CONSTRUCTION OF VECTOR VALUED WAVELET PACKETS ON $\mathbb{R}_+$
USING WALSH-FOURIER TRANSFORM

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In this paper, the concept of vector-valued wavelet packets in space $L^2(\mathbb{R}_+, \mathbb{C}^N)$ is introduced. Some properties of vector-valued wavelets packets are studied and orthogonality formulas of these wavelets packets are obtained. New orthonormal basis of $L^2(\mathbb{R}_+, \mathbb{C}^N)$ is obtained by constructing a series of subspaces of vector-valued wavelet packets.

**Key words**: Vector-valued multiresolution analysis on $\mathbb{R}_+$; Walsh function; Walsh-Fourier transform; vector-valued wavelet packets.

1. INTRODUCTION

Wavelet packet analysis is an important generalization of wavelet analysis. Wavelet packets retain many of the orthogonality, smoothness and localization properties of their parent wavelets. In order to improve the localization of the frequency field of wavelet bases, Coifman [2, 3] introduced the notion of orthogonal wavelet packets. Wavelet packet functions comprise a rich family of building block functions and offers more flexibility than wavelets in representing different types of signals [4, 14].

Mallat [9] gave the remarkable idea of multiresolution analysis (MRA) which deals with a general algorithm for the construction of an orthonormal wavelet basis. Farkov [5, 6] has given
the general construction of all compactly supported orthogonal \( p \)-wavelets in \( L^2(\mathbb{R}_+) \). Protasov and Farkov [11] have constructed dyadic compactly supported wavelets in \( L^2(\mathbb{R}_+) \). Xia and Suter [15] have introduced the notion of vector-valued multiresolution analysis on real line \( \mathbb{R} \). Chen and Cheng [1] have given an algorithm for construction of vector-valued wavelets. Sun and Cheng [13] have investigated the construction of a class of compactly supported orthogonal vector-valued wavelets. Han and Chen [8] have presented the procedure for construction of multiple vector-valued wavelet packets. The aim of this paper is to construct vector-valued wavelet packets on positive half line and study its properties.

The paper is organized as follows: The remainder of this section is devoted to the Walsh-Fourier analysis. In Section 2, we study vector-valued multiresolution analysis on positive half line and its associated wavelets on positive half line. We construct vector-valued wavelet packets on positive half line and its properties are obtained in section 3. In section 4, we study the decomposition of the space \( L^2(\mathbb{R}_+, \mathbb{C}^N) \).

1.1 Walsh-Fourier Analysis

All the definitions and properties in this section can also be found in [7, 12]. Let \( p \) be a fixed natural number greater than 1. As usual, let \( \mathbb{R}_+ = [0, \infty) \) and \( \mathbb{Z}_+ = \{0, 1, \ldots\} \). Denote by \([x]\) the integer part of \( x \). For \( x \in \mathbb{R}_+ \) and for any positive integer \( j \), we set

\[
x_j = [p^j x](\text{mod } p), \quad x_{-j} = [p^{-j} x](\text{mod } p)
\]

where \( x_j, x_{-j} \in \{0, 1, \ldots, p - 1\} \). Consider on \( \mathbb{R}_+ \) the addition defined as follows:

\[
x \oplus y = \sum_{j<0} \xi_j p^{-j-1} + \sum_{j>0} \xi_j p^{-j}
\]

with

\[
\xi_j = x_j + y_j(\text{mod } p), \quad j \in \mathbb{Z} \setminus \{0\},
\]

where \( \xi_j \in \{0, 1, 2, \ldots, p - 1\} \) and \( x_j, y_j \) are calculated by (1.1). As usual, we write \( z = x \odot y \) if \( z \oplus y = x \), where \( \odot \) denotes subtraction modulo \( p \) in \( \mathbb{R}_+ \).

For \( x \in [0, 1) \), let \( r_0(x) \) is given by

\[
r_0(x) = \begin{cases} 
1, & x \in [0, 1/p) \\
\varepsilon_j^p, & x \in [jp^{-1}, (j+1)p^{-1}), j = 1, 2, \ldots, p - 1 
\end{cases}
\]
where \( \varepsilon_p = \exp \left( \frac{2\pi i}{p} \right) \). The extension of the function \( r_0 \) to \( \mathbb{R}_+ \) is defined by the equality \( r_0(x+1) = r_0(x), x \in \mathbb{R}_+ \). Then the generalized Walsh functions \( \{ \omega_m(x) \}_{m \in \mathbb{Z}_+} \) are defined by

\[
\omega_0(x) = 1, \quad \omega_m(x) = \prod_{j=0}^{k} (r_0(p^j x))^{\mu_j},
\]

where \( m = \sum_{j=0}^{k} \mu_j p^j \), \( \mu_j \in \{0, 1, 2, \ldots, p-1\} \), \( \mu_k \neq 0 \).

