ON APPLICATION OF EULER’S DIFFERENTIAL METHOD TO A CONTINUED FRACTION DEPENDING ON PARAMETER

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In this paper we apply Euler’s differential method, which was not used by mathematicians for a long time, to derive a new formula for a certain kind irregular continued fraction depending on a parameter. This formula is in the form of the ratio of two integrals. In case of integer values of the parameter, the formula reduces to the ratio of two finite sums. Asymptotic behavior of this continued fraction is investigated numerically and it is shown that it increases in the same rate as the root function.

Key words: Continued fraction; Euler’s differential method.

1. INTRODUCTION

A continued fraction is an expression of the form

\[ a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \ldots}}}, \]

where \(a_0, a_1, a_2, \ldots\) and \(b_1, b_2, b_3, \ldots\) are two sequences of real or complex numbers. In case if these sequences are finite, the continued fraction is said to be finite. If they are infinite, then
the continued fraction is said to be infinite. It is useful to use for the above continued fraction the symbols
\begin{equation}
    a_0 + K_{k=1}^m \left[ \frac{b_k}{a_k} \right] \quad \text{and} \quad a_0 + K_{k=1}^{\infty} \left[ \frac{b_k}{a_k} \right],
\end{equation}
depending on whether it is finite or infinite, or the Rogers’ symbols (see Roger [9])
\[ a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \cdots + a_m}} \quad \text{and} \quad a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \cdots}}} . \]

In this regard, \( c_m = a_0 + K_{k=1}^m \left[ \frac{b_k}{a_k} \right] \) is called an \( m \)-th convergent of \( a_0 + K_{k=1}^{\infty} \left[ \frac{b_k}{a_k} \right] \). If \( \lim c_m \) exists and equals to \( c \), then it is natural to call the continued fraction \( a_0 + K_{k=1}^{\infty} \left[ \frac{b_k}{a_k} \right] \) to be convergent and write
\[ c = a_0 + K_{k=1}^{\infty} \left[ \frac{b_k}{a_k} \right] . \]

Otherwise, \( a_0 + K_{k=1}^{\infty} \left[ \frac{b_k}{a_k} \right] \) is said to be divergent.

A fractional symbol can be used for the continued fraction in (1) as well. If the sequences \( \{P_m\} \) and \( \{Q_m\} \) are defined recursively by
\[ P_{-1} = 1, \quad P_0 = a_0, \quad Q_{-1} = 0, \quad Q_0 = 1 \]
and
\[ P_m = a_m P_{m-1} + b_m P_{m-2}, \quad Q_m = a_m Q_{m-1} + b_m Q_{m-2}, \] (2)
then the \( m \)-th convergent \( c_m \) of the continued fraction in (1) has the fractional representation
\[ c_m = \frac{P_m}{Q_m} . \]

Moreover, the following relation holds:
\[ P_m Q_{m-1} - P_{m-1} Q_m = (-1)^{m-1} b_1 b_2 \cdots b_m . \] (3)

The proof of these formulas due to Euler–Wallis can be found in Jones and Thron [4], p. 20, and Khrushchev [5], p. 12.

The continued fraction in (1) is said to be regular if \( b_1 = b_2 = \cdots = 1 \), \( a_0 \) is an integer and \( a_1, a_2, \ldots \) are positive integers. It is remarkable that rational and irrational numbers can be clearly distinguished by continued fractions: a real number is rational if and only if it can be
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represented as a finite regular continued fraction. At the same time, a real number is irrational if and only if it has an infinite convergent regular continued fraction representation. This fact motivates the study of continued fractions in more general cases.

Continued fractions became a subject area in mathematics since John Wallis and Lord Brouncker. The great mathematicians such as Karl Jacobi, Oscar Perron, Charles Hermit, Karl Friderich Gauss, Augustin Cauchy, Thomas Stieltjes etc. have contributed to the theory. But much in the theory of continued fractions has been done in the Euler’s 1737 work *De Fractionibus Continuis*. Extraordinary contributions to continued fractions has been made by Indian mathematician Srinivasa Ramanujan (see Berndt [1]). Presently, there are many books devoted to continued fractions, for example, Cuyt *et al.* [2], Jones and Thron [4], Khrushchev [5], Lorentzen and Waadeland [7], Olds [8], Wall [10], etc.

The object of study in this paper is the continued fraction $K_{k=1}^{\infty} \frac{k + t}{k}$, which depends on the parameter $t$. This continued fraction was studied by Euler. We study this continued fraction for $-1 < t < \infty$ and establish some of its properties.

