THE SHEPHARD TYPE PROBLEMS AND MONOTONICITY FOR $L_p$-MIXED CENTROID BODY

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Lutwak and Zhang proposed the notion of $L_p$-centroid body. Further, Ma gave the definition of $L_p$-mixed centroid body, and obtained affirmative form for the Shephard type problems of $L_p$-mixed centroid body. In this article, we first give another affirmative form of the Shephard type problems for $L_p$-mixed centroid body, meanwhile, obtain its negative form. Next, we also give an extension of the generalized Funk’s section theorem for $L_p$-mixed centroid body. Finally, we establish two monotonicity inequalities of $L_p$-mixed centroid body.

Key words : $L_p$-mixed centroid body; Shephard type problems; monotonicity.

1. INTRODUCTION AND MAIN RESULTS

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors, the set of convex bodies whose centroid lie at the origin and the set of origin-symmetric convex bodies in $\mathbb{R}^n$, we write $\mathcal{K}_o^n$, $\mathcal{K}_c^n$ and $\mathcal{K}_{os}^n$, respectively. $\mathcal{S}_o^n$ and $\mathcal{S}_{os}^n$ respectively denote the set of star bodies (about the origin) and the set of origin-symmetric star bodies in $\mathbb{R}^n$. Let $\mathcal{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$ and $V(K)$ denote the $n$-dimensional volume of a body $K$. For the standard unit ball $B$ in $\mathbb{R}^n$, its volume is written by $\omega_n = V(B)$. 
In 1997, Lutwak and Zhang in [5] introduced the concept of $L_p$-centroid body as follows: For each compact star-shaped about the origin $K \subset \mathbb{R}^n$, real number $p \geq 1$, the $L_p$-centroid body, $\Gamma_p K$, of $K$ is the origin-symmetric convex body whose support function is defined by

$$h_{\Gamma_p K}^p(u) = \frac{1}{c_{n,p} V(K)} \int_K |u \cdot x|^p dx,$$

(1.1)

for any $u \in S^{n-1}$. Here the integration is with respect to Lebesgue and $c_{n,p} = \omega_{n+p}/\omega_2 \omega_n \omega_{p-1}$.

Using polar coordinates in (1.1), we easily get

$$h_{\Gamma_p K}^p(u) = \frac{1}{(n+p)c_{n,p} V(K)} \int_{S^{n-1}} |u \cdot v|^p \rho_K(v)^{n+p} dS(v),$$

(1.2)

for any $u \in S^{n-1}$.

Lutwak, Yang and Zhang have made a series of studies for $L_p$-centroid body, and many scholars were attracted. The more results about $L_p$-centroid body may be seen from these article (see [2, 4-7, 11, 14, 16, 17, 20]).

Based on the definition of $L_p$-centroid body, Ma in [13] gave the definition of $L_p$-mixed centroid body as follows: For $K \in S^n_o$, $p \geq 1$ and any real number $i$, the $L_p$-mixed centroid body of $K$, $\Gamma_{p,i} K$, is defined by

$$h_{\Gamma_{p,i} K}^p(u) = \frac{1}{(n+p)c_{n,p} \tilde{W}_i(K)} \int_{S^{n-1}} |u \cdot v|^p \rho_K(v)^{n+p-i} dS(v).$$

(1.3)

From definitions (1.2) and (1.3), we easily get for $i = 0$

$$\Gamma_p K = \Gamma_{p,0} K.$$

With regard to the study of $L_p$-mixed centroid body, Ma in [13] have got some results. Particularly, the affirmative form for the Shephard type problems of $L_p$-mixed centroid body was given as follows:

**Theorem 1.A** — For $K \in S^n_o$, $L \in \Pi_{p,j}^*$, $j = 0, 1, \ldots, n-1$, $1 \leq p < \infty$, $i \in \mathbb{R}$, $i \neq n$, $i \neq n + p$, if $\Gamma_{p,i} K \subseteq \Gamma_{p,i} L$, then for $i < n$

$$\tilde{W}_i(K) \leq \tilde{W}_i(L);$$

(1.4)

for $n < i < n + p$

$$\tilde{W}_i(K) \geq \tilde{W}_i(L).$$

(1.5)
If $L \in S^0_p$, $K \in \Pi^*_p$, $j = 0, 1, \ldots, n - 1$, $1 \leq p < \infty$, and $\Gamma_{p,i}K \subseteq \Gamma_{p,i}L$, then for $i > n + p$, (1.5) also holds. With equality in every inequality if and only if $K = L$. Here $\Pi^*_p$ denotes the set of polar of $L_p$-mixed projection body.

In this article, we further research $L_p$-mixed centroid body. Firstly, we give another form of Theorem 1.A, and also call it for the affirmative form of the Shephard type problems of $L_p$-mixed centroid body (see Theorem 1.1). What we need to note is the main difference of this theorem with Theorem 1.A lie in being used an important result (see Lemma 4.1), whereas Theorem 1.A was obtained by the key fact which actually is the Lemma 5.2 of this paper.

**Theorem 1.1** — For $K \in S^0_o$, $L \in Z^*_p$, and $p \geq 1$, if $\Gamma_{p,i}K \subseteq \Gamma_{p,i}L$, then for $i < n$

\[
\widetilde{W}_i(K) \leq \widetilde{W}_i(L),
\]

for $i > n + p$

\[
\widetilde{W}_i(K) \geq \widetilde{W}_i(L).
\]

For $L \in S^0_o$, $K \in Z^*_p$, and $p \geq 1$, if $\Gamma_{p,i}K \subseteq \Gamma_{p,i}L$, then for $n < i < n + p$, the inequality (1.7) holds as well. With equality in every inequality if and only if $K = L$. Here $Z^*_p$ denotes the set of polar of $L_p$-mixed centroid body.

Moreover, the following theorem provides the negative form of the Shephard type problems for $L_p$-mixed centroid body.