For \( x, w \in \mathbb{R}_+ \), let

\[
\chi(x, w) = \exp \left( \frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j w_{-j} + x_{-j} w_j) \right), \quad (1.5)
\]

where \( x_j, w_j \) are calculated by (1.1). If \( x, y, w \in \mathbb{R}_+ \) and \( x \oplus y \) is \( p \)-adic irrational, then

\[
\chi(x \oplus y, w) = \chi(x, w)\chi(y, w), \quad \chi(x \ominus y, w) = \chi(x, w)\overline{\chi(y, w)}, \quad (1.6)
\]

The **Walsh-Fourier transform** of a function \( f \in L^2(\mathbb{R}_+) \) is defined by

\[
\hat{f}(w) = \int_{\mathbb{R}_+} f(x)\overline{\chi(x, w)}dx, \quad (1.7)
\]

where \( \chi(x, w) \) is given by (1.5). The properties of Walsh-Fourier transform are quite similar to the classical Fourier transform [7, 12]. It is known that systems \( \{ \chi(\alpha, \cdot) \}_{\alpha=0}^{\infty} \) and \( \{ \chi(\cdot, \alpha) \}_{\alpha=0}^{\infty} \) are orthonormal bases in \( L^2(0, 1) \).

According to Golubov et al. [7] for any \( \phi \in L^2(\mathbb{R}_+) \), we have

\[
\int_{\mathbb{R}_+} \phi(t)\overline{\phi(t \oplus k)} dt = \int_{\mathbb{R}_+} \phi(w)\overline{\phi(w)}\chi(k, w) dw, \quad k \in \mathbb{Z}_+. \quad (1.8)
\]

2. **Vector-Valued Multiresolution Analysis on \( \mathbb{R}_+ \)**

We use the following notations. Let \( \mathbb{C} \) be the set of all complex numbers, \( I_N \) and \( O \) represent \( N \times N \) identity matrix and the zero matrix respectively. \( L^2(\mathbb{R}_+, \mathbb{C}^N) \) represents the set of square integrable vector-valued functions \( \mathbf{F}(t) \) on positive half line \( \mathbb{R}_+ \) i.e.

\[
L^2(\mathbb{R}_+, \mathbb{C}^N) = \left\{ \mathbf{F}(t) = \left( f_1(t), f_2(t), \ldots, f_N(t) \right)^T \right\}
\]
where \( t \in \mathbb{R}_+ \), \( f_v(t) \in L^2(\mathbb{R}_+) \), \( v = 1, 2, \ldots N \) and \( T \) denotes transpose. For \( F \in L^2(\mathbb{R}_+, \mathbb{C}^N) \), 
\[ \|F\|_{L^2(\mathbb{R}_+, \mathbb{C}^N)} \] is the norm of the function \( F \), i.e., 
\[ \|F\|_{L^2(\mathbb{R}_+, \mathbb{C}^N)} = \sqrt{\sum_{v=1}^N \int_{\mathbb{R}_+} |f_v(t)|^2 dt}. \]
The integration and Walsh-Fourier transform of \( F(t) \) are defined, respectively as:

\[
\int_{\mathbb{R}_+} F(t) dt = \left( \int_{\mathbb{R}_+} f_1(t) dt, \int_{\mathbb{R}_+} f_2(t) dt, \ldots, \int_{\mathbb{R}_+} f_N(t) dt \right)^T,
\]

\[
\hat{F}(w) = \int_{\mathbb{R}_+} F(t) \overline{\chi(k,w)} dt.
\]

For \( F, H \in L^2(\mathbb{R}_+, \mathbb{C}^N) \), their symbol inner product is defined by

\[
\langle F, H \rangle_{L^2(\mathbb{R}_+, \mathbb{C}^N)} = \int_{\mathbb{R}_+} F(t)H(t)^* dt,
\]

where \( \langle \cdot, \cdot \rangle \) means complex conjugate and transpose.

**Definition 2.1** — We say that \( F(t) \in U \subseteq L^2(\mathbb{R}_+, \mathbb{C}^N) \) is an orthogonal vector-valued function in \( U \) if its translations \( \{F(t \circ k)\}_{k \in \mathbb{Z}_+} \) satisfy

\[
\langle F(t \circ k), F(t \circ n) \rangle = \delta_{k,n} I_N, \quad k, n \in \mathbb{Z}_+,
\]

where \( \delta_{k,n} = 1 \) when \( k = n \) and \( \delta_{k,n} = 0 \) when \( k \neq n \).

**Definition 2.1** — A sequence \( \{F_k(t)\}_{k \in \mathbb{Z}_+} \in U \subseteq L^2(\mathbb{R}_+, \mathbb{C}^N) \) is called an orthonormal basis of \( U \) if it satisfies (2.1), and for any \( H(t) \in U \), there exists a unique sequence of \( N \times N \) constant matrices \( \{A_k\}_{k \in \mathbb{Z}_+} \) such that

\[
H(t) = \sum_{k \in \mathbb{Z}_+} A_k F_k(t).
\]

Now, we introduce vector-valued multiresolution analysis on positive half line and give the definition of associated orthogonal vector-valued wavelets.

