Carriazo [2] defined the notion of bi-slant submanifolds of an almost Hermitian manifold. As a special case of these submanifolds, he introduced anti-slant submanifolds which he, later called as pseudo-slant submanifolds [2]. The purpose of the present paper is to study the pseudo-slant submanifolds of a Sasakian manifold. In this paper we work out integrability conditions of distributions on these submanifolds and also, obtain a few interesting results of this setting.

Key Words: Sasakian Manifold; Slant Submanifold; Pseudo-Slant Submanifold

1. INTRODUCTION

The geometry of slant immersions was initiated by Chen [4] as a natural generalization of both holomorphic and totally real immersions. Many authors have studied slant immersions in almost Hermitian manifolds. Lotta [3] introduced the notion of slant immersions in contact manifolds. Cabrerizo et al. [6] studied and characterized slant submanifolds of $K$-contact and Sasakian manifolds and have given several examples of such immersions. Recently, Carriazo [2] defined and studied bi-slant immersions in almost Hermitian manifolds and simultaneously gave the notion of pseudo-slant submanifold in almost Hermitian manifolds. The purpose of the present paper is to define and study the contact version of pseudo-slant submanifolds. In section 2 we review and collect some necessary results. In section 3 we define pseudo-slant submanifolds of contact manifolds. In particular, we study the pseudo-slant submanifolds in the setting of Sasakian manifolds and work out integrability conditions of distributions involved in the definition of pseudo-slant submanifolds and have also obtained some geometrically significant results of this setting.
2. Preliminaries

Let \((\bar{M}, g)\) be an odd dimensional Riemannian manifold, \(T\bar{M}\) the Lie algebra of vector fields in \(\bar{M}\). Then \(\bar{M}\) is said to be an almost contact metric manifold [5], if there exist on \(\bar{M}\) a tensor field \(\phi\) of type \((1, 1)\) and a global vector field \(\xi\) (known as structure vector field) such that, if \(\eta\) is the dual 1-form of \(\xi\) then

\[
\phi^2 X = -X + \eta(X)\xi, \quad g(X, \xi) = \eta(X)
\]
\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).
\]

From the above relations, it also follows that

\[
g(\phi X, Y) = -g(X, \phi Y)
\]  

for any \(X, Y \in T\bar{M}\). Let \(\Phi\) denote a 2-form in \(\bar{M}\) given by \(\Phi(X, Y) = g(X, \phi Y)\) for all \(X, Y \in T\bar{M}\). The 2-form \(\Phi\) is then called the fundamental 2-form on \(\bar{M}\). The manifold \(\bar{M}\) is said to be a contact metric manifold if \(\Phi = d\eta\).

If \(\xi\) is a Killing vector field with respect to \(g\), the contact metric structure is called \(K\)-contact structure. It is known that a contact metric manifold is \(K\)-contact if and only if \(\nabla_X \xi = -\phi X\), for any \(X \in T\bar{M}\), where \(\nabla\) denotes the Levi-Civita connection on \(\bar{M}\). The almost contact structure of \(\bar{M}\) is said to be normal if \([\phi, \phi] + 2d\eta \otimes \xi = 0\), where \([\phi, \phi]\) is the Nijenhuis tensor of \(\phi\). A Sasakian manifold is a normal contact metric manifold. Every Sasakian manifold is a \(K\)-contact manifold. It is known that an almost contact metric manifold is a Sasakian manifold if and only if

\[
(\nabla_X \phi) Y = g(X, Y)\xi - \eta(Y)X.
\]

Moreover, on a Sasakian manifold \(\bar{M}\)

\[
\nabla_X \xi = -\phi X
\]  

for any \(X, Y \in T\bar{M}\) and \(\xi\) is the structure vector field.