2. Euler’s Differential Method

In the sequel, we will use the Euler’s differential method. Therefore, in this section we briefly mention its basic idea.

In [3] Euler considered a continued fraction of the form

$$K(t) = \frac{t}{1 + \frac{t + 1}{2 + \frac{t + 2}{3 + \frac{t + 3}{4 + \ldots}}}.$$  

In the above notation, $K(t) = K_{k=1}^{\infty} \frac{k + t - 1}{k}$. Clearly, at $t = 0, -1, -2, \ldots$ the right hand side of (4) is a finite continued fraction and, hence, $K(t)$ can be calculated straightforwardly. But at $t = 1, 2, \ldots$ it is an infinite continued fraction and straightforward calculation of $K(t)$ becomes complicated. For this, Euler applied a differential method. The idea of the differential method consists in application of the following result.

**Theorem 1** — *Let $R$ and $P$ be two real-valued functions on the interval $[0, 1]$, which are positive on $(0, 1)$, let $a$, $b$ and $c$ be any real numbers, and let $\alpha$, $\beta$ and $\gamma$ be any positive*
numbers. If for \( n = 0, 1, 2 \ldots \),
\[
(a + n\alpha) \int_0^1 PR^n \, dx = (b + n\beta) \int_0^1 PR^{n+1} \, dx + (c + n\gamma) \int_0^1 PR^{n+2} \, dx,
\]
then
\[
\frac{\int_0^1 PR \, dx}{\int_0^1 P \, dx} = \frac{a}{b} + \frac{(a + \alpha)c}{b + \beta} + \frac{(a + 2\alpha)(c + \gamma)}{b + 2\beta} + \frac{(a + 3\alpha)(c + 2\gamma)}{b + 3\beta} + \ldots.
\]

Moreover, (5) holds if \( R \) and \( P \) satisfy the differential relations
\[
\begin{align*}
R \, ds + S \, dr &= (bR + cR^2 - a)P \, dx, \\
S \, dr &= (\beta R + \gamma R^2 - \alpha)P \, dx,
\end{align*}
\]
on \((0, 1)\) for some function \( S \) on \([0, 1]\) such that \( R^{n+1} S \) vanishes at 0 and 1.

**Proof**: The proof of this theorem can be found in Khrushchev [5], p. 184.

Using this theorem, Euler transferred the continued fraction in (4) to the following continued fraction
\[
K(t) = \frac{t}{2 + \frac{t - 2}{3 + \frac{t + 1}{4 + \frac{t - 3}{5 + \frac{t + 2}{6 + \frac{t - 4}{7 + \frac{t + 3}{8 + \ldots}}}}}}},
\]
which is finite at all integer values of \( t \) except \( t = 1 \). Based on this, he calculated \( K(2) = 1 \), \( K(3) = \frac{4}{3} \) etc. and obtained that \( K(t) \) is rational for all integer values of \( t \) except \( t = 1 \). It is remarkable that \( K(1) = (e - 1)^{-1} \) is irrational, the fact again proved by Euler (see Khrushchev [5], p. 161).

### 3. Some Properties of \( K(t) \)

Below we study some properties of the function \( K(t) \). Technically, it is convenient to consider \( f(t) = K(t + 1) \). Therefore, we let
\[
f(t) = K(t + 1) = \prod_{k=1}^{\infty} \left[ \frac{k + t}{k} \right], -1 < t < \infty.
\]

In this section we study some properties of this function.

#### 3.1 The analyticity

Although the analyticity of \( f(t) \) can be deduced by use of the parabola theorem (see Jones and Thron [4], p. 99), we are intending to give an alternative (purely real) proof of this property without referring to complex numbers. For this, note that the \( m \)th convergent \( c_m(t) = \ldots
\[ K^m_{k=1} \left[ \frac{k + t}{k} \right] \text{ of } K^\infty_{k=1} \left[ \frac{k + t}{k} \right] \] is a rational function of \( t \). Hence, we can represent it in the form

\[ c_m(t) = \frac{P_m(t)}{Q_m(t)} \]

for some polynomials \( P_m \) and \( Q_m \). One can also observe that \( Q_m \) is a positive function on \((-1, \infty)\). Consequently, \( c_m \) is an analytic function on \((-1, \infty)\) for all \( m = 1, 2, \ldots \).

**Lemma 1** — The functions

\[ g_m(t) = \frac{Q_{m+1}(t)}{Q_m(t)}, m = 1, 2, \ldots, \]

are positive and strictly increasing on \((-1, \infty)\).