**Definition 2.3** — A vector-valued multiresolution analysis in \( L^2(\mathbb{R}_+, \mathbb{C}^N) \) is a nested sequence of closed subspaces \( \{V_j\}_{j \in \mathbb{Z}} \) of \( L^2(\mathbb{R}_+, \mathbb{C}^N) \) such that following hold:

(a) \( V_j \subset V_{j+1}, j \in \mathbb{Z} \).
(b) $\bigcup_j V_j$ is dense in $L^2(\mathbb{R}_+, \mathbb{C}^N)$ and $\bigcap_j V_j = \{0\}$, where 0 is the zero vector of $L^2(\mathbb{R}_+, \mathbb{C}^N)$.

(c) $F(t) \in V_j$ if and only if $F(pt) \in V_{j+1}$.

(d) $F(t) \in V_0 \Rightarrow F(t \odot k) \in V_0$ for all $k \in \mathbb{Z}_+$.

(e) there exists a function called vector-valued scaling function $\Phi \in V_0$ such that its translations $\Phi_k(t) = \Phi(t \odot k), k \in \mathbb{Z}_+$, form an orthonormal basis for $V_0$.

The space $V_j$ is defined by

$$V_j = \text{clos}_{L^2(\mathbb{R}_+, \mathbb{C}^N)}(\text{span}\{\Phi(p^j t \odot k) : k \in \mathbb{Z}_+\}), \ j \in \mathbb{Z}. $$

Since $\Phi \in V_0 \subset V_1$, by definitions (2.1) and (2.2), there exists a finitely supported sequence of constant $N \times N$ matrices $\{R_k^{(0)}\}_{k \in \mathbb{Z}_+}$ such that

$$\Phi(t) = \sum_{k \in \mathbb{Z}_+} R_k^{(0)} \Phi(pt \odot k), \ t \in \mathbb{R}_+. \quad (2.2)$$

By Walsh Fourier transform, we have

$$\hat{\Phi}(w) = R^{(0)}(w/p) \hat{\Phi}(w/p), \ w \in \mathbb{R}_+. \quad (2.3)$$

where

$$R^{(0)}(w) = \frac{1}{p} \sum_{k \in \mathbb{Z}_+} R_k^{(0)} \overline{\chi(k, w)}. \quad (2.4)$$

Noting that $\chi(k, w + l) = \chi(k, w), k, l \in \mathbb{Z}_+$, so $R^{(0)}(w)$ is 1-periodic function of $w$. Let $W_j, j \in \mathbb{Z}$ denote the orthocomplement subspace of $V_j$ in $V_{j+1}$ and there exists $p - 1$ vector-valued functions $\Psi_\nu(t) \in L^2(\mathbb{R}_+, \mathbb{C}^N), \nu \in \Lambda$, where $\Lambda = \{1, 2, \ldots, p - 1\}$, such that their translations and dilations form Riesz basis of $W_j$, i.e.,

$$W_j = \text{clos}_{L^2(\mathbb{R}_+, \mathbb{C}^N)}(\text{span}\{\Psi_\nu(p^j t \odot k) : \nu \in \Lambda, k \in \mathbb{Z}_+\}), j \in \mathbb{Z}. \quad (2.5)$$

For each $\nu \in \Lambda$, $\Psi_\nu(t) \in W_0 \subset V_1$, there exists $p - 1$ sequences of $N \times N$ constant matrices $\{R_k^{(\nu)}\}_{k \in \mathbb{Z}_+}$ such that

$$\Psi_\nu(t) = \sum_{k \in \mathbb{Z}_+} R_k^{(\nu)} \Phi(pt \odot k), \ \nu \in \Lambda, t \in \mathbb{R}_+. \quad (2.6)$$
By taking Walsh-Fourier transform, the refinement equation (2.6) becomes

\[ \tilde{\Psi}_\nu(w) = \mathcal{R}^{(\nu)}(w/p)\Phi(w/p), \quad w \in \mathbb{R}_+, \nu \in \Lambda \]

(2.7)

where

\[ \mathcal{R}^{(\nu)}(w) = \frac{1}{p} \sum_{k \in \mathbb{Z}_+} R_k^{(\nu)} \chi(k, w). \]

(2.8)

We say \( p - 1 \) vector-valued functions \( \Psi_1(t), \Psi_2(t), \ldots, \Psi_{p-1}(t) \) are orthogonal vector-valued wavelet functions associated with the orthogonal vector-valued scaling function \( \Phi(t) \), if they satisfy

\[ \langle \Psi_\nu(t) \Phi(t \otimes k) \rangle = O, \quad \nu \in \Lambda, k \in \mathbb{Z}_+, \]

(2.9)

and the family \( \{ \Psi_\nu(t \otimes k), \nu \in \Lambda, k \in \mathbb{Z}_+ \} \) is an orthonormal basis of the subspace \( W_0 \). So we have

\[ \langle \Psi_\mu(t), \Psi_\nu(t \otimes k) \rangle = \delta_{0,k} \delta_{\mu,\nu} I_N, \quad \mu, \nu \in \Lambda, k \in \mathbb{Z}_+. \]

(2.10)

The following lemma gives a characterization in the frequency domain of an orthogonal vector-valued function \( \mathbf{F}(t) \).