**Theorem 1.2** — For $L \in S^0_o$, and $p \geq 1$, if $L$ is not origin-symmetric star body, then there exists $K \in S^0_o$, such that

\[
\Gamma_{p,i}K \subseteq \Gamma_{p,i}L,
\]

but for $i < n$ or $i > n + p$

\[
\widetilde{W}_i(K) > \widetilde{W}_i(L);
\]

for $n < i < n + p$

\[
\widetilde{W}_i(K) < \widetilde{W}_i(L).
\]

An extended Funk’s section theorem (see [12]) was further generalized by Yuan and Chueng (see [21]). However, recalling $I_pK = c\Gamma^*_pK$ (For $K \in S^0_o$ and $p < 1$, $I_pK$ denotes the $L_p$-intersection body of $K$ (see [21]), and $c$ is a scale constant (see [3])), we easily see that the following theorem is an extension of the generalized Funk’s section theorem.
Theorem 1.3 — For $K \in S^n_0$, $L \in S^n_{0+}$, $p \geq 1$, if $\Gamma_{p,i}K = \Gamma_{p,i}L$, then for $i < n$

$$\widetilde{W}_i(K) \leq \widetilde{W}_i(L);$$

(1.8)

for $i > n + p$

$$\widetilde{W}_i(K) \geq \widetilde{W}_i(L).$$

(1.9)

For $K \in S^n_{0+}$, $L \in S^n_0$, $p \geq 1$, if $\Gamma_{p,i}K = \Gamma_{p,i}L$, then for $n < i < n + p$, the inequality (1.9) also holds. With equality in every inequality if and only if $K = L$.

Finally, associated with $L_p$-mixed centroid body, we study the following monotonicity inequalities.

Theorem 1.4 — For $K, L \in S^n_0$ and $p \geq 1$, if $K \subseteq L$, then for $i < n$ or $n < i < n + p$

$$\frac{\widetilde{W}_i(\Gamma_{p,i}K)^{\frac{p}{n-i}}}{\widetilde{W}_i(K)} \geq \frac{\widetilde{W}_i(\Gamma_{p,i}L)^{\frac{p}{n-i}}}{\widetilde{W}_i(L)};$$

(1.10)

for $i > n + p$

$$\frac{\widetilde{W}_i(\Gamma_{p,i}K)^{\frac{p}{n-i}}}{\widetilde{W}_i(K)} \leq \frac{\widetilde{W}_i(\Gamma_{p,i}L)^{\frac{p}{n-i}}}{\widetilde{W}_i(L)},$$

(1.11)

with equality in every inequality if and only if $K = L$.

Theorem 1.5 — For $K, L \in S^n_0$, $p \geq 1$ and $j = 0, 1, \cdots, n - 1$, if $K \subseteq L$, then for $i < n + p$ and $i \neq n$

$$\frac{W_j(\Gamma_{p,i}K)^{-\frac{p}{n-j}}}{W_i(K)} \geq \frac{W_j(\Gamma_{p,i}L)^{-\frac{p}{n-j}}}{W_i(L)};$$

(1.12)

for $i > n + p$

$$\frac{W_j(\Gamma_{p,i}K)^{-\frac{p}{n-j}}}{W_i(K)} \leq \frac{W_j(\Gamma_{p,i}L)^{-\frac{p}{n-j}}}{W_i(L)},$$

(1.13)

with equality in every inequality if and only if $K = L$.

2. Preliminaries

2.1 Support Function, Radial Function and Polar of Convex Bodies

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K, \cdot) : \mathbb{R}^n \to (-\infty, \infty)$, is defined by (see [1, 15])

$$h(K, x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$  

(2.1)
where $x \cdot y$ denotes the standard inner product of $x$ and $y$.

If $K$ is a compact star-shaped (about the origin) set in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$, is defined by (see[1, 15])

$$
\rho(K, u) = \max\{\lambda \geq 0 : \lambda \cdot u \in K\}, \ u \in S^{n-1}.
$$

(2.2)

If $\rho_K$ is continuous and positive, then $K$ will be called a star body. Two star bodies $K$, $L$ are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If $K \in \mathcal{K}^n_0$, the polar body , $K^*$, of $K$ is defined by (see[1, 15])

$$
K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in K\}.
$$

(2.3)

From (2.3), we easily have $(K^*)^* = K$ and

$$
h_{K^*} = \frac{1}{\rho_K}, \ \rho_{K^*} = \frac{1}{h_K}.
$$

(2.4)

For $K, L \in \mathcal{K}^n_0$ and $\lambda > 0$, it follows that from (2.4)

$$
K \subseteq \lambda L \iff K^* \supseteq \frac{1}{\lambda} L^*.
$$

(2.5)

2.2 $L_p$-Mixed Quermassintegrals

For $K \in \mathcal{K}^n$, $i = 0, 1, \cdots, n - 1$, the quermassintegrals, $W_i(K)$, of $K$ are defined by (see [1, 15])

$$
W_i(K) = \frac{1}{n} \int_{S^{n-1}} h(K, u)dS_i(K, u),
$$

(2.6)

where $S_i(K, \cdot)$ is a classical positive Borel measure on $S^{n-1}$.

Obviously,

$$
W_0(K) = \frac{1}{n} \int_{S^{n-1}} h(K, u)dS(K, u) = V(K).
$$

(2.7)

For $K, L \in \mathcal{K}^n_0$, $p \geq 1$ and $\varepsilon > 0$, the Firey $L_p$-combination, $K +_p \varepsilon \cdot L \in \mathcal{K}^n_0$, of $K$ and $L$ is defined by (see [8])

$$
h(K +_p \varepsilon \cdot L, \cdot)^p = h(K, \cdot)^p + \varepsilon h(L, \cdot)^p.
$$

(2.8)
Associated with the Firey $L_p$-combination, Lutwak in [8] defined $L_p$-mixed quermassintegrals (also called mixed $p$-quermassintegrals) as follows:

