ABSOlUTE RIESZ SUMMABILITY FACTORS FOR A FOURIER SERIES 
AND ITS CONJUGATE SERIES 

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(Received 10 September 1979; after revision 7 June 1980) 

In this paper the author discusses two general theorems on the absolute Riesz summability factors for a Fourier series and its conjugate series and obtains appropriate conditions under which the above summability can be ensured. The theorems exhibit the competency of a Riesz method as an instrument to study absolute Cesàro summability. Results due to Mohanty (1950) follow as corollaries to his theorems. 

1. Definitions and Notations 

Let \( \lambda = \lambda(\omega) \) be a differentiable, monotonic-increasing function of \( \omega \) tending to infinity with \( \omega \). A given infinite series \( \Sigma a_n \) is said to summable \( (R, \lambda, r) \), \( r > 0 \) and we write \( \Sigma a_n \in (R, \lambda, r) \), if 

\[
\int_{A}^{\infty} \frac{\lambda' (\omega)}{[\lambda(\omega)]^{r+1}} \left| \sum_{n < \omega} \{\lambda(\omega) - \lambda(n)\}^{r-1} \lambda_n a_n \right| d\omega < \infty
\]

where \( A \) is a finite positive number (Obrechkoff 1928, 1929; Mohanty 1951). It is known (Hyslop 1936) that \( |R, \omega, k| \sim |C, k| \). 

Let \( f \in L (-\pi, \pi) \) be a 2\( \pi \)-periodic function. Let the Fourier series of \( f \) at a point \( x \), be 

\[
\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x). 
\]

Then the conjugate series of the above Fourier series is 

\[
\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x). 
\]

Throughout this paper we shall use the following notations: 

\[
\phi(t) = \frac{1}{2} \{f(x + t) + f(x - t)\} 
\]

\[
\phi(t) = \frac{1}{2} \{f(x + t) - f(x - t)\} 
\]

\[
\phi^*(t) = \phi(t) - \phi(+0) 
\]
\[ e(\omega) = \exp((\log \omega)^{1+\delta}), \quad \delta \geq 0 \]

\[ g(\omega, t) = \sum_{n<\omega} \{e(\omega) - e(n)\}^{\beta-1} e(n) \ n^{\alpha-1}(\log n)^{-\alpha \delta} \cos nt \]

\[ g(\omega, t) = \sum_{n<\omega} \{e(\omega) - e(n)\}^{\beta-1} e(n) \ n^{\alpha-1}(\log n)^{-\alpha \delta} \sin nt \]

\[ Q(\omega) = \{e(\omega) - e(m)\}^{\beta-1} e(m) \ m^{\alpha-1}(\log m)^{-\alpha \delta} \]

where \( m \) is an integer such that \( 0 < \omega - m \leq 1 \).

\( k \) denotes a suitable constant chosen for convenience in analysis. \( K, K_1, K_2 \) and \( K_3 \) stand for absolute constants, possibly different each time.

2. Introduction

Mohanty (1950) established the following two results pertaining to absolute Cesàro summability factors for a Fourier series and its conjugate series.

**Theorem A** — Let \( 0 < \alpha < 1 \). Then \( \int_0^{\pi} t^{-\alpha} \ | \ d\phi(t) \ | < \infty \) implies that

\[ \sum n^{\alpha} A_n(x) \in [C, \beta] \ , \ \beta > \alpha. \]

**Theorem B** — Let \( 0 < \alpha < 1 \). Then \( \psi(+0) = 0 \) and \( \int_0^{\pi} t^{-\alpha} \ | \ d\psi(t) \ | < \infty \) imply that \( \sum n^{\alpha} B_n(x) \in [C, \beta] \ , \ \beta > \alpha. \)

The purpose of this paper is to establish the following two general results on the absolute Riesz summability factors for a Fourier series and its conjugate series.