**Lemma 2.1** — Let \( \mathbf{F}(t) \in L^2(\mathbb{R}_+, \mathbb{C}^N) \). Then \( \mathbf{F}(t) \) is an orthogonal vector-valued function if and only if

\[ \sum_{l \in \mathbb{Z}_+} \mathbf{F}(w + l) \mathbf{F}(w + l)^* = I_N, \quad w \in \mathbb{R}_+. \]

(2.11)

The following theorem proved in [10] gives the existence of vector-valued wavelets associated with vector-valued MRA on positive half line.

**Theorem 2.1** — Let \( \Phi(t) \in L^2(\mathbb{R}_+, \mathbb{C}^N) \) defined in (2.2) be an orthogonal vector-valued scaling function. Then \( \Psi_\nu(t) \in L^2(\mathbb{R}_+, \mathbb{C}^N), \nu \in \Lambda \) defined by (2.6) are orthogonal vector-valued wavelet functions associated with \( \Phi(t) \) if and only if

\[ \sum_{l=0}^{p-1} R^{(0)} \left( \frac{w + l}{p} \right) R^{(\nu)} \left( \frac{w + l}{p} \right)^* = O, \quad \nu \in \Lambda, w \in \mathbb{R}_+, \]

(2.12)

\[ \sum_{l=0}^{p-1} R^{(\mu)} \left( \frac{w + l}{p} \right) R^{(\nu)} \left( \frac{w + l}{p} \right)^* = \delta_{\mu,\nu} I_N, \quad \mu, \nu \in \Lambda, w \in \mathbb{R}_+. \]

(2.13)
Let $\Lambda_0 = \{0, 1, \ldots, p - 1\}$. By Walsh-Fourier analysis and (2.4),(2.8), identities (2.12), (2.13) become
\[
\sum_{l \in \mathbb{Z}_+} R_l^{(\mu)}(R_{l \oplus pk}^{(\nu)})^* = p\delta_{\mu, \nu}\delta_{0, k}I_N, \quad \mu, \nu \in \Lambda_0, k \in \mathbb{Z}_+. \tag{2.14}
\]

**Theorem 2.2** — Let $\Phi \in L^2(\mathbb{R}_+, \mathbb{C}^N)$ be an orthogonal vector-valued function. Define $V = \text{clos}_{L^2(\mathbb{R}_+, \mathbb{C}^N)}(\text{span}\{\Phi(pt \oplus k) : k \in \mathbb{Z}_+\})$. Let $\mathcal{R}^{(0)}(w), \mathcal{R}^{(\nu)}(w), \nu \in \Lambda$ are defined by (2.4) and (2.8) respectively, satisfy (2.12),(2.13). Define
\[
\tilde{\Psi}_l(w) = \mathcal{R}^{(l)}(w/p)\tilde{\Phi}(w/p), \quad l \in \Lambda_0, \tag{2.15}
\]
then $\{\Psi_l(t \oplus k) : l \in \Lambda_0, k \in \mathbb{Z}_+, t \in \mathbb{R}_+\}$ is an orthonormal system. Moreover, this system is an orthonormal basis for $V$.

**Proof:** Since $\Phi \in L^2(\mathbb{R}_+, \mathbb{C}^N)$ is an orthogonal vector-valued function, By Lemma 2.1, we have
\[
\sum_{l \in \mathbb{Z}_+} \hat{\Phi}(w + l)\hat{\Phi}(w + l)^* = I_N. \tag{2.16}
\]

For $\mu, \nu \in \Lambda_0$ and $k \in \mathbb{Z}_+$, By (2.15) we have
\[
\langle \Psi_\mu(t), \Psi_\nu(t \oplus k) \rangle = \sum_{l \in \mathbb{Z}_+} \int_{\mathbb{R}_+} \mathcal{R}^{(\mu)}(w/p)\tilde{\Phi}(w/p)\tilde{\Phi}^*(w/p)\chi(k, w)dw
\]
\[
= \int_{\mathbb{R}_+} \sum_{l \in \mathbb{Z}_+} \mathcal{R}^{(\mu)}(w/p)\hat{\Phi}(w/p)\hat{\Phi}(w/p)^*\mathcal{R}^{(\nu)}(w/p)^*\chi(k, w)dw
\]
\[
= \int_{\mathbb{R}_+} \sum_{m=0}^{p-1} \mathcal{R}^{(\mu)}(w/p)(\sum_{l \in \mathbb{Z}_+} \tilde{\Phi}(w/m + l)\hat{\Phi}(w/m + l)^*)
\]
\[
\times \left(\mathcal{R}^{(\nu)}(w/m)^*\chi(k, w)\right)dw
\]
By (2.12),(2.13) and (2.16), we get
\[
\langle \Psi_\mu(t), \Psi_\nu(t \oplus k) \rangle = \delta_{\mu, \nu}\delta_{0, k}I_N.
\]
Therefore \( \{ \Psi_l(t \odot k) : l \in \Lambda_0, k \in \mathbb{Z}_+, t \in \mathbb{R}_+ \} \) is an orthonormal system. Let \( F(t) \in V \), there exists constant matrix sequence \( \{ A_k \}_{k \in \mathbb{Z}_+} \) such that
\[
F(t) = \sum_{k \in \mathbb{Z}_+} A_k \Phi(pt \odot k) \tag{2.17}
\]