For $K, L \in K^n_0$, and real $p \geq 1$, $L_p$-mixed quermassintegrals $W_{p,i}(K, L)(i = 0, 1, \cdots, n-1)$ are defined as

$$\frac{n-i}{p} W_{p,i}(K, L) = \lim_{\varepsilon \to 0^+} \frac{W_i(K + p \varepsilon \cdot L) - W_i(K)}{\varepsilon}. \quad (2.9)$$

Obviously, for $p = 1$, $W_{1,i}(K, L)$ is just the classical mixed quermassintegrals $W_i(K, L)$ (see [8]). For $i = 0$, $L_p$-mixed quermassintegrals $W_{p,0}(K, L)$ is shown, by (2.7) and (2.9), to be

$$\frac{n}{p} W_{p,0}(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K + p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

It is just $L_p$-mixed volume $V_p(K, L)$ (see [8, 9]), namely,

$$W_{p,0}(K, L) = V_p(K, L).$$

In [8], Lutwak has shown that for $p \geq 1, i = 0, 1, \cdots, n-1$, and each $K \in K^n_0$, there exists a positive Borel measure $S_{p,i}(K, \cdot)$ on $S^{n-1}$, such that $L_p$-mixed quermassintegrals $W_{p,i}(K, L)$ has the following integral representation:

$$W_{p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} h^p(L, u) dS_{p,i}(K, u), \quad (2.10)$$

for all $L \in K^n_0$. It turns out that the measure $S_{p,i}(K, \cdot)(i = 0, 1, \cdots, n-1)$ on $S^{n-1}$ is absolutely continuous with respect to $S_i(K, \cdot)$, and has the Randon-Nikodym derivative

$$\frac{dS_{p,i}(K, \cdot)}{dS_i(K, \cdot)} = h^{1-p}(K, \cdot). \quad (2.11)$$

The case $i = 0, S_{p,0}(K, \cdot)$ is just the $L_p$-surface area measure $S_p(K, \cdot)$ of $K$.

From the definition of $L_p$-mixed quermassintegrals, it follows immediately that, for each $K \in K^n_0$,

$$W_{p,i}(K, K) = W_i(K), \quad (2.12)$$

for all $p \geq 1$.

The Minkowski’s inequality for $L_p$-mixed quermassintegrals is that (see [8])
Theorem 2.1 — For $K, L \in K_n^e$, and $p \geq 1$, $0 \leq i < n$, then
\[
W_{p,i}(K, L)^{n-i} \geq W_i(K)^{n-i-p}W_i(L)^p,
\]
(2.13)

with equality if and only if $K$ and $L$ are dilates.

Recently, Wang in [19] gave the notion of $L_p$-mixed projection body as follows. For each $K \in K^n$, real $p \geq 1$ and $i = 0, 1, \ldots, n - 1$, the $L_p$-mixed projection body, $\Pi_{p,i}K$, of $K$ is an origin-symmetric convex body whose support function is defined by
\[
h_{\Pi_{p,i}K}(u) = \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |u \cdot v|^p dS_{p,i}(K, v),
\]
where $u, v \in S^{n-1}$, the positive Borel measure $S_{p,i}(K, \cdot)$ on $S^{n-1}$ is absolutely continuous with respect to $S_i(K, \cdot)$, and has the Radon-Nikodym derivative.

2.2 $L_p$-Dual Mixed Quermassintegrals

For $K, L \in S_o^n$, $p \geq 1$ and $\varepsilon > 0$, the $L_p$-harmonic radial combination, $K + \varepsilon \circ L \in S_o^n$, of $K$ and $L$ is defined by (see [9])
\[
\rho(K + \varepsilon \circ L, \cdot)^{-p} = \rho(K, \cdot)^{-p} + \varepsilon \rho(L, \cdot)^{-p}.
\]
(2.14)

For $K \in S_o^n$ and any real $i$, the dual quermassintegrals, $\widetilde{W}_i(K)$, of $K$ are defined by (see [10])
\[
\widetilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i}dS(u).
\]
(2.15)

Obviously,
\[
\widetilde{W}_0(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n}dS(u) = V(K).
\]
(2.16)

Associated with the $L_p$-harmonic radial combination of star bodies, Wang and Leng (see [18]) introduced the notion of $L_p$-dual mixed quermassintegrals as follows: For $K, L \in S_o^n$, $p \geq 1$, $\varepsilon > 0$, real $i \neq n$, the $L_p$-dual mixed quermassintegrals, $\widetilde{W}_{-p,i}(K, L)$, of the $K$ and $L$ is defined by (see [18])
\[
\frac{n-i}{-p} \widetilde{W}_{-p,i}(K, L) = \lim_{\varepsilon \to 0^+} \frac{\widetilde{W}_i(K + \varepsilon \circ L) - \widetilde{W}_i(K)}{\varepsilon}.
\]
(2.17)
The definition above and Hospital’s rule give the following integral representation of the $L_p$-dual mixed quermassintegrals (see [18]):

$$\widetilde{W}_{-p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p-i}(u)\rho_L^{-p}(u)dS(u),$$

(2.18)

where the integration is with respect to spherical Lebesgue measure $S$ on $S^{n-1}$. From formula (2.18) and definition (2.15), we get

$$\widetilde{W}_{-p,i}(K, K) = \widetilde{W}_i(K).$$

(2.19)

The Minkowski’s inequality for $L_p$-dual mixed quermassintegrals is that (see [18])

**Theorem 2.3** — Let $K, L \in S^n_0$, $p \geq 1$, and real $i \neq n$, then for $i < n$ or $i > n + p$,

$$\widetilde{W}_{-p,i}(K, L) \geq \widetilde{W}_i(K)^{n+p-i\over n-i} \widetilde{W}_i(L)^{-p\over n-i};$$

(2.20)

for $n < i < n + p$, inequality (2.20) is reverse. Equality holds in every inequality if and only if $K$ and $L$ are dilates. For $i = n + p$, (2.20) is identic.

3. $L_p$-Mixed Harmonic Blaschke Combination

In order to obtain the negative form of the Shephard type problems for $L_p$-mixed centrod body, we need to introduce the concept of $L_p$-mixed harmonic blashcke combination.