In what follows we prove the following:

**Theorem 1** — Let \( 0 < \alpha < 1 \) and \( \delta \geq 0 \). Then \( \int_0^{\pi} t^{-\alpha} \ | \ d\phi(t) \ | < \infty \) implies that

\[ \sum_{n=2}^{\infty} n^{\alpha} A_n(x) (\log n)^{-\alpha \delta} \in [R, \exp(\log \omega)^{1+\delta}, \beta] \ , \ \beta > \alpha. \]

**Theorem 2** — Let \( 0 < \alpha < 1 \) and \( \delta \geq 0 \). Then \( \psi(+0) = 0 \) and \( \int_0^{\pi} t^{-\alpha} \ | \ d\psi(t) \ | < \infty \) imply that \( \sum_{n=2}^{\infty} n^{\alpha} B_n(x) (\log n)^{-\alpha \delta} \in [R, \exp(\log \omega)^{1+\delta}, \beta] \ , \ \beta > \alpha. \)

3. Estimates

We require the following order estimates to prove our theorems.

For \( 0 < \alpha < \beta \leq 1 \),
\( g(\omega, t) = \int O(\omega^\alpha e^{\beta(\omega)} (\log \omega)^{-\epsilon(\alpha+1)}) + Q(\omega), \)
\( \bar{g}(\omega, t) = \int O(t^{-\beta} \omega^{\alpha-\beta} e^{\beta(\omega)} (\log \omega)^{\epsilon(\beta-\alpha-1)}) + Q(\omega). \)

We give a proof for the estimates for \( g \) only, the proof for \( \bar{g} \) is similar.

**Proof:**

\[
g(\omega, t) \leq \sum_{n<\omega} \{ e(\omega) - e(n) \}^{\beta-1} e(n) n^{\alpha-1}(\log n)^{-\alpha \epsilon} \\
= \sum_{n<\omega} + \sum_{\sqrt{\omega} < n < \omega} = R_1 + R_2, \text{ say.}
\]

\[
R_1 \leq \{ e(\omega) - e(\sqrt{\omega}) \}^{\beta-1} e(\sqrt{\omega}) (\sqrt{\omega})^\alpha (\log \sqrt{\omega})^{-\alpha \epsilon} \\
= \{ (\omega - \sqrt{\omega}) e'(\omega) \}^{\beta-1} e(\sqrt{\omega}) (\sqrt{\omega})^\alpha (\log \sqrt{\omega})^{-\alpha \epsilon}, \quad (\sqrt{\omega} < \omega_1 < \omega), \\
\leq K \omega^{\alpha/2} e^{\beta(\omega)} (\log \omega)^{\epsilon(\beta-\alpha-1)};
\]

\[
R_2 \leq \int_{\sqrt{\omega}}^{\omega} \{ e(\omega) - e(u) \}^{\beta-1} e(u) u^{\alpha-1}(\log u)^{-\alpha \epsilon} \, du + Q(\omega) \\
\leq K \omega^{\alpha}(\log \sqrt{\omega})^{-\epsilon(\alpha+1)} \{ e(\omega) - e(\sqrt{\omega}) \}^{\beta} + Q(\omega) \\
\leq K \omega^{\alpha}(\log \omega)^{-\epsilon(\alpha+1)} e^{\beta(\omega)} + Q(\omega).
\]

To prove the second estimate we put \( \omega_1 = \left\lfloor \frac{\omega - \frac{k}{t}}{t} \right\rfloor \). Now

\[
g(\omega, t) = \{ \sum_{n<\omega_1} + \sum_{\omega_1+1}^{m} \} \{ e(\omega) - e(n) \}^{\beta-1} e(n) n^{\alpha-1}(\log n)^{-\alpha \epsilon} \cos nt \\
= R_3 + R_4, \text{ say.}
\]