We claim that
\[
\Phi(pt \odot k) = \frac{1}{p^2} \sum_{\nu=0}^{p-1} \sum_{l \in \mathbb{Z}_+} (R_{k \otimes pt}^{(\nu)})^* \Psi_{\nu}(t \odot l). \tag{2.18}
\]

Consider
\[
\frac{1}{p^2} \sum_{\nu=0}^{p-1} \sum_{l \in \mathbb{Z}_+} (R_{k \otimes pt}^{(\nu)})^* \Psi_{\nu}(t \odot l) = \frac{1}{p^2} \sum_{\nu=0}^{p-1} \sum_{l \in \mathbb{Z}_+} (R_{k \otimes pt}^{(\nu)})^* \sum_{m \in \mathbb{Z}_+} R_{m \otimes pt}^{(\nu)} \Phi(pt \odot pl \odot m)
\]
\[
= \frac{1}{p^2} \sum_{n \in \mathbb{Z}_+} \left\{ \sum_{\nu=0}^{p-1} \sum_{l \in \mathbb{Z}_+} (R_{k \otimes pt}^{(\nu)})^* R_{n \otimes pt}^{(\nu)} \right\} \Phi(pt \odot n)
\]
\[
= \Phi(pt \odot k)
\]

In view of (2.17) and (2.18), we arrive at the system \( \{ \Psi_l(t \odot k) : l \in \Lambda_0, k \in \mathbb{Z}_+, t \in \mathbb{R}_+ \} \) is an orthonormal basis of \( V \).

Using Theorem 2.2, we can split vector-valued function space into mutually orthogonal subspaces.

3. Construction of Vector-Valued Wavelet Packets on \( \mathbb{R}_+ \)

Let \( \{ V_j \}_{j \in \mathbb{Z}} \) be a vector-valued multiresolution analysis on \( \mathbb{R}_+ \) with the orthogonal vector-valued scaling function \( \Phi \). Applying the Theorem 2.2 to the space \( V_1 \), we get functions \( G_\nu, \nu \in \Lambda_0 \), where
\[
\tilde{G}_\nu(w) = R^{(\nu)}(w/p)\tilde{\Phi}(w/p), \tag{3.1}
\]
such that \( \{ G_\nu(t \odot k) : \nu \in \Lambda_0, k \in \mathbb{Z}_+, t \in \mathbb{R}_+ \} \) forms an orthonormal basis for \( V_1 \). Here \( G_0 = \Phi \), the vector-valued scaling function and \( G_l, 1 \leq l \leq p - 1 \) are the vector-valued wavelets.
Definition 3.1 — The collection of vector-valued functions \( \{G_{\mu \nu}(t), \mu \in \mathbb{Z}_+, \nu \in \Lambda_0\} \) is called vector-valued wavelet packets associated with vector-valued MRA on \( \mathbb{R}_+ \), where
\[
G_{\mu \nu}(t) = \sum_{k \in \mathbb{Z}_+} R_k^{(\nu)} G_\mu(pt \oplus k), \quad \nu \in \Lambda_0. \tag{3.2}
\]

By Walsh-Fourier transform, we get
\[
\tilde{G}_{\mu \nu}(w) = \mathcal{R}^{(\nu)}(w/p) \tilde{G}_\mu(w/p), \quad w \in \mathbb{R}_+. \tag{3.3}
\]

In the following, we will characterize the properties of the vector-valued wavelet packets on \( \mathbb{R}_+ \).

**Theorem 3.1** — If \( \{G_\mu(t), \mu \in \mathbb{Z}_+\} \) are vector-valued wavelet packets associated with vector-valued MRA on \( \mathbb{R}_+ \), then for every \( \mu \in \mathbb{Z}_+ \), we have
\[
\langle G_\mu(t), G_\mu(t \ominus k) \rangle = \delta_{0,k} I_N, \quad k \in \mathbb{Z}_+. \tag{3.4}
\]

**Proof:** For \( \mu = 0 \), we have \( \langle G_0(t), G_0(t \ominus k) \rangle = \langle \Phi(t), \Phi(t \ominus k) \rangle = \delta_{0,k} I_N \). Assume that (3.4) follows if \( 0 \leq \mu \leq p^m \), \( m \) is a fixed positive integer. For \( p^m \leq \mu \leq p^{m+1} \), we have \( p^{m-1} \leq |\mu/p| \leq p^m \). We set \( \mu = p[\mu/p] + \nu, \nu \in \Lambda_0 \). In view of induction hypothesis and Lemma 2.1, we have
\[
\langle G_{[\mu/p]}(t), G_{[\mu/p]}(t \ominus k) \rangle = \delta_{0,k} I_N \iff \sum_{l \in \mathbb{Z}_+} \tilde{G}_{[\mu/p]}(w+l) \tilde{G}_{[\mu/p]}(w+l)^* = I_N. \tag{3.5}
\]