For $K, L \in S^n_0$, $p \geq 1$, $i \neq n, n + p$, and $\lambda, \mu \geq 0$ (not both zero), the $L_p$-mixed harmonic Blaschke combination, $\lambda \ast K^{\hat{+},p,i} \mu \ast L \in S^n_0$, of $K$ and $L$ is defined by

$$\frac{\rho(\lambda \ast K^{\hat{+},p,i} \mu \ast L, \cdot)^{n+p-i}}{\widetilde{W}_i(\lambda \ast K^{\hat{+},p,i} \mu \ast L)} = \lambda \frac{\rho(K, \cdot)^{n+p-i}}{\widetilde{W}_i(K)} + \mu \frac{\rho(L, \cdot)^{n+p-i}}{\widetilde{W}_i(L)}.$$

(3.1)

Taking $\lambda = \mu = \frac{1}{2}$, $L = -K$ in $\lambda \ast K^{\hat{+},p,i} \mu \ast L$, then $L_p$-mixed harmonic Blaschke body is defined by

$$\hat{\nabla}_{p,i}K = \frac{1}{2} \ast K^{\hat{+},p,i} \frac{1}{2} \ast (-K).$$

(3.2)

Obviously, $L_p$-mixed harmonic Blaschke body $\hat{\nabla}_{p,i}K$ is origin-symmetric.
**Theorem 3.1** — If $K, L \in S^n_\sigma$, $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero), then for $i < n$ or $i > n + p$

$$
\hat{\tilde{W}}_i(\lambda * K + p_i \mu * L)^{\frac{p}{n-i}} \geq \lambda \hat{\tilde{W}}_i(K)^{\frac{p}{n-i}} + \mu \hat{\tilde{W}}_i(L)^{\frac{p}{n-i}}; \quad (3.3)
$$

for $n < i < n + p$

$$
\hat{\tilde{W}}_i(\lambda * K + p_i \mu * L)^{\frac{p}{n-i}} \leq \lambda \hat{\tilde{W}}_i(K)^{\frac{p}{n-i}} + \mu \hat{\tilde{W}}_i(L)^{\frac{p}{n-i}}, \quad (3.4)
$$

with equality in every inequality if and only if $K$ and $L$ are dilates.

**Proof:** From (2.18), (2.20) and (3.1), we have for any $Q \in S^n_\sigma$ and $i < n$ or $i > n + p$,

$$
\frac{\hat{\tilde{W}}_{-p,i}(\lambda * K + p_i \mu * L, Q)}{\hat{\tilde{W}}_i(\lambda * K + p_i \mu * L)} = \lambda \frac{\hat{\tilde{W}}_{-p,i}(K, Q)}{\hat{\tilde{W}}_i(K)} + \mu \frac{\hat{\tilde{W}}_{-p,i}(L, Q)}{\hat{\tilde{W}}_i(L)} \geq [\lambda \hat{\tilde{W}}_i(K)^{\frac{p}{n-i}} + \mu \hat{\tilde{W}}_i(L)^{\frac{p}{n-i}}] \hat{\tilde{W}}_i(Q)^{-\frac{p}{n-i}}. \quad (3.5)
$$

Taking $Q = \lambda * K + p_i \mu * L$ in (3.5), and from (2.19), it follows that

$$
\hat{\tilde{W}}_i(\lambda * K + p_i \mu * L)^{\frac{p}{n-i}} \geq \lambda \hat{\tilde{W}}_i(K)^{\frac{p}{n-i}} + \mu \hat{\tilde{W}}_i(L)^{\frac{p}{n-i}}.
$$

For $n < i < n + p$, similar to the above way, we easily get

$$
\hat{\tilde{W}}_i(\lambda * K + p_i \mu * L)^{\frac{p}{n-i}} \leq \lambda \hat{\tilde{W}}_i(K)^{\frac{p}{n-i}} + \mu \hat{\tilde{W}}_i(L)^{\frac{p}{n-i}}.
$$

Associated with the equality condition of (2.20), we see that equality respectively holds in (3.3) and (3.4) if and only if $K$ and $L$ are dilates. \(\Box\)

Taking $\lambda = \mu = \frac{1}{2}$, $L = -K$ in (3.3) and (3.4) respectively, we easily get the following result.

**Corollary 3.1** — If $K \in S^n_\sigma$, $p \geq 1$, then for $i < n$ or $i > n + p$

$$
\hat{\tilde{W}}_i(\nabla_{p,i}K) \geq \hat{\tilde{W}}_i(K), \quad (3.6)
$$

for $n < i < n + p$

$$
\hat{\tilde{W}}_i(\nabla_{p,i}K) \leq \hat{\tilde{W}}_i(K), \quad (3.7)
$$

with equality in every inequality if and only if $K$ is an origin-symmetric body.
4. THE SHEPARD TYPE PROBLEMS

In this section, we first complete the proofs of Theorems 1.1-1.2. Next, we give an equivalence theorem about $L_p^*$-mixed centroid body (i.e., Theorem 4.1), and as the application of this theorem, we give the proof of Theorem 1.3.

Lemma 4.1 — If $p \geq 1$, $i \in \mathbb{R}$, then for $K, L \in S^*_0$

\[
\frac{\widetilde{W}_{-p,i}(K, \Gamma_{p,i}^* L)}{\widetilde{W}_i(K)} = \frac{\widetilde{W}_{-p,i}(L, \Gamma_{p,i}^* K)}{\widetilde{W}_i(L)}. \tag{4.1}
\]

