by above methods, and it is not easy to established some simple and explicit formulae. And it is still open to present some conditions for the existence and the representations of $A_{\leq}^{(1,2)}$ and $A_{\geq}^{(1,2)}$, so, further research will be needed.

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