By (2.13), (3.3) and (3.5), we have
\[
\langle G_\mu(t), G_\mu(t \ominus k) \rangle
= \int_{\mathbb{R}_+} \tilde{G}_\mu(w) \tilde{G}_\mu(w)^* \chi(k,w) dw
= \int_0^1 \sum_{l \in \mathbb{Z}_+} \mathcal{R}^{(\nu)}(w/l) \tilde{G}_\mu^{h}(w/l) \chi(k,w) dw
= \int_0^1 \mathcal{R}^{(\nu)} \left( \sum_{l \in \mathbb{Z}_+} \tilde{G}_\mu^{h}(w/l) \chi(k,w) \right) dw
\]
\[ = \int_0^1 \sum_{m=0}^{p-1} R^{(\nu)} \left( \frac{w + m}{p} \right) R^{(\nu)} \left( \frac{w + m}{p} \right) \chi(k, w) dw \]
\[ = \delta_{0,k} I_N. \]

**Theorem 3.2** — If \( \{G_\mu(t), \mu \in \mathbb{Z}_+\} \) are vector-valued wavelet packets associated with vector-valued MRA on \( \mathbb{R}_+ \), then for every \( \mu \in \mathbb{Z}_+ \), we have

\[ \langle G_{p\nu + \nu}(t), G_{p\nu + \lambda}(t \ominus k) \rangle = \delta_{\nu, \lambda} \delta_{0,k} I_N, \quad \nu, \lambda \in \Lambda_0, k \in \mathbb{Z}_+. \tag{3.6} \]

**Proof:** By formulas (2.13), (3.3) and (3.5) we have

\[
\langle G_{p\nu + \nu}(t), G_{p\nu + \lambda}(t \ominus k) \rangle \\
= p \sum_{l \in \mathbb{Z}_+} \int_0^{l+1} R^{(\nu)}(w) \tilde{G}_\mu(w) \tilde{G}_\mu(w) R^{(\lambda)}(w) \chi(k, pw) \chi(k, pw) dw \\
= p \int_0^1 R^{(\nu)}(w) \left( \sum_{l \in \mathbb{Z}_+} \tilde{G}_\mu(w + l) \tilde{G}_\mu(w + l) \chi(k, pw) \chi(k, pw) \right) dw \\
= p \int_0^{1/p} \sum_{m=0}^{p-1} R^{(\nu)}(w + m/p) R^{(\lambda)}(w + m/p) \chi(k, pw) \chi(k, pw) dw \\
= p \int_0^{1/p} \delta_{\nu, \lambda} I_N \chi(k, pw) \chi(k, pw) \chi(k, pw) \chi(k, pw) dw = \delta_{\nu, \lambda} \delta_{0,k} I_N. \tag{\Box} \]

**Theorem 3.3** — For any \( \mu, \nu \in \mathbb{Z}_+ \) and \( k \in \mathbb{Z}_+ \), we have

\[ \langle G_\mu(t), G_\nu(t \ominus k) \rangle = \delta_{\mu, \nu} \delta_{0,k} I_N \tag{3.7} \]

**Proof:** For \( \mu = \nu \), (3.7) follows by Theorem 3.1. If \( \mu \neq \nu \), let \( \mu = p\mu_1 + \rho_1, \nu = p\nu_1 + s_1 \), where \( \rho_1, s_1 \in \Lambda_0 \).

(i) If \( \mu_1 = \nu_1 \), then \( \rho_1 \neq s_1 \), formula (3.7) follows from (2.13),(3.3),(3.5).

(ii) If \( \mu_1 \neq \nu_1 \), then set \( \mu_1 = p\mu_2 + \rho_2, \nu_1 = p\nu_2 + s_2, \rho_2, s_2 \in \Lambda_0 \). If \( \mu_2 = \nu_2 \), (3.7) follows from case (i). If \( \mu_2 \neq \nu_2 \), then we order \( \mu_2 = p\mu_3 + \rho_3, \nu_2 = p\nu_3 + s_3, \rho_3, s_3 \in \Lambda_0 \). Thus after taking finite \( \tau \) steps, we get \( \mu_\tau, \nu_\tau \in \Lambda_0 \) and \( \rho_\tau, s_\tau \in \Lambda_0 \). If \( \mu_\tau = \nu_\tau \), then \( \rho_\tau \neq s_\tau \),
result (3.7) follows similar to case (i). If \( \mu_r \neq \nu_r \), then \( G_{\mu_r}, G_{\nu_r} \) are vector-valued wavelets. By Lemma 2.1, we have
\[
\langle G_{\mu_r}(t), G_{\nu_r}(t \odot k) \rangle = O \Leftrightarrow \sum_{l \in \mathbb{Z}^+} \tilde{G}_{\mu_r}(w + l) \tilde{G}_{\nu_r}(w + l)^* = O, \quad w \in \mathbb{R}_+.
\]
Thus
\[
\langle G_{\mu}(t), G_{\nu}(t \odot k) \rangle
\]
\[
= \int_{\mathbb{R}_+} \prod_{l=1}^{\tau} \mathcal{R}_{(\rho)} \left( \frac{w}{p^l} \right) \tilde{G}_{\mu_r} \left( \frac{w}{p^l} \right) \tilde{G}_{\nu_r} \left( \frac{w}{p^l} \right)^* \left( \prod_{l=1}^{\tau} \mathcal{R}_{(s)} \left( \frac{w}{p^l} \right) \right)^* \chi(k, w) dw
\]
\[
= \sum_{m \in \mathbb{Z}_+} \int_{p^m}^{p^{m+1}} \prod_{l=1}^{\tau} \mathcal{R}_{(\rho)} \left( \frac{w}{p^l} \right) \tilde{G}_{\mu_r} \left( \frac{w}{p^l} \right) \tilde{G}_{\nu_r} \left( \frac{w}{p^l} \right)^* \left( \prod_{l=1}^{\tau} \mathcal{R}_{(s)} \left( \frac{w}{p^l} \right) \right)^* \chi(k, w) dw
\]
\[
= \int_{0}^{p^\tau} \prod_{l=1}^{\tau} \mathcal{R}_{(\rho)} \left( \frac{w}{p^l} \right) \left( \sum_{m \in \mathbb{Z}_+} \tilde{G}_{\mu_r} \left( \frac{w}{p^l} + m \right) \tilde{G}_{\nu_r} \left( \frac{w}{p^l} + m \right)^* \left( \prod_{l=1}^{\tau} \mathcal{R}_{(s)} \left( \frac{w}{p^l} \right) \right)^* \chi(k, w) dw = O. \quad \Box
\]