**Proof**: The positiveness of \( g_m \) follows from the positiveness of \( Q_m \) for all \( m = 1, 2, \ldots \).

To prove that \( g_m \) is strictly increasing, we will verify the condition \( g'_m(t) > 0 \) for \( m = 1, 2, \ldots \).

By (2),

\[ \frac{Q_m(t)}{Q_{m-1}(t)} = m + \frac{m + t}{Q_{m-1}(t)/Q_{m-2}(t)}, \]

implying

\[ g_m(t) = m + 1 + \frac{m + 1 + t}{g_{m-1}(t)}, \]

where \( g_0(t) = 1 \). Hence, for \( m = 1 \),

\[ g'_1(t) = \left( 2 + \frac{2 + t}{1} \right)' = 1 > 0, \]

and for \( m = 2 \),

\[ g'_2(t) = \left( 3 + \frac{3 + t}{g_1(t)} \right)' = -\frac{1}{(4 + t)^2} > 0. \]

Assume that \( g'_n(t) > 0 \) for all \( n = 1, \ldots, m - 1 \). Then

\[ g'_m(t) = \left( m + 1 + \frac{m + 1 + t}{g_{m-1}(t)} \right)' = g_{m-1}(t) - (m + 1 + t)g'_{m-1}(t) \]

\[ = g_{m-1}(t)^{-2} \left( m + \frac{m + t}{g_{m-2}(t)} - (m + 1 + t)g_{m-2}(t) - (m + t)g'_{m-2}(t) \right) \]

\[ = g_{m-1}(t)^{-2} \left( m - \frac{1}{g_{m-2}(t)} + (m + 1 + t)(m + t)g'_{m-2}(t) \right). \]

Since \( g_{m-2}(t) > m - 1 \), we obtain \( g'_m(t) > 0 \). By induction, \( g'_m(t) > 0 \) for all \( m = 1, 2, \ldots \). \( \square \)
Lemma 2 — The functions

\[ h_m(t) = |c_{m-1}(t) - c_m(t)|, \quad m = 1, 2, \ldots \]

are positive and strictly increasing on \((-1, \infty)\).

**Proof:** If \( m = 1 \), then

\[ h_1(t) = |c_2(t) - c_1(t)| = \frac{(1 + t)(2 + t)}{4 + t}. \]

One can verify that \( h_1(t) > 0 \) and \( h'_1(t) > 0 \). Assume that \( h_{m-1} \) is positive and strictly increasing. By (3),

\[ \frac{P_m(t)}{Q_m(t)} - \frac{P_{m-1}(t)}{Q_{m-1}(t)} = \frac{(-1)^{m-1}(1 + t)(2 + t) \cdots (m + t)}{Q_m(t)Q_{m-1}(t)}, \]

implying

\[ h_m(t) = \frac{(1 + t)(2 + t) \cdots (m + 1 + t)}{Q_{m+1}(t)Q_m(t)} = \frac{(1 + t)(2 + t) \cdots (m + t)}{Q_m(t)Q_{m-1}(t)} \cdot \frac{(m + 1 + t)Q_{m-1}(t)}{Q_{m+1}(t)}. \]

Hence, \( h_m \) is positive since all \( Q_m(t) \) are positive. Moreover, by (2),

\[ Q_{m+1}(t) = (m + 1)Q_m(t) + (m + 1 + t)Q_{m-1}(t), \]

producing

\[ h_m(t) = h_{m-1}(t) \cdot \frac{Q_{m+1}(t) - (m + 1)Q_m(t)}{Q_{m+1}(t)} = h_{m-1}(t) \left( 1 - \frac{m + 1}{g_m(t)} \right). \]

Thus, by Lemma 1, \( h_m \) equals to the product of two positive strictly increasing functions. Hence, \( h_m \) is strictly increasing. By induction, \( h_m \) is strictly increasing for all \( m = 1, 2, \ldots \). \( \square \)

**Theorem 2** — The convergents of the continued fraction in (8) converge uniformly on every compact subinterval of \((-1, \infty)\) and the limit function \( f \) is analytic on \((-1, \infty)\).