Now, let \(M\) be a submanifold immersed in \(\bar{M}\); we denote by the same symbol \(g\) the induced metric on \(M\). Let \(TM\) be the Lie-algebra of vector fields on \(M\) and \(T^\perp M\) the set of all vector fields normal to \(M\). Then the Gauss and Weingarten formulas are given by

\[
\nabla_X Y = \nabla_X Y + h(X, Y)
\]
\[
\nabla_X V = -A_V X + \nabla^\perp_X V
\]

for any \(X, Y \in TM\) and \(V \in T^\perp M\), where \(\nabla^\perp\) is the connection in the normal bundle, \(h\) is the second fundamental form of \(M\) and \(A_V\) is the shape operator associated with \(V\). The second fundamental form \(h\) and the shape operator \(A_V\) are related by

\[
g(A_V X, Y) = g(h(X, Y), V).
\]
For any $X \in TM$ and $V \in T^\perp M$, we write
\begin{align}
\phi X &= TX + NX \\
\phi V &= tV + nV
\end{align}
where $TX$ (resp. $tV$) denotes the tangential part of $\phi X$ (resp. $\phi V$) and $NX$ (resp. $nV$) denotes the normal part of $\phi X$ (resp. $\phi V$). The covariant derivative of $T$ and $N$ are defined as
\begin{align}
(\bar{\nabla}_X T)Y &= \nabla_X TY - T\nabla_X Y \\
(\bar{\nabla}_X N)Y &= \nabla^\perp_X NY - N\nabla_X Y.
\end{align}
The submanifold $M$ is invariant if $N$ is identically zero, that is, $\phi X \in TM$, for any $X \in TM$. On the other hand, $M$ is an anti-invariant submanifold if $T$ is identically zero, that is, $\phi X \in T^\perp M$, for any $X \in TM$. The distribution spanned by the structure vector field $\xi$ is denoted by $\langle \xi \rangle$. By applying equation (4) and (7) on (3), it follows that
\begin{align}
(a) \quad \nabla_X \xi &= -TX \\
(b) \quad h(X, \xi) &= -NX
\end{align}
for all $X \in TM$.

3. PSEUDO-SLANT SUBMANIFOLDS OF ALMOST HERMITIAN AND ALMOST CONTACT METRIC MANIFOLDS

In this section we study pseudo-slant submanifolds of Sasakian manifold and obtain integrability conditions of the distributions on pseudo-slant submanifolds. To begin with, we show how to obtain pseudo-slant submanifolds of almost Hermitian manifolds by slant submanifold of contact manifolds.

Let $M$ be a Riemannian manifold, isometrically immersed in an almost contact metric manifold $(\tilde{M}, \phi, \xi, \eta, g)$. From now on, we suppose that the structure vector field $\xi$ is tangent to $M$. Hence, if we denote by $D$ the orthogonal distribution to $\xi$ in $TM$, then
\[ TM = D \oplus \langle \xi \rangle. \]

In this setting, for each nonzero vector $X$ tangent to $M$ at $x$, such that $X$ is not proportional to $\xi_x$, we denote by $\theta(X)$ the angle between $\phi X$ and $D_x$. Then, $M$ is said to be slant [3] if the angle $\theta(X)$ is constant, which is independent of the choice of $x \in M$ and $X \in T_x M - \langle \xi_x \rangle$. The angle $\theta$ of the slant immersion is called the slant angle of the immersion. Invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant immersion which is neither invariant nor anti-invariant is called a proper slant immersion. In the setting of submanifolds of an almost Hermitian manifolds, slant distribution are defined on the same lines. They are defined as follows.
Given a submanifold $S$, isometrically immersed in an almost Hermitian manifold $(\bar{S}, J, g_1)$, a differentiable distribution $\nu$ on $S$ is said to be a slant distribution, if for any nonzero $X \in \nu_x, x \in S$, the angle between $JX$ and the vector space $\nu_x$ is constant, that is, it is independent of choice of $x \in S$ and $X \in \nu_x$. This constant angle is called the slant angle of the slant distribution $\nu$.

The following theorem provides a useful characterization for the existence of a slant distribution on a contact metric manifold.

**Theorem 1** [7] — Let $\nu$ be a distribution on $\bar{M}$, orthogonal to $\xi$. Then, $\nu$ is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that $(PT)^2X = -\lambda X$, for any $X \in \nu$, where $P$ denotes the orthogonal projection on $\nu$. Furthermore, in this case $\lambda = \cos^2 \theta_\nu$.