As \( \{ (e(\omega) - e(n)) \}^{\beta-1} e(n) n^{\alpha-1}(\log n)^{-\alpha \epsilon} \} \) is ultimately monotonic-increasing in \( n < \omega, \)

\[
R_3 = O[(e(\omega) - e(\omega_1))^{\beta-1} e(\omega_1) \omega_1^{\alpha-1} (\log \omega_1)^{-\alpha \epsilon} \max_{2 \leq a < b < \omega_1} \frac{b}{a} \Sigma \cos nt |] \\
= O\{t^{-\beta} \omega^{\alpha-\beta} e^{\beta(\omega)} (\log \omega)^{\epsilon(\beta-\alpha-1)}\},
\]

and similarly we can have

\[
R_4 = O\{t^{-\beta} \omega^{\alpha-\beta} e^{\beta(\omega)} (\log \omega)^{\epsilon(\beta-\alpha-1)}\}.
\]

4. **Proof of Theorem 1**

Without any loss of generality we assume \( 0 < \alpha < \beta < 1 \). Integrating by parts, we have

\[
A_\alpha(x) = \frac{2}{\pi} \int_0^\pi \phi(t) \cos nt \, dt = -\frac{2}{\pi} \int_0^\pi \frac{\sin nt}{n} \, d\phi(t).
\]
The series \( \sum_{n=1}^{\infty} n^\alpha A_n(x) (\log n)^{-\alpha \xi} \in | R \ e(\omega), \beta |, \) iff

\[
I = \int_2^\infty \frac{(\log \omega)^\xi}{\omega e^\beta(\omega)} \left| \int_0^\pi \sum_{n<\omega} (e(\omega) - e(n))^{\beta-1} e(n) n^\alpha (\log n)^{-\alpha \xi} \phi(t) \cos nt \, dt \right| \, d\omega
\]

\[
= \int_2^\infty \frac{(\log \omega)^\xi}{\omega e^\beta(\omega)} \left| \int_0^\pi d\phi(t) \, \bar{g}(\omega, t) \, dt \right| \, d\omega
\]

is convergent.

We observe that in order to prove our theorem it is enough to show that

\[
\int_0^\pi d\phi(t) | p(t) | < \infty, \quad \ldots (4.1)
\]

where

\[
p(t) = \int_2^\infty \frac{(\log \omega)^\xi}{\omega e^\beta(\omega)} \left| \bar{g}(\omega, t) \right| \, d\omega.
\]

On writing \( T = \frac{k}{t} (\log k/t)^\xi \) and subdividing the range of integration in \( p(t) \), we have

\[
p(t) \leq K_1 \int_2^T \omega^{\alpha-1} (\log \omega)^{-\alpha \xi} \, d\omega + K_2 t^{-\beta} \int_T^\infty \omega^{\alpha-1} (\log \omega)^{\xi(\beta-\alpha)} \, d\omega
\]

\[
+ K_3 \int_2^\infty \frac{(\log \omega)^\xi}{\omega e^\beta(\omega)} \, Q(\omega) \, d\omega
\]

\[
\leq K_1 T^{\alpha-1} (\log T)^{-\alpha \xi} + K_2 t^{-\beta} T^{\alpha-\beta} (\log T)^{\xi(\beta-\alpha)}
\]

\[
+ K_3 \sum_{m=2}^{\infty} \int_{m-1}^m \frac{\{e(\omega) - e(m)\}^{\beta-1}}{\omega e^\beta(\omega)} \left( \frac{e(m) (\log \omega)^\xi m^{\alpha-1}}{\omega e^\beta(\omega) (\log m)^{\alpha \xi}} \right) \, d\omega
\]

\[
\leq K_1 t^{-\alpha} \left( \log \frac{k}{t} \right)^{\alpha \xi} \left\{ \log \frac{k}{t} + \delta \log \log \frac{k}{t} \right\}^{-\alpha \xi}
\]

\[
+ K_2 t^{-\alpha} \left( \log \frac{k}{t} \right)^{\xi(\alpha-\beta)} \left\{ \log k/t + \delta \log \log k/t \right\}^{\xi(\beta-\alpha)}
\]