**Theorem 3.4** — Let \( \{ G_{\mu} : \mu \in \mathbb{Z}_+ \} \) be the vector-valued wavelet packets associated with the vector-valued multiresolution analysis on \( \mathbb{R}_+ \). Then

(i) \( \{ G_{\mu}(t \odot k) : p^j \leq \mu \leq p^{j+1} - 1, k \in \mathbb{Z}_+ \} \) is an orthonormal basis of \( W_j, j \geq 0 \).

(ii) \( \{ G_{\mu}(t \odot k) : 0 \leq \mu \leq p^j - 1, k \in \mathbb{Z}_+ \} \) is an orthonormal basis of \( V_j, j \geq 0 \).

(iii) \( \{ G_{\mu}(t \odot k) : \mu \geq 0, k \in \mathbb{Z}_+ \} \) is an orthonormal basis of \( L^2(\mathbb{R}_+, \mathbb{C}^N) \).

**Proof** : We prove the theorem by induction on \( j \). For \( j = 0 \), we have \( \{ G_{\mu} : 1 \leq \mu \leq p - 1 \} \), which are vector-valued wavelets form orthonormal basis for \( W_0 \). Now assume that result holds for \( m \), i.e., \( \{ G_{\mu}(t \odot k) : p^m \leq \mu \leq p^{m+1} - 1, k \in \mathbb{Z}_+ \} \) is an orthonormal basis for \( W_m \). Therefore \( \{ G_{\mu}(pt \odot k) : p^m \leq \mu \leq p^{m+1} - 1, k \in \mathbb{Z}_+ \} \) is an orthonormal basis for \( W_{m+1} \).

Define \( \mathcal{F}_\mu = \text{clos}_{L^2(\mathbb{R}_+, \mathbb{C}^N)}(\text{span}\{ G_{\mu}(pt \odot k) : k \in \mathbb{Z}_+ \}) \) such that
\[
W_{m+1} = \bigoplus_{\mu = p^m}^{p^{m+1}-1} \mathcal{F}_\mu \quad (3.8)
\]
Applying Theorem 2.2 to $F_{\mu}$, we get the functions $H_{\mu,l}$, $l \in \Lambda_0$ defined by

$$\hat{H}_{\mu,l}(w) = R^{(l)}(w/p)\hat{G}_{\mu}(w/p), \quad l \in \Lambda_0$$

(3.9)

such that $\{H_{\mu,l}(t \otimes k) : 0 \leq l \leq p - 1, k \in \mathbb{Z}_+\}$ is an orthonormal basis of $F_{\mu}$. By (3.3), we have $H_{\mu,l} = G_{\mu p^l}$.

Since

$$\{l + p\mu : 0 \leq l \leq p - 1, p^m \leq \mu \leq p^{m+1} - 1\} = \{\mu : p^{m+1} \leq \mu \leq p^{m+2} - 1\}$$

Thus $\{G_{\mu}(t \otimes k) : p^{m+1} \leq \mu \leq p^{m+2} - 1, k \in \mathbb{Z}_+\}$ is an orthonormal basis of $W_{m+1}$ and the induction is complete.