A submanifold $S$ of an almost Hermitian manifolds $\bar{S}$ is said to be pseudo-slant submanifold if there exist on $S$ two orthogonal distributions $D_1$ and $D_2$ such that $TS = D_1 \oplus D_2$, where $D_1$ is totally real distribution (i.e., $JD_1 \subset T^\perp S$) and $D_2$ is slant distribution with the slant angle $\theta \neq \pi/2$ (see [2]). In particular, if dim $D_1 = 0$ and $\theta \in (0, \pi/2)$, then $S$ is a proper slant submanifold of almost Hermitian manifold $\bar{S}$ introduced by Chen [4].

In the following paragraphs, we show that there is a relationship between slant submanifolds of almost contact metric manifolds and pseudo-slant submanifolds of almost Hermitian manifolds.

Let $(\bar{M}, \phi, \xi, \eta, g)$ be an almost contact metric manifold. Then we consider the manifold $\bar{M} \times R$ and denote by $(X, f \frac{d}{dt})$ a vector field of $\bar{M} \times R$, where $X$ is tangent to $\bar{M}$, $t$ is the coordinate of $R$ and $f$ is a differentiable function on $\bar{M} \times R$. An almost complex structure $J$ on this manifold is defined as

$$J \left(X, f \frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X) \frac{d}{dt}\right).$$

(12)

It is well known that $(\bar{M} \times R, J, g_1)$ is an almost Hermitian manifold [9], where $g_1$ denotes the product metric given by

$$g_1 \left(\left(X, f \frac{d}{dt}\right), \left(Y, h \frac{d}{dt}\right)\right) = g(X, Y) + fh.$$

We now recall the following important result due to Lotta.

**Theorem 2** [3] — Let $M$ be a slant submanifold of dimension $n$ of an almost contact metric manifold $\bar{M}$ with slant angle $\theta \neq \pi/2$. Then we have

- $n$ is even $\iff$ $\xi$ is orthogonal to $M$
- $n$ is odd $\iff$ $\xi$ is tangent to $M$.

The following result provides a method to obtain a pseudo-slant submanifold of $\bar{M} \times R$ from slant submanifold of $\bar{M}$.

**Theorem 3** — Let $M$ be a non anti-invariant even dimensional slant submanifold of an almost contact metric manifold $\bar{M}$. Then $\bar{M} \times R$ is a pseudo-slant submanifold of the almost Hermitian manifold $\bar{M} \times R$, with totally real distribution $D_1 = \left\{\left(0, \frac{d}{dt}\right)\right\}$ and slant distribution
\[ D_2 = \{(X, 0) | X \in D \}. \]

**Proof:** It is clear that distribution \( D_1 \) and \( D_2 \) are orthogonal and \( T(M \times R) = D_1 \oplus D_2 \).

Moreover, by virtue of equation (12)
\[
J \left( 0, \frac{d}{dt} \right) = -(\xi, 0).
\]

Hence, \( D_1 \) is a totally real distribution in view of Theorem 2. Finally, it is easy to see that \( D_2 \) is a slant distribution in the sense of Papaghiuc [9].

To introduce pseudo-slant submanifold of an almost contact metric manifold; first we recall the definition of Bi-slant submanifold.

**Definition 1[7] —** \( M \) is said to be a Bi-slant submanifold of an almost contact metric manifold \( \bar{M} \) if there exist two orthogonal distributions \( D_1 \) and \( D_2 \) on \( M \) such that
(i) \( TM \) admits the orthogonal direct decomposition \( TM = D_1 \oplus D_2 \oplus \langle \xi \rangle \)
(ii) The distribution \( D_1 \) is slant with angle \( \theta_1 \)
(iii) The distribution \( D_2 \) is slant with angle \( \theta_2 \)

**Definition 2 —** We say that \( M \) is a pseudo-slant submanifold of an almost contact metric manifold \( \bar{M} \) if there exist two orthogonal distributions \( D_1 \) and \( D_2 \) on \( M \) such that
(i) \( TM \) admits the orthogonal direct decomposition \( TM = D_1 \oplus D_2 \oplus \langle \xi \rangle \)
(ii) The distribution \( D_1 \) is anti-invariant i.e., \( \phi D_1 \subseteq T^\perp M \)
(iii) The distribution \( D_2 \) is slant with slant angle \( \theta \neq \pi/2 \)