**Proof:** At first, note that the continued fraction \( K_{k=1}^{\infty} \left[ \frac{k + t}{k} \right] \) converges at every \( t > -1 \). This follows from the theorem of Pringsheim (see Khrushchev [6]) since

\[ \sum_{k=1}^{\infty} \sqrt{\frac{(k-1)k}{k + t}} = +\infty \]
for $t > -1$. Next, let us show that this convergence is uniform on every compact subinterval $[a, b]$ of $(-1, \infty)$. Since the sequence $\{c_m(b)\}$ converges, it is a Cauchy sequence. Hence, for given $\varepsilon > 0$, there is $N$ such that for all $m > N$,

$$|c_{m+1}(b) - c_m(b)| < \varepsilon.$$ 

By Lemma 2, for all $m > N$,

$$\max_{t \in [a, b]} |c_{m+1}(t) - c_m(t)| = |c_{m+1}(b) - c_m(b)| < \varepsilon.$$ 

By Theorem 1.7 from Khrushchev [5], p. 14,

$$c_2(b) < c_4(b) < \cdots < c_{2k}(b) < \cdots < c_{2k+1}(b) < \cdots < c_3(b) < c_1(b).$$

Hence, for all $n > m > N$,

$$\max_{t \in [a, b]} |c_n(t) - c_m(t)| \leq \max_{t \in [a, b]} |c_{m+1}(t) - c_m(t)| = |c_{m+1}(b) - c_m(b)| < \varepsilon.$$ 

This means that the sequence of functions $\{c_m\}$ is uniformly Cauchy on $[a, b]$. Hence, it converges uniformly on $[a, b]$. Finally, since all terms of the sequence $\{c_m\}$ are analytic functions on $[a, b]$, the limit function $f$ is also analytic on $[a, b]$. From the analyticity on every compact subinterval of $(-1, \infty)$, we obtain that $f$ is analytic on $(-1, \infty)$. 

3.2 The representation by integrals

In this subsection we represent the function $f$, defined by (8), as a ratio of two integrals. We will use the following.

**Lemma 3** — The integral

$$\int_0^1 (1 - x)^{b-1} x^{a-1} e^x \, dx$$

is well-defined for $a > 0$ and $b > 0$. Otherwise, it diverges to $\infty$.

**Proof**: Since the exponential function is bounded positive on $[0, 1]$, the lemma follows from comparison of the above integral with the Euler’s integral

$$\int_0^1 (1 - x)^{b-1} x^{a-1} \, dx,$$

which is well-defined for $a > 0$ and $b > 0$, and diverges to $\infty$ whenever $a \leq 0$ or $b \leq 0$. $\square$
Theorem 3 — The function \( f, \) defined by (8), has the representation
\[
f(t) = \frac{\int_0^1 (1-x)^{p-t} \frac{d^p}{dx^p} (x^{t+1}e^x) \, dx}{\int_0^1 (1-x)^{p-t} \frac{d^p}{dx^p} (x^t e^x) \, dx}, \quad p-1 < t \leq p+1,
\] (9)
where \( p = 0, 1, 2, \ldots. \)

**Proof:** Let \( p = 0 \) and prove that
\[
f(t) = \frac{\int_0^1 (1-x)^{-t} x^{t+1}e^x \, dx}{\int_0^1 (1-x)^{-t} x^t e^x \, dx}, \quad -1 < t < 1. \] (10)

It is easily seen that (10) is the same as (6) for the selection \( a = t + 1, \ b = c = \alpha = \beta = 1, \ \gamma = 0, \ R(x) = x \) and \( P(x) = x^t (1-x)^{-t} e^x. \) Therefore, it suffices to verify the condition in (7) of Theorem 1 for a suitable choice of \( S. \) Let \( S(x) = -x^t (1-x)^{1-t} e^x. \) Then \( R^{n+1}(x)S(x) = -x^{t+n+1}(1-x)^{1-t} e^x, \) implying \( R^{n+1}(0)S(0) = R^{n+1}(1)S(1) = 0. \) For our choice of functions \( P, \ R \) and \( S, \) the equations in (7) have the form
\[
\begin{cases}
  x \, dS + S \, dx = (x + x^2 - t - 1)P \, dx, \\
  S \, dx = (x - 1)P \, dx,
\end{cases}
\]
which can be verified easily.

The formula in (10) does not work for \( t \geq 1 \) since in view of Lemma 3 it produces \( \infty \) in its right hand side. Next, we let \( p = 1 \) in (9) and and prove
\[
f(t) = \frac{\int_0^1 (1-x)^{1-t} (x^{t+1}e^x)' \, dx}{\int_0^1 (1-x)^{1-t} (x^t e^x)' \, dx}, \quad 0 < t < 2. \] (11)

For \( 0 < t < 1, \) formula (11) can be deduced by the application of the integration by parts formula to the integrals in (10). The right hand side of (11) is analytic on \( (0, 2). \) By Theorem 2, \( f \) is analytic on \( (-1, \infty). \) Therefore, by the uniqueness theorem of analytic functions, (11) holds for \( 0 < t < 2. \) By multiple application of this procedure, we can prove (9) for every \( p = 0, 1, 2, \ldots. \)

**Corollary 1** — \( f(0) = K(1) = (e - 1)^{-1}, \) where \( K(t) \) is defined by (4).

**Proof:** Letting \( t = 0 \) in (10), we can calculate
\[
f(0) = K(1) = \frac{\int_0^1 xe^x \, dx}{\int_0^1 e^x \, dx} = \frac{1}{e - 1}.
\]
This proves the corollary.