\[
+ K_3 \sum_{m=2}^{\infty} m^{\alpha-1} e^{-\beta(m)} (\log m)^{-\alpha \xi} \{e(m+1) - e(m)\}^\beta
\]
\[ \leq K_1 t^{-\alpha} \left\{ 1 + \frac{\delta \log \log k/t}{\log k/t} \right\}^{-\alpha \delta} + K_2 t^{-\alpha} \left\{ 1 + \frac{\delta \log \log k/t}{\log k/t} \right\}^{\varepsilon(\beta - \alpha)} \]

\[ + K_3 \sum_{m=2}^{\infty} m^{\alpha - 1} e^{-\beta(m)} (\log m)^{-\alpha \delta} \left[ \frac{e(m_1)(1 + \delta)(\log m_1^\varepsilon)}{m_1} \right]^{\beta} \]

for \( m < m_1 < m + 1, \)

\[ \leq K_1 t^{-\alpha} + K_2 t^{-\alpha} + K_3 \sum_{2}^{\infty} \frac{(\log m)^{\delta(\beta - \alpha)}}{m^{1+\beta - \alpha}} \]

\[ \leq K t^{-\alpha} + K. \]

Hence uniformly in \( 0 < t < \pi, \)

\[ p(t) = O(t^{-\alpha}). \]

Thus from (4.1) and the hypothesis

\[ \int_{0}^{\pi} t^{-\alpha} | d\psi(t) | < \infty. \]

This completes the proof of Theorem 1.

5. PROOF OF THEOREM 2

Integrating by parts and remembering that \( \psi(+0) = 0, \) we have

\[ B_n(x) = -\frac{2}{\pi} \psi(\pi) \cos \frac{n\pi}{n} + \frac{2}{\pi} \int_{0}^{\pi} \frac{\cos nt}{n} d\psi(t). \]

The proof of this theorem is almost similar to that of Theorem 1 except that in place of the estimates for \( g(\omega, t) \) we have to use the estimates for \( \bar{g}(\omega, t) \) and the hypotheses of this theorem.

6. COROLLARIES

Mohanty (1951) established that if \( \alpha > 0, \eta > 0, \) then necessary and sufficient conditions that \( \int_{0}^{\eta} t^{-\alpha} | d\psi(t) | < \infty, \) and \( \psi(+0) = 0, \) are that

(i) \( t^{-\alpha} \psi(t) \in BV(0, \eta), \)

and

(ii) \( \frac{|\psi(t)|}{t^{\alpha+1}} \in L(0, \eta). \)

Replacing \( \psi(t) \) in the hypothesis of theorem 1 by \( \psi^*(t) = \phi(t) - \phi(+0), \) so that \( \phi^*(+0) = 0, \) and using the above result along with \( \phi^*(t) \) in place of \( \psi(t); \) and \( \delta = 0, \) we see that Theorems 1 and 2 are respectively equivalent to Corollaries 1 and 2.
Corollary 1 — Let $0 < \alpha < 1$. Then

(i) $t^{-\alpha} \varphi^*(t) \in BV(0, \pi)$ and

(ii) $\frac{|\phi^*(t)|}{t^{\alpha+1}} \in L(0, \pi)$ imply that $\sum_{n=2}^{\infty} n^\alpha A_n(x) \in |C, \beta|$, $\beta > \alpha$.

Corollary 2 — Let $0 < \alpha < 1$. Then

(i) $t^{-\alpha} \psi(t) \in BV(0, \pi)$ and

(ii) $\frac{|\psi(t)|}{t^{\alpha+1}} \in L(0, \pi)$ imply that $\sum_{n=2}^{\infty} n^\alpha B_n(x) \in |C, \beta|$, $\beta > \alpha$.

ACKNOWLEDGEMENT

The author is thankful to Dr S. N. Bhatt and Dr R. N. Singh for their kind encouragement and would like to express his thanks to the referee for his helpful suggestions to improve the presentation of this paper.

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