Proof: For $p \geq 1$, and any real number $i$, then we get from (1.3), (2.4), (2.18) and Fubini’s theorem

\[
\frac{\widetilde{W}_{-p,i}(K, \Gamma_{p,i}^* L)}{\widetilde{W}_i(K)} = \frac{1}{n\widetilde{W}_i(K)} \int_{S^{n-1}} \rho_{K}^{n+p-i}(u)\rho_{\Gamma_{p,i}^* L}^p(u)dS(u)
\]

\[
= \frac{1}{n\widetilde{W}_i(K)} \int_{S^{n-1}} \rho_{K}^{n+p-i}(u)h_{\Gamma_{p,i}^* L}(u)dS(u)
\]

\[
= \frac{1}{n\widetilde{W}_i(K)} \int_{S^{n-1}} \rho_{K}^{n+p-i}(u)\left[ \frac{1}{(n+p)c_{n,p}\widetilde{W}_i(L)} \int_{S^{n-1}} |u \cdot v|^p \rho_{\Gamma_{p,i}^* L}(v)dS(v) \right]dS(u)
\]

\[
= \frac{1}{n\widetilde{W}_i(L)} \int_{S^{n-1}} \rho_{L}^{n+p-i}(v)\left[ \frac{1}{(n+p)c_{n,p}\widetilde{W}_i(K)} \int_{S^{n-1}} |u \cdot v|^p \rho_{\Gamma_{p,i}^* K}(u)dS(u) \right]dS(v)
\]

\[
= \frac{1}{n\widetilde{W}_i(L)} \int_{S^{n-1}} \rho_{L}^{n+p-i}(v)\rho_{\Gamma_{p,i}^* K}(v)dS(v) = \frac{\widetilde{W}_{-p,i}(L, \Gamma_{p,i}^* K)}{\widetilde{W}_i(L)}. \tag{4.2}
\]

Proof of Theorem 1.1: For $p \geq 1$ and $M \in S^*_0$, it follows that from Lemma 4.1

\[
\frac{\widetilde{W}_{-p,i}(K, \Gamma_{p,i}^* M)}{\widetilde{W}_i(K)} = \frac{\widetilde{W}_{-p,i}(M, \Gamma_{p,i}^* K)}{\widetilde{W}_i(M)}. \tag{4.3}
\]

Since $\Gamma_{p,i} K \subseteq \Gamma_{p,i} L$, then $\Gamma_{p,i}^* K \supseteq \Gamma_{p,i}^* L$, hence we have for all $u \in S^{n-1}$

\[
\rho_{\Gamma_{p,i}^* K}(u) \leq \rho_{\Gamma_{p,i}^* L}(u). \tag{4.4}
\]
From (4.2), (4.3), and (4.4), we easily get
\[
\frac{\tilde{W}_{-p,i}(K, \Gamma_{p,i}^* M)}{\tilde{W}_i(K)} \leq \frac{\tilde{W}_{-p,i}(L, \Gamma_{p,i}^* M)}{\tilde{W}_i(L)}. \tag{4.5}
\]

For \(i < n\), since \(L \in Z_{p,i}^*\), and taking \(\Gamma_{p,i}^* M\) for \(L\) in (4.5), then we can get from (2.19) and (2.20)
\[
\tilde{W}_i(K) \geq \tilde{W}_{-p,i}(K, L) \geq \tilde{W}_i(K)^{\frac{n+p-i}{n-i}} \tilde{W}_i(L)^{-\frac{p}{n-i}}, \tag{4.6}
\]
i.e.,
\[
\tilde{W}_i(K) \leq \tilde{W}_i(L).
\]

For \(i > n + p\), then \(-(p/n - i) > 0\), thus from (4.6), we easily get
\[
\tilde{W}_i(K) \geq \tilde{W}_i(L).
\]

For \(n < i < n + p\), likewise \(-(p/n - i) > 0\), if \(K \in Z_{p,i}^*, L \in S_{\circ}^n\), and taking \(\Gamma_{p,i}^* M\) for \(K\) in (4.5), thus it follows that from (2.19) and the reverse form of (2.20)
\[
\tilde{W}_i(L) \leq \tilde{W}_{-p,i}(L, K) \leq \tilde{W}_i(L)^{\frac{n+p-i}{n-i}} \tilde{W}_i(K)^{-\frac{p}{n-i}}, \tag{4.7}
\]
i.e.,
\[
\tilde{W}_i(K) \geq \tilde{W}_i(L).
\]

According to the equality condition of (2.20), we see that equality holds in (1.6) if and only if \(K\) and \(L\) are dilates, thus taking \(K = cL(c > 0)\), and from \(\tilde{W}_i(K) = \tilde{W}_i(L)\), easily get \(c = 1\), that is \(K = L\). In turn, when \(K = L\), then \(\tilde{W}_i(K) = \tilde{W}_i(L)\) is apparent. Likewise, equality holds in (1.7) if and only if \(K = L\). Therefore, equality respectively holds in (1.6) and (1.7) if and only if \(K = L\).

\[
\text{Lemma 4.2}^{[13]} \quad \text{For } K \in S_{\circ}^n, p \geq 1, i \in \mathbb{R}, \text{ then}
\]
\[
\Gamma_{p,i}(\lambda K) = \lambda \Gamma_{p,i} K. \tag{4.8}
\]

\[
\text{Lemma 4.3} \quad \text{For } K \in S_{\circ}^n, p \geq 1, i \in \mathbb{R}, \text{ then}
\]
\[
\Gamma_{p,i}(\tilde{\nabla}_{p,i} K) = \Gamma_{p,i} K. \tag{4.9}
\]

\textsc{Proof}: From definitions (1.3) and (3.2), we get for \(i \in \mathbb{R}\)
\[
h^p(\Gamma_{p,i}(\tilde{\nabla}_{p,i} K), u) = \frac{1}{(n+p)c_{n,p}} \int_{S_{\circ}^{n-1}} |u \cdot v|^p \frac{\rho(\tilde{\nabla}_{p,i} K, v)^{n+p-i}}{\tilde{W}_i(\tilde{\nabla}_{p,i} K)} dS(v)
\]
\[
\begin{align*}
&= \frac{1}{(n+p)\alpha_{n,p}} \int_{S^{n-1}_+} [u \cdot v]^p \left[ \frac{1}{2} \frac{\rho(K,v)^{n+p-i}}{\tilde{W}_i(K)} + \frac{1}{2} \frac{\rho(-K,v)^{n+p-i}}{\tilde{W}_i(-K)} \right] dS(v) \\
&= \frac{1}{2} h_i^p(\Gamma_{p,i}K, u) + \frac{1}{2} h_i^p(\Gamma_{p,i}(-K), u). \quad (4.10)
\end{align*}
\]