As we have $V_j = V_0 \oplus W_0 \oplus \ldots \oplus W_{j-1}$, part (ii) follows. Since, we know

$$L^2(\mathbb{R}_+, \mathbb{C}^N) = V_0 \oplus \bigoplus_{j \geq 0} W_j$$

(3.10)

This completes the proof. \(\square\)

4. THE DIRECT DECOMPOSITION FOR SPACE $L^2(\mathbb{R}_+, \mathbb{C}^N)$

In this section, we will decompose subspaces $V_j$ and $W_j$ by virtue of a series of subspaces of vector-valued wavelets packets. Furthermore, we present the direct decomposition for space $L^2(\mathbb{R}_+, \mathbb{C}^N)$. For $\mu \in \mathbb{Z}_+$ and $j \in \mathbb{Z}$, define the subspaces

$$U_j^\mu = \text{clos}_{L^2(\mathbb{R}_+, \mathbb{C}^N)}(\text{span}\{G_{\mu}(p^j t \otimes k) : k \in \mathbb{Z}_+\}).$$

(4.1)

We observe that $U_0^0 = V_j, W_j = \bigoplus_{r=1}^{p-1} U_j^r$, so that orthogonal decomposition $V_{j+1} = V_j \oplus W_j$ can be written as

$$U_{j+1}^0 = \bigoplus_{r=0}^{p-1} U_j^r.$$  

(4.2)

**Theorem 4.1** — If $\mu \in \mathbb{Z}_+$ and $j \in \mathbb{Z}$, we have

$$U_{j+1}^\mu = \bigoplus_{\nu=0}^{p-1} U_j^{\nu+p\mu}.$$  

(4.3)
CONSTRUCTION OF VECTOR VALUED WAVELET PACKETS

PROOF: By definition

$$U_{j+1}^\mu = \text{clos}_{L^2(\mathbb{R}^+, C^N)}(\text{span}\{G_\mu(p^{j+1}t \otimes k) : k \in \mathbb{Z}_+\}) \quad (4.4)$$

Suppose $H_k(t) = G_\mu(p^{j+1}t \otimes k), k \in \mathbb{Z}_+$, then $\{H_k : k \in \mathbb{Z}_+\}$ is an orthonormal basis for $U_{j+1}^\mu$. Define

$$\Upsilon_m^{(\nu)}(t) = \sum_{k \in \mathbb{Z}_+} R_k^{(\nu)} H_m(t), \quad \nu \in \Lambda_0, m \in \mathbb{Z}_+, \quad (4.5)$$

and $\mathcal{F}_\nu = \text{clos}_{L^2(\mathbb{R}^+, C^N)}(\text{span}\{\Upsilon_m^{(\nu)} : m \in \mathbb{Z}_+\})$. Clearly, we have $\bigoplus_{\nu=0}^{p-1} \mathcal{F}_\nu \subset U_{j+1}^\mu$. Now

$$\Upsilon_m^{(\nu)}(t) = \sum_{k \in \mathbb{Z}_+} R_k^{(\nu)} G_\mu(p^{j+1}t \otimes k \otimes pm)$$

$$= \sum_{k \in \mathbb{Z}_+} R_k^{(\nu)} G_\mu(p(p^j t \otimes m) \otimes k)$$

$$= G_{p\mu+\nu}(p^j t \otimes m).$$

Therefore, we have $\mathcal{F}_\nu = U_j^{\nu-p\mu}$. Now consider

$$\frac{1}{p^2} \sum_{\nu=0}^{p-1} \sum_{m \in \mathbb{Z}_+} (R_k^{(\nu)})^* G_{p\mu+\nu}(p^j t \otimes m) = \frac{1}{p^2} \sum_{\nu=0}^{p-1} \sum_{m \in \mathbb{Z}_+} (R_k^{(\nu)})^*$$

$$\sum_{l \in \mathbb{Z}_+} R_l^{(\nu)} G_\mu(p^{j+1}t \otimes pm \otimes l) = \frac{1}{p^2} \sum_{n \in \mathbb{Z}_+} \left\{ \sum_{\nu=0}^{p-1} \sum_{m \in \mathbb{Z}_+} (R_k^{(\nu)})^* R_k^{(\nu)} \right\} G_\mu(p^{j+1}t \otimes n)$$

By (2.14), we get

$$G_\mu(p^{j+1}t \otimes n) = \frac{1}{p^2} \sum_{\nu=0}^{p-1} \sum_{m \in \mathbb{Z}_+} (R_k^{(\nu)})^* G_{p\mu+\nu}(p^j t \otimes m) \quad (4.6)$$

By (4.6), we have $U_{j+1}^\mu \subset \bigoplus_{\nu=0}^{p-1} \mathcal{F}_\nu$. Therefore

$$U_{j+1}^\mu = \bigoplus_{\nu=0}^{p-1} \mathcal{F}_\nu = \bigoplus_{\nu=0}^{p-1} U_j^{\nu+p\mu}. \quad \square$$
If \( j \geq 0 \), by above Theorem, we have

\[
W_j = \bigoplus_{r=1}^{p-1} U_j^r = \bigoplus_{r=p}^{p^2-1} U_j^r = \ldots = \bigoplus_{r=p^m}^{p^{m+1}-1} U_{j-m}^r, \quad m \leq j
\]

\[
= \bigoplus_{r=p^j}^{p^{j+1}-1} U_j^r.
\]

Therefore, we can obtain new orthonormal basis of \( L^2(\mathbb{R}_+, \mathbb{C}^N) \) by constructing a series of subspaces of vector-valued wavelet packets.

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