From the above definition it is clear that if \( \theta = 0 \), then the pseudo-slant submanifold is a semi-invariant submanifold. On the other hand if we denote the dimension of \( D_i \) by \( d_i \), for \( i = 1, 2 \), then we find the following cases
(a) If \( d_2 = 0 \) then \( M \) is an anti-invariant submanifold.
(b) If \( d_1 = 0 \) and \( \theta = 0 \), then \( M \) is an invariant submanifold.
(c) If \( d_1 = 0 \) and \( \theta \neq 0 \), then \( M \) is a proper slant submanifold, with slant angle \( \theta \).

A pseudo-slant submanifold is proper if \( d_1 d_2 \neq 0 \) and \( \theta \neq 0 \).

If we put \( \theta_1 = \pi/2 \) and \( \theta_2 = \theta \in [0, \pi/2) \) in the following example of 5-dimensional bi-slant submanifold \( M \) with slant angles \( \theta_1 \) and \( \theta_2 \) (see [7])
\[
x(u, v, w, s, t) = 2(u, 0, w, 0, v \cos \theta_1, v \sin \theta_1, s \cos \theta_2, s \sin \theta_2, t) \tag{13}
\]
then it becomes a pseudo-slant submanifold in \( R \) [9]. Furthermore, it is easy to see that
\[
e_1 = 2 \left( \frac{\partial}{\partial x^1} + y^1 \frac{\partial}{\partial Z} \right), \quad e_2 = 2 \frac{\partial}{\partial y^2}, \quad e_3 = 2 \left( \frac{\partial}{\partial x^3} + y^3 \frac{\partial}{\partial Z} \right), \quad e_4 = 2 \cos \theta \frac{\partial}{\partial y^3} + 2 \sin \theta \frac{\partial}{\partial y^4}, \quad e_5 = 2 \frac{\partial}{\partial Z} = \xi
\]
form a local orthonormal frame of $TM$. We define, the distributions $D_1 = \langle e_1, e_2 \rangle$ and $D_2 = \langle e_3, e_4 \rangle$. Then it is clear that $TM = D_1 \oplus D_2 \oplus \langle \xi \rangle$, in which $D_1$ is anti-invariant distribution and $D_2$ is slant distribution with slant angle $\theta$.

Suppose $M$ to be a pseudo-slant submanifold of an almost contact metric manifold $\bar{M}$. Then, for any $X \in TM$, put

$$X = P_1 X + P_2 X + \eta(X) \xi$$

(14)

where $P_i$ ($i = 1, 2$) are projection maps on the distributions $D_1$ and $D_2$. Now operating $\phi$ on both sides of equation (14)

$$\phi X = NP_1 X + TP_2 X + NP_2 X.$$  

(15)

It is easy to see that

$$TX = TP_2 X, \quad NX = NP_1 X + NP_2 X$$

(16)

and,

$$\phi P_1 X = NP_1 X, \quad TP_1 X = 0;$$

(17)

$$TP_2 X \in D_2.$$  

(18)

Since $D_2$ is slant distribution, by Theorem 1

$$T^2 X = - \cos^2 \theta X$$

(19)

for any $X \in D_2$.

Now, we have the following theorem:

**Theorem 4** — Let $M$ be a submanifold of an almost contact metric manifold $\bar{M}$, such that $\xi \in TM$. Then $M$ is a pseudo-slant submanifold if and only if there exists a constant $\lambda \in (0, 1]$ such that

(i) $D = \{X \in TM \mid T^2 X = -\lambda X\}$ is a distribution on $M$

(ii) For any $X \in TM$, orthogonal to $D$, $TX = 0$.

Furthermore, in this case $\lambda = \cos^2 \theta$, where $\theta$ denotes the slant angle of $D$.

**PROOF:** Set $\lambda = \cos^2 \theta$, then it follows from (17) and (18) that $D = D_2$. Conversely, consider the orthogonal direct decomposition $TM = D \oplus D^\perp \oplus \langle \xi \rangle$. It is evident that $TD \subseteq D$. Hence, by using statement (ii) it is clear that $D^\perp$ is an anti-invariant distribution. Finally, Theorem 1 and statement (i) imply that $D$ is a slant distribution, with slant angle $\theta$ satisfying $\lambda = \cos^2 \theta$.