From (1.3), we easily know \( \Gamma_{p,i}(-K) = \Gamma_{p,i}K \), so combining with (4.10), it follows that for any \( u \in S^{n-1} \)
\[
h_i^p(\Gamma_{p,i}(\tilde{\nabla}_{p,i} K), u) = h_i^p(\Gamma_{p,i}K, u),
\]
i.e.
\[
\Gamma_{p,i}(\tilde{\nabla}_{p,i} K) = \Gamma_{p,i}K. \quad \square
\]

**Proof of Theorem 1.2:** Since \( L \) is not an origin-symmetric, so from Corollary 3.1, we know for \( i < n \) or \( i > n + p \)
\[
\tilde{W}_i(\tilde{\nabla}_{p,i} L) > \tilde{W}_i(L);
\]
for \( n < i < n + p \)
\[
\tilde{W}_i(\tilde{\nabla}_{p,i} L) < \tilde{W}_i(L).
\]
Choose \( \varepsilon > 0 \), such that for \( i < n \) or \( i > n + p \)
\[
\tilde{W}_i((1 - \varepsilon) \tilde{\nabla}_{p,i} L) > \tilde{W}_i(L);
\]
for \( n < i < n + p \)
\[
\tilde{W}_i((1 - \varepsilon) \tilde{\nabla}_{p,i} L) < \tilde{W}_i(L).
\]
Therefore, let \( K = (1 - \varepsilon) \tilde{\nabla}_{p,i} L \), then for \( i < n \) or \( i > n + p \)
\[
\tilde{W}_i(K) > \tilde{W}_i(L);
\]
for \( n < i < n + p \)
\[
\tilde{W}_i(K) < \tilde{W}_i(L).
\]

But from Lemma 4.2 and Lemma 4.3, we can get
\[
\Gamma_{p,i}K = \Gamma_{p,i}((1 - \varepsilon) \tilde{\nabla}_{p,i} L) = (1 - \varepsilon) \Gamma_{p,i}(\tilde{\nabla}_{p,i} L) = (1 - \varepsilon) \Gamma_{p,i}L \subset \Gamma_{p,i}L. \quad \square
\]

**Theorem 4.1** — For \( K, L \in S^n_0 \), \( p \geq 1 \), and \( i \in \mathbb{R} \), then \( \Gamma_{p,i}K = \Gamma_{p,i}L \) if and only if for any \( Q \in S^n_0 \),
\[
\frac{\tilde{W}_{-p,i}(K, Q)}{\tilde{W}_i(K)} = \frac{\tilde{W}_{-p,i}(L, Q)}{\tilde{W}_i(L)}. \quad (4.11)
\]
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PROOF: From (2.4) and (2.18), we get

$$\frac{\widetilde{W}_{-p,i}(K, Q)}{\widetilde{W}_{i}(K)} = \frac{1}{nW_{i}(K)} \int_{S^{n-1}} \rho_K(v)^{n+p-i} q(v)^{-p} dS(v)$$

$$= \frac{1}{nW_{i}(K)} \int_{S^{n-1}} \rho_K(v)^{n+p-i} h_{Q^*}^p(v) dS(v). \quad (4.12)$$

Since $Q \in S_{os}^n$, so taking $Q^* = [-u, u]$ for any $u \in S^{n-1}$, then we know for every $v \in S^{n-1}$,

$$h(Q^*, v) = |u \cdot v|.$$ 

From (1.3) and (4.12), we have

$$h_{p,i,K}^p = \frac{1}{(n+p)c_{n,p} W_{i}(K)} \int_{S^{n-1}} |u \cdot v|^p \rho_K(v)^{n+p-i} dS(v)$$

$$= \frac{n\widetilde{W}_{-p,i}(K, [-u, u]^*)}{(n+p)c_{n,p} W_{i}(K)}. \quad (4.13)$$

Therefore, if $K, L \in S^n$ and any $Q \in S_{os}^n$,

$$\frac{\widetilde{W}_{-p,i}(K, Q)}{\widetilde{W}_{i}(K)} = \frac{\widetilde{W}_{-p,i}(L, Q)}{\widetilde{W}_{i}(L)},$$

then we have

$$\Gamma_{p,i} K = \Gamma_{p,i} L.$$ 

In turn, according to (4.9), we may assume that $K, L \in S_{os}^n$, because we can replace $K$ and $L$ by $\tilde{\nabla}_{p,i} K$ and $\tilde{\nabla}_{p,i} L$, respectively. Thus from (1.3), we know for $i \in \mathbb{R}$

$$h_{p,i,K}^p = \frac{1}{(n+p)c_{n,p} W_{i}(K)} \int_{S^{n-1}} |u \cdot v|^p \rho_K(v)^{n+p-i} dS(v). \quad (4.14)$$

Let

$$f_i(v) = \frac{1}{W_{i}(K)} \rho_K(v)^{n+p-i},$$

since $K \in S_{os}^n$, thus $\rho_K(v) = \rho_K(-v)$ for any $v \in S^{n-1}$, this gives $f_i(v)$ is a finite even Borel measure on $S^{n-1}$. From (4.14), we have

$$h_{p,i,K}^p = \frac{1}{(n+p)c_{n,p}} \int_{S^{n-1}} |u \cdot v|^p f_i(v) dS(v), \quad (4.15)$$

for any $u \in S^{n-1}$. 

Similarly, if \( L \in S^n_{\text{o}_n} \), then for any \( u \in S^{n-1} \),
\[
h^p_{\Gamma p, i L}(u) = \frac{1}{(n+p)c_{n,p}} \int_{S^{n-1}} |u \cdot v|^p g_i(v) dS(v), \tag{4.16}
\]
where
\[
g_i(v) = \frac{1}{W_i(L)} \rho_L(v)^{n+p-i}
\]
is also a finite even Borel measure on \( S^{n-1} \).