3.3 The representation by finite sums

Now we are interested in integer values of \( t \).

**Theorem 4** — For \( p = 1, 2, \ldots \),

\[
f(p) = (p + 1) \sum_{k=0}^{p-1} \frac{\frac{a_{p,k}}{p - k + 1}}{\sum_{k=0}^{p-1} a_{p,k}},
\]

where

\[
a_{p,k} = \binom{p}{k} \cdot \frac{1}{(p - k - 1)!}.
\]

**Proof**: Substituting \( t = p \) in (11) and integrating, we obtain

\[
f(p) = \left. \frac{dp^{-1}}{dx^p} (x^{p+1} e^x) \right|_{x=1}.
\]

By Leibnitz’s formula for the higher order derivatives of product function,

\[
\left. \frac{dp^{-1}}{dx^p} (x^{p+1} e^x) \right|_{x=1} = \sum_{k=0}^{p-1} \binom{p-1}{k} e^x (x^{p+1})^{(k)} \bigg|_{x=1}
\]

\[
= e \sum_{k=0}^{p-1} \frac{(p - 1)!}{k!(p - k - 1)!} \cdot (p + 1)! \cdot \frac{(p - k + 1)!}{(p - k - 1)!}
\]

\[
= e \sum_{k=0}^{p-1} \frac{(p - 1)!}{k!(p - k - 1)!} \cdot (p + 1)! \cdot \frac{(p - 1)!}{(p - k + 1)(p - k)!}
\]

\[
= e(p + 1) \sum_{k=0}^{p-1} \binom{p}{k} \frac{(p - 1)!}{(p - k + 1)(p - k - 1)!}
\]

\[
= e(p + 1)(p - 1)! \sum_{k=0}^{p-1} \frac{a_{p,k}}{p - k + 1}.
\]

In a similar way,

\[
\left. \frac{dp^{-1}}{dx^p} (x^p e^x) \right|_{x=1} = e(p - 1)! \sum_{k=0}^{p-1} a_{p,k}.
\]
A substitution the calculated expressions in (13) produces the formula in (12).

Using Theorem 4, one can easily recalculate \( f(1) = K(2) = 1, f(2) = K(3) = \frac{4}{3} \), etc.

3.4 The asymptotic behavior

The function \( f \), defined by (8), has a very interesting asymptotic behavior. If

\[
\sigma(t) = \sqrt{t} - f(t), \quad 0 < t < \infty,
\]

then \( \sigma \) a slowly increasing function. The values of \( \sigma \), calculated by use of Wolfram Mathematica Software, are presented in Table 1 for \( t \) changing from \( t = 10^4 \) to \( t = 10^{15} \) with different steps. This table shows that the difference between \( \sqrt{t} \) and \( f(t) \) continuously increases but the values of increase became tiny in comparison to the change of \( t \). From \( t = 10^{12} \) to \( t = 10^{15} \) the program calculates the value 0.25. This allows to conjecture whether \( \sigma \) has a horizontal asymptote. If yes, then it becomes interesting to prove whether \( \sigma_\infty = \lim_{t \to \infty} \sigma(t) = 0.25 \). Theorem 4 may be used for this purpose since \( \sigma_\infty \) can be evaluated by giving \( t \) integer values \( p \) in the limiting process. The formula in (12) is heavily based on factorials. Therefore, the Stirling’s approximate formula for factorials may be efficient to solve this conjecture.

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4. Conclusion

The Euler’s differential method, which was forgotten for a long time, is applied to a continued fraction depending a parameter. A new formula, different from the Euler’s one, derived for this continued fraction. In case of integer values of the parameter, this formula takes a simple form. Asymptotic behavior of this continued fraction is investigated numerically.

REFERENCES


3. L. Euler, Observationes circa fractiones continuas in hac forma contentas \( S = n/(1+(n+1)/(2+(n+2)/(3+(n+3)/(4+\text{ etc.})))) \), *Mémoires de l’Académie des Sciences de St Petersbourg*, 4, (presented on 18 November 1779), 52-74, (Opera Omnia, Ser. 1, 16, 139-161).