Now, we will discuss the integrability of distributions involved in a pseudo-slant submanifold of a Sasakian manifold.

If $\mu$ is the invariant subspace of the normal bundle $T^\perp M$ then, in the case of pseudo-slant submanifold, the normal bundle $T^\perp M$ can be decomposed as follows

$$T^\perp M = \mu \oplus N D_1 \oplus N D_2.$$  

(20)
As $D_1$ and $D_2$ are orthogonal distributions on $M$, $g(Z, X) = 0$ for each $Z \in D_1$ and $X \in D_2$. Thus, by equation (7) and (1), we may write

$$g(NZ, NX) = g(\phi Z, \phi X) = g(Z, X) = 0.$$ 

That means the distributions $ND_1$ and $ND_2$ are mutually perpendicular. In fact, the decomposition (20) is an orthogonal direct decomposition.

For a pseudo-slant submanifold of a Sasakian manifold the following lemmas play an important role in working out the integrability conditions of the distributions involved in this setting.

**Lemma 1** — Let $M$ be a pseudo-slant submanifold of a Sasakian manifold $\bar{M}$. Then

$$A_{\phi Y}X = A_{\phi X}Y$$

for all $X, Y \in D_1$.

**Proof:** For any $X, Y$ in $D_1$ and $Z$ in $TM$, using (6), (1), (3) and (4) we find that

$$g(A_{\phi Y}X, Z) = -g(\phi \nabla_Z X - \phi \nabla_Z Y)$$

$$= -g(\phi \nabla_Z X, Y)$$

i.e.,

$$g(A_{\phi Y}X, Z) = -g(\nabla_Z \phi X - (\nabla_Z \phi)X, Y).$$

On applying equations (2) and (5) the above equation yields

$$g(A_{\phi Y}X, Z) = g(A_{\phi X}Y, Z)$$

The result follows from the above equation.

**Lemma 2** — Let $M$ be a pseudo-slant submanifold of a Sasakian manifold $\bar{M}$, then

$$[X, \xi] \in D_1$$

for all $X \in D_1$.

**Proof:** For any $X \in D_1$ and $Z \in D_2$

$$g([X, \xi], TZ) = g(\bar{\nabla}_\xi TZ, X).$$

On using equations (2) & (7) we have

$$\bar{\nabla}_\xi TZ = -\bar{\nabla}_\xi NZ + \phi \bar{\nabla}_\xi Z.$$ 

Applying the above formula in (23) and using (6), (7), (4) (5) and (11)(b), we get

$$g([X, \xi], TZ) = -g(NX, NZ) + g(NX, NZ) = 0.$$
This proves the lemma completely.

**Lemma 3** — Let $M$ be a pseudo-slant submanifold of a Sasakian manifold $\tilde{M}$ then, for any $X, Y \in D_1 \oplus D_2$

$$g([X, Y], \xi) = 2g(X, TY). \quad (24)$$

The proof of equation (24) is straightforward and may be obtained by using (11)(a).

**Proposition 1** — Let $M$ be a pseudo-slant submanifold of a Sasakian manifold $\tilde{M}$. Then, anti-invariant distribution $D_1$ is integrable.

**Proof:** For any $X, Y \in D_1$ and $Z \in D_2$, by (14)

$$g([X, Y], TP_2Z) = -g(\phi[X, Y], P_2Z)$$

Now, using (2) and (5), we find

$$g([X, Y], TP_2Z) = g(A_{\phi Y}X - A_{\phi X}Y, P_2Z).$$

Now, the integrability of the distribution $D_1$ follows on using equations (21) and (24).

**Corollary 1** — On a pseudo-slant submanifold $M$ of a Sasakian manifold $\tilde{M}$, the distribution $D_1 \oplus \langle \xi \rangle$ is integrable.

The corollary follows from Proposition (1) and equation (22).

**Lemma 4** — Let $M$ be a pseudo-slant submanifold of a Sasakian manifold $\tilde{M}$. Then, the slant distribution $D_2$ is not integrable.

By the definition of pseudo-slant submanifold and in view of equation (24) the result follows.