Therefore, if \( \Gamma_{p,i} K = \Gamma_{p,i} L \), then by (4.15) and (4.16), we obtain
\[
\frac{1}{(n+p)c_{n,p}} \int_{S^{n-1}} |u \cdot v|^p [f_i(v) - g_i(v)] dS(v) = 0. \tag{4.17}
\]

Let \( \mu_i(v) = f_i(v) - g_i(v) \), then (4.17) may be written as
\[
\frac{1}{(n+p)c_{n,p}} \int_{S^{n-1}} |u \cdot v|^p \mu_i(v) dS(v) = 0. \tag{4.18}
\]

Notice that \( \mu_i(v) \) is a continuous finite even Borel measure on \( S^{n-1} \), therefore together with (4.18), we obtain \( \mu_i(v) = 0 \), i.e., \( f_i(v) - g_i(v) = 0 \). This show that for any \( v \in S^{n-1} \),
\[
\frac{1}{W_i(K)} \rho_K(v)^{n+p-i} = \frac{1}{W_i(L)} \rho_L(v)^{n+p-i}. \tag{4.19}
\]

But we know for any \( Q \in S^n_{\text{o}_n} \),
\[
\frac{\overline{W}_{-p,i}(K,Q)}{\overline{W}_i(K)} = \frac{1}{nW_i(K)} \int_{S^{n-1}} \rho_K(v)^{n+p-i} \rho_Q^{-p}(v) dS(v),
\]
\[
\frac{\overline{W}_{-p,i}(L,Q)}{\overline{W}_i(L)} = \frac{1}{nW_i(L)} \int_{S^{n-1}} \rho_L(v)^{n+p-i} \rho_Q^{-p}(v) dS(v).
\]

Hence, associated with (4.19), we have that for any \( Q \in S^n_{\text{o}_n} \),
\[
\frac{\overline{W}_{-p,i}(K,Q)}{\overline{W}_i(K)} = \frac{\overline{W}_{-p,i}(L,Q)}{\overline{W}_i(L)}. \tag{4.20}
\]

**Proof of Theorem 1.3:** Since \( \Gamma_{p,i} K = \Gamma_{p,i} L \) and \( Q \in S^n_{\text{o}_n} \), then it follows that from Theorem 4.1

\[
\frac{\overline{W}_{-p,i}(K,Q)}{\overline{W}_i(K)} = \frac{\overline{W}_{-p,i}(L,Q)}{\overline{W}_i(L)}, \tag{4.20}
\]
Due to \( L \in \mathcal{S}_o^n \), thus taking \( Q = L \) in (4.20), together with (2.19) and (2.20), easily show that for \( i < n \)

\[
\widetilde{W}_i(K) = \widetilde{W}_{-p,i}(K, L) \geq \widetilde{W}_i(K)^{\frac{n+p-i}{n-i}} \widetilde{W}_i(L)^{-\frac{i}{n-i}}. \tag{4.21}
\]

That is

\[
\widetilde{W}_i(K) \leq \widetilde{W}_i(L).
\]

For \( i > n + p \), then \(-(p/n - i) > 0\), thus from (4.21), we easily obtain

\[
\widetilde{W}_i(K) \geq \widetilde{W}_i(L).
\]

For \( n < i < n + p \), likewise \(-(p/n - i) > 0\), if \( K \in \mathcal{S}_o^n \), \( L \in \mathcal{S}_o^n \), then let \( Q = K \) in (4.20), thus it follows that from (2.19) and the reverse form of (2.20)

\[
\widetilde{W}_i(L) = \widetilde{W}_{-p,i}(L, K) \leq \widetilde{W}_i(L)^{\frac{n+p-i}{n-i}} \widetilde{W}_i(K)^{-\frac{i}{n-i}}. \tag{4.22}
\]

That is

\[
\widetilde{W}_i(K) \geq \widetilde{W}_i(L).
\]

For the equality conditions above equalities, similar to the proof of Theorem 1.1, equality respectively holds in (1.8) and (1.9) if and only if \( K = L \). \( \square \)

5. The Proofs of Monotonicity

In this section, we prove Theorems 1.4-1.5. Here the proof of Theorem 1.4 require a Lemma as follows.

**Lemma 5.1** \[^{[13]}\] — For \( K, L \in \mathcal{S}_o^n \), \( p \geq 1 \), \( i \in \mathbb{R} \) and \( i \neq n, n + p \), then

\[
\widetilde{W}_{-p,i}(K, Q) = \widetilde{W}_{-p,i}(L, Q)
\]

for any \( Q \in \mathcal{S}_o^n \) if and only if \( K = L \).

**Proof of Theorem 1.4** : Since \( p \geq 1 \), and \( K \subseteq L \), then we have for any \( M \in \mathcal{S}_o^n \)

\[
\widetilde{W}_{-p,i}(K, \Gamma_{p,i}^* M) \leq \widetilde{W}_{-p,i}(L, \Gamma_{p,i}^* M). \tag{5.1}
\]

Associated with Lemma 4.1 and (5.1), we get

\[
\frac{\widetilde{W}_i(K)\widetilde{W}_{-p,i}(M, \Gamma_{p,i}^* K)}{\widetilde{W}_i(M)} \leq \frac{\widetilde{W}_i(L)\widetilde{W}_{-p,i}(M, \Gamma_{p,i}^* L)}{\widetilde{W}_i(M)}. \tag{5.2}
\]
Taking $M$ for $\Gamma^*_{p,i} L$ in (5.2), and from (2.19) and (2.20), we obtain for $i < n$

$$\widetilde{W}_i(L) \widetilde{W}_i(\Gamma^*_{p,i} L) \geq \widetilde{W}_i(K) \widetilde{W}_{-p,i}(\Gamma^*_{p,i} L, \Gamma^*_{p,i} K)$$

$$\geq \frac{\widetilde{W}_i(K)^{\frac{p}{n-i}}}{\widetilde{W}_i(L)} \frac{\widetilde{W}_i(\Gamma^*_{p,i} L)^{\frac{p}{n-i}}}{\widetilde{W}_i(K)^{\frac{p}{n-i}}}.$$  (5.3)

i.e.,

$$\frac{\widetilde{W}_i(\Gamma^*_{p,i} K)^{\frac{p}{n-i}}}{\widetilde{W}_i(K)} \geq \frac{\widetilde{W}_i(\Gamma^*_{p,i} L)^{\frac{p}{n-i}}}{\widetilde{W}_i(L)}.$$

For $n < i < n + p$, and taking $M$ for $\Gamma^*_{p,i} K$ in (5.2), and from (2.19) and the reverse form of (2.20), we obtain

$$\widetilde{W}_i(K) \widetilde{W}_i(\Gamma^*_{p,i} K) \leq \widetilde{W}_i(L) \widetilde{W}_{-p,i}(\Gamma^*_{p,i} K, \Gamma^*_{p,i} L)$$

$$\leq \frac{\widetilde{W}_i(L)\widetilde{W}_i(\Gamma^*_{p,i} K)^{\frac{p}{n-i}}}{\widetilde{W}_i(L)} \frac{\widetilde{W}_i(\Gamma^*_{p,i} L)^{\frac{p}{n-i}}}{\widetilde{W}_i(L)}.$$  (5.4)

i.e.,

$$\frac{\widetilde{W}_i(\Gamma^*_{p,i} K)^{\frac{p}{n-i}}}{\widetilde{W}_i(K)} \leq \frac{\widetilde{W}_i(\Gamma^*_{p,i} L)^{\frac{p}{n-i}}}{\widetilde{W}_i(L)}.$$

For $i > n + p$, then $n + p - i < 0$, thus we get

$$\widetilde{W}_{-p,i}(K, \Gamma^*_{p,i} M) \geq \widetilde{W}_{-p,i}(L, \Gamma^*_{p,i} M).$$  (5.5)

Associated with Lemma 4.1 and (5.5), we get

$$\frac{\widetilde{W}_i(K)\widetilde{W}_{-p,i}(M, \Gamma^*_{p,i} K)}{\widetilde{W}_i(M)} \geq \frac{\widetilde{W}_i(L)\widetilde{W}_{-p,i}(M, \Gamma^*_{p,i} L)}{\widetilde{W}_i(M)}.$$  (5.6)

Taking $M$ for $\Gamma^*_{p,i} K$ in (5.6), similar to the above way, we easily obtain

$$\frac{\widetilde{W}_i(K)^{\frac{p}{n-i}}}{\widetilde{W}_i(K)} \leq \frac{\widetilde{W}_i(\Gamma^*_{p,i} L)^{\frac{p}{n-i}}}{\widetilde{W}_i(L)}.$$  

From Lemma 5.1 and the equality condition of (2.20), we know equality respectively holds in (1.10) and (1.11) if and only if $K = L$. 

\[ \square \]

**Lemma 5.2** — If $K \in K^n_p$, $p \geq 1$, $i \neq n$, $i \neq n + p$, and $j = 0, 1, \cdots, n - 1$, then for all $Q \in S^n_o$

$$\widetilde{W}_{-p,i}(Q, \Pi^*_{p,j} K) = \frac{\widetilde{W}_i(Q)}{\omega_n} W_{p,j}(K, \Gamma_{p,i} Q).$$  (5.7)
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PROOF OF THEOREM 1.5: For $p \geq 1$, and $j = 0, 1, \cdots, n - 1$, if $K \subseteq L$ and for any $M \in K^*\_o$, then we have from $i < n + p$ and $i \neq n$

$$
\bar{W}_{-p,i}(K, \Pi^*_p, M) \leq \bar{W}_{-p,i}(L, \Pi^*_p, M),
$$

(5.8)

Combining with Lemma 5.2 and (5.8), we obtain

$$
\bar{W}_i(K)W_{p,j}(M, \Gamma_{p,i}K) \leq \bar{W}_i(L)W_{p,j}(M, \Gamma_{p,i}L).
$$

(5.9)

Taking $M$ for $\Gamma_{p,i}L$ in (5.9), together with (2.13), we have

$$
W_j(\Gamma_{p,i}L)\bar{W}_i(L) \geq \bar{W}_i(K)W_{p,j}(\Gamma_{p,i}L, \Gamma_{p,i}K)
$$

$$
\geq \bar{W}_i(K)W_j(\Gamma_{p,i}L)^{\frac{n-p}{n-j}}W_j(\Gamma_{p,i}K)^{\frac{n}{n-j}}.
$$

(5.10)

i.e.,

$$
\frac{W_j(\Gamma_{p,i}K)^{-\frac{p}{n-j}}}{\bar{W}_i(K)} \geq \frac{W_j(\Gamma_{p,i}L)^{-\frac{p}{n-j}}}{\bar{W}_i(L)}.
$$

For $i > n + p$, then $n + p - i < 0$, thus we know

$$
\bar{W}_{-p,i}(K, \Pi^*_p, M) \geq \bar{W}_{-p,i}(L, \Pi^*_p, M),
$$

(5.11)

Similarly, together with Lemma 5.2 and (5.11), we obtain

$$
\bar{W}_i(K)W_{p,j}(M, \Gamma_{p,i}K) \geq \bar{W}_i(L)W_{p,j}(M, \Gamma_{p,i}L).
$$

(5.12)

Taking $M$ for $\Gamma_{p,i}K$ in (5.12), similar to the above way, this yields

$$
\frac{W_j(\Gamma_{p,i}K)^{-\frac{p}{n-j}}}{\bar{W}_i(K)} \leq \frac{W_j(\Gamma_{p,i}L)^{-\frac{p}{n-j}}}{\bar{W}_i(L)}.
$$

According to Lemma 5.1 and the equality condition of (2.13), we see that equality respectively holds in (1.12) and (1.13) if and only if $K = L$.

☐

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