**Proposition 2** — Let $M$ be a pseudo-slant submanifold of $\tilde{M}$ then the distribution $D_2 \oplus \langle \xi \rangle$ is integrable if and only if

$$h(Z, TW) - h(W, TZ) + \nabla^\perp_Z NW - \nabla^\perp_W NZ$$

lies in $ND_2$ for each $Z, W \in D_2 \oplus \langle \xi \rangle$.

**Proof:** Making use of equations (7), (2), (4) and (5), we obtain

$$g(N[Z, W], NX) = g(h(Z, TW) - h(W, TZ) + \nabla^\perp_Z NW - \nabla^\perp_W NZ, NX)$$

for each $X \in D_1$ and $Z, W \in D_2$. The result follows on using the fact that $ND_1$ and $ND_2$ are mutually perpendicular.

Cabrerizo et al. [6] have obtained the following expression for $\nabla T$ in case of some slant submanifolds of Sasakian manifolds

$$(\nabla_X T)Y = \cos^2 \theta (g(X, Y)\xi - \eta(Y)X) \quad (25)$$

for any $X, Y \in TM$. 
Note — 1 Formula (25) provides a sufficient condition for a submanifold to be a proper slant submanifold of a \( K \)-contact manifold (see [6]).

Consider the 5-dimensional pseudo-slant submanifold of \( R^9 \) given by equation (13) i.e.,
\[
x(u, v, w, s, t) = 2(u, 0, w, 0, v, s \cos \theta, s \sin \theta, t)
\]
where \( \theta \in (0, \pi/2) \) is the slant angle of the slant distribution \( D_2 \). Then it is easy to see that
\[
(\nabla_X T)Y = \cos^2 \theta(g(P_2X, Y)\xi - \eta(Y)P_2X)
\]
for any \( X, Y \in TM \). If we take \( X = P_2X + \eta(X)\xi \) and \( Y = P_2Y + \eta(Y)\xi \) then (26) implies that
\[
(\nabla_X T)Y = \cos^2 \theta(g(X, Y)\xi - \eta(Y)X)
\]
Thus, in view of note (1), the distribution \( D_2 \oplus \langle \xi \rangle \) works as the tangent bundle of proper slant submanifold. On the other hand if \( X, Y \in D_1 \oplus \langle \xi \rangle \) then (26) implies
\[
(\nabla_X T)Y = 0
\]
which shows that anti-invariant submanifolds satisfy the above equation. We want to realize that (26) is a natural condition for pseudo-slant submanifolds of a Sasakian manifold analogous to slant and semi-slant submanifolds of Sasakian manifolds worked out by Cabreroiz et al. (see [6, 7]. Now, we have the following theorem

**Theorem 5** — Let \( M \) be a proper pseudo-slant submanifold with angle \( \theta \), of a Sasakian manifold \( \tilde{M} \). Then, for any \( X, Y \in TM \).
\[
(\nabla_X T)Y = A_{NP_1Y}X + A_{NP_2Y}X + th(X, Y) + g(X, Y)\xi - \eta(Y)X.
\]
Hence, \( M \) satisfies (26) if and only if
\[
A_{NY}X = A_{NX}Y + \eta(Y)P_1X - \eta(X)P_1Y - \sin^2 \theta(\eta(X)P_2Y - \eta(Y)P_2X)
\]
where \( NX = NP_1X + NP_2X \)

**Proof** : For any \( X, Y \in TM \)
\[
\tilde{\nabla}_X \phi Y = (\tilde{\nabla}_X \phi)Y + \phi \tilde{\nabla}_X Y
\]
Now using (2), (7), (4), (5) and (9) and comparing tangential parts we obtain (27). Suppose \( M \) is a proper pseudo-slant submanifold satisfying (26). Then, by (27)
\[
\cos^2 \theta(g(P_2X, Y)\xi - \eta(Y)P_2X) = A_{NP_1Y}X + A_{NP_2Y}X + th(X, Y) + g(P_1X, Y)\xi
\]
\[
+ g(P_2X, Y)\xi - \eta(Y)P_1X - \eta(Y)P_2X
\]
or,
\[ A_{NP_1}X + A_{NP_2}Y = -\text{th}(X, Y) - g(P_1 X, Y)\xi + \eta(Y)P_1 X \]
\[ - \sin^2 \theta(g(P_2 X, Y)\xi - \eta(Y)P_2 X). \]

Similarly,
\[ A_{NP_1}X + A_{NP_2}Y = -\text{th}(X, Y) - g(P_1 Y, X)\xi + \eta(X)P_1 Y \]
\[ - \sin^2 \theta(g(P_2 Y, X)\xi - \eta(X)P_2 Y), \]

Finally, we have
\[ A_{NY}X = A_{NX}Y + \eta(Y)P_1 X - \eta(X)P_1 Y - \sin^2 \theta(\eta(X)P_2 Y - \eta(Y)P_2 X) \]

Conversely, suppose (28) holds. Then for any \( Z \in TM \)
\[ g(A_{NY}X, Z) = -g(\text{th}(Y, Z), X) + \eta(Y)g(P_1 X, Z) - \eta(X)g(P_1 Y, Z) \]
\[ - \sin^2 \theta(\eta(X)g(P_2 Y, Z) - \eta(Y)g(P_2 X, Z)). \]

Interchanging \( X \) and \( Z \), and making use of the fact that \( g(P_1 X, Y) = g(X, P_1 Y) \), for each \( X, Y \in TM \), we get
\[ g(A_{NY}X, Z) = -g(\text{th}(Y, Z), X) + \eta(Y)g(P_1 X, Z) - \eta(Z)g(P_1 Y, X) \]
\[ - \sin^2 \theta(\eta(Z)g(P_2 Y, X) - \eta(Y)g(P_2 X, Z)). \]

Taking account of equation (14), the above equation yields
\[ g(A_{NY}X, Z) = -g(\text{th}(Y, Z), X) + \eta(Y)g(X, Z) - \eta(Z)g(X, Y) \]
\[ + \cos^2 \theta(\eta(Z)g(P_2 Y, X) - \eta(Y)g(P_2 X, Z)). \]

Now, by using equation (27), we obtain
\[ (\nabla_X T)Y = \cos^2 \theta(g(P_2 X, Y)\xi - \eta(Y)P_2 X) \]
which proves the assertion.

**Theorem 6** — Let \( M \) be a proper pseudo-slant submanifold of a Sasakian manifold \( \bar{M} \) with slant angle \( \theta \), then
(i) \( M \) satisfies (26) if and only if
\[ (\nabla_X TP_2)Y = \cos^2 \theta(g(P_2 X, Y)\xi - \eta(Y)P_2 X) \]
(ii) If \( M \) satisfies (26), then
\[
\nabla_X Z \in D_1 \oplus \langle \xi \rangle, \quad \nabla_X W \in D_2 \oplus \langle \xi \rangle
\]
for any \( X, Y \in TM, Z \in D_1 \) and \( W \in D_2 \).

**Proof:** For any pseudo-slant submanifold of Sasakian manifold \( \bar{M} \),
\( T = TP_2 \). Then statement
(i) follows from (26) and (16).

Suppose that \( M \) satisfies (26). Then by statement (i)
\[
(\nabla_X TP_2)Y = \cos^2 \theta (g(P_2X, Y)\xi - \eta(Y)P_2X).
\]
From above equation it is evident that
\[
P_1 \nabla_X TP_2 Y = 0 \text{ i.e., } \nabla_X TP_2 Y \in D_2 \oplus \langle \xi \rangle
\]
or, equivalently
\[
\nabla_X Z \in D_2 \oplus \langle \xi \rangle
\]
for any \( X \in TM \) and \( Z \in D_2 \). Then (ii) follows from (30).

Now, we have the following corollary:

**Corollary 2** — Let \( M \) be a pseudo-slant submanifold of a Sasakian manifold \( \bar{M} \) such that \( M \)
satisfies (26). Then \( D_2 \oplus \langle \xi \rangle \) is integrable and leaves of distributions \( D_1 \oplus \langle \xi \rangle \) and \( D_2 \oplus \langle \xi \rangle \) are
totally geodesic in \( M \).

The corollary follows by statement (ii) of Theorem 6 and (11)(a).

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**References**


