INTERPOLATION BY GENERALIZED POLYNOMIALS

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In this paper, the author studies the cubic spline interpolation and approximation by generalized polynomials with four terms. A theorem on existence and uniqueness is proved.

1. Introduction


Let \( a = x_0 < x_1 < \ldots < x_n = b \) be a partition on \([a, b]\). Let \( f \) be a function defined on \([a, b]\). A proof of the following theorem may be found in Rivlin (1969, pp. 106–107).

**Theorem 1.1** — For given \( U_0, U_n \), there exists a unique function \( s(x) \in C^2 [a, b] \) such that in each interval \([x_{i-1}, x_i], i = 1, 2, \ldots, n \), \( s(x) \) agrees with a polynomial of degree at most 3 and \( s'(x) \) satisfies

\[
s(x_i) = f(x_i), \quad i = 0, 1, \ldots, n
\]

and

\[
s'(x_i) = u_i, \quad i = 0, n.
\]

Throughout this paper, \( n, a_0, a_1, \ldots, a_n \) will denote integers such that \( n > 1 \) and \( 0 \leq a_0 < a_1 < \ldots < a_n \).

**Definition 1.2** — Let \( \{g_\alpha\}_{\alpha=0}^{\infty} \) be a sequence of functions, real-valued, non-negative and continuous on \([0, R]\) and analytic on \((0, R]\) where \( R \) is a positive constant. Further suppose that \( g_\alpha \) is not a constant function if \( \alpha \geq 1 \), \( g_0 \) is not identically zero and \( g_\alpha(0) = 0 \) unless \( g_\alpha \) is a constant. Then \( \{g_\alpha\}_{\alpha=0}^{\infty} \) is said to have property \( \mathcal{D} \) if and only if the following hold:

(i) For every set of non-zero real numbers \( \{C_0, C_1, \ldots, C_n\} \) and for every choice of integers \( \{a_0, a_1, \ldots, a_n\} \) the number of zeros, counted with due regard to

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multiplicity in \((0, R]\), of the sum \(\sum_{k=0}^{n} C_k g_{a_k}\) is at most equal to the number of variations of sign in the sequence \(\{C_0, C_1, \ldots, C_n\}\).

(ii) For every set of non-zero numbers \(\{C_0, C_1, \ldots, C_n\}\) and for every choice of integers \(\{a_0, a_1, \ldots, a_n\}\) the number of zeros, counted with due regard to multiplicity in \((0, R]\) of the sum \(\sum_{k=0}^{n} C_k g'_{a_k}\) is at most equal to the number of variations of sign in the sequence \(\{C_0, C_1, \ldots, C_n\}\). (cf. Bell and Shah 1975).

**Definition 1.3** — Let \(\{g_{a}\}_{a=0}^{\infty}\) be a sequence of functions with property \(\mathcal{D}\). For any set of non-zero finite real numbers \(\{C_0, C_1, \ldots, C_n\}\), \(\sum_{k=0}^{n} C_k g_{a_k}\) is said to be a generalized polynomial of \((n + 1)\) terms.

The purpose of this paper is to replace the piecewise cubic polynomials in Theorem 1.1 by generalized polynomials of four terms, and to prove a theorem on existence and uniqueness.

2. **INTERPOLATION THEOREMS BY GENERALIZED POLYNOMIALS WITH FOUR TERMS**

Let \(\{g_{a}\}_{a=0}^{\infty}\) be a sequence of functions with property \(\mathcal{D}\) as defined in Definition 1.2. Let

\[
V(x) = [g_{a_0}(x) \quad g_{a_1}(x) \quad g_{a_2}(x) \quad g_{a_3}(x)]
\]

be a row vector with components \(g_{a_k}(x)\), \(g_{a_k} \in \{g_{a}\}_{a=0}^{\infty}\), \(k = 0, 1, 2, 3\).

Similarly, define row vectors \(V_i\), \(V'_i\) as follows,

\[
V_i = [g_{a_0}(x_i) \quad g_{a_1}(x_i) \quad g_{a_2}(x_i) \quad g_{a_3}(x_i)], \quad i = 0, 1, \ldots, n,
\]

\[
V'_i = [g'_{a_0}(x_i) \quad g'_{a_1}(x_i) \quad g'_{a_2}(x_i) \quad g'_{a_3}(x_i)], \quad i = 0, 1, \ldots, n,
\]

where \(0 < x_0 < \ldots < x_n \leq R\).

Let

\[
\begin{vmatrix}
V(x) \\
V_i \\
V'_{i-1} \\
V'_i
\end{vmatrix}
\]

be the determinant with row vectors \(V(x)\), \(V_i\), \(V'_{i-1}\), \(V'_i\) defined as above.
*Lemma 2.1* — If \( x_{i-1} \) and \( x_i \) are not both double zeros for the same function
\[
g_{a_k}, \ k = 0, 1, 2, 3; \tag{1}
\]
then
\[
\begin{vmatrix}
V_{i-1} \\
V_i \\
V_{i-1}' \\
V_i'
\end{vmatrix} \neq 0 \quad \text{for } i = 1, 2, \ldots, n.
\]

**Proof:** By the property \( D \) in Definition 1.2 there exist no common zeros in \((0, R]\) for any two functions in the sequence \( \{g_a\}_{a=0}^{\infty} \). Otherwise the following generalized polynomial with no variation of sign
\[
g_{a_i}(x) + g_{a_k}(x)
\]
would have a zero in \((0, R]\) if \( g_{a_i}(x) \) and \( g_{a_k}(x) \) have a common zero in \((0, R]\). Thus we may assume that
\[
g_{a_0}(x_{i-1}) \neq 0.
\]
The property \( D \) also implies that the determinant
\[
\begin{vmatrix}
g_{a_0}(x_{i-1}) & g_{a_k}(x_{i-1}) \\
g_{a_0}(x_i) & g_{a_k}(x_i)
\end{vmatrix} = 0
\]
if and only if
\[
g_{a_k}(x_{i-1}) = 0 \quad \text{and} \quad g_{a_k}(x_i) = 0, \ k = 1, 2, 3. \tag{2}
\]
Moreover, there exists at most one function \( g_{a_k} \), \( 0 < k \leq 3 \) such that (2) holds, since there exist no common zeros in \((0, R]\) for any two members in the sequence \( \{g_a\}_{a=0}^{\infty} \). Thus we may assume that
\[
\begin{vmatrix}
g_{a_0}(x_{i-1}) & g_{a_i}(x_{i-1}) \\
g_{a_0}(x_i) & g_{a_i}(x_i)
\end{vmatrix} \neq 0 \quad \text{for } l = 1, 2.
\]
Similar consideration as above leads to the conclusion that
\[
\begin{vmatrix}
g_{\alpha_0}(x_{i-1}) & g_{\alpha_1}(x_{i-1}) & g_{\alpha_k}(x_{i-1}) \\
g_{\alpha_0}(x_i) & g_{\alpha_1}(x_i) & g_{\alpha_k}(x_i) \\
g'_{\alpha_0}(x_{i-1}) & g'_{\alpha_1}(x_{i-1}) & g'_{\alpha_k}(x_{i-1}) 
\end{vmatrix} = 0, \quad 1 < k \leq 3
\]

if and only if
\[
g_{\alpha_k}(x_{i-1}) = 0, \quad g_{\alpha_k}(x_i) = 0 \quad \text{and} \quad g'_{\alpha_k}(x_{i-1}) = 0, \quad 1 < k \leq 3. \quad \text{...(3)}
\]

Also, there exists at most one \( g_{\alpha_k} \) \( 0 < k \leq 3 \) satisfying (3).

If there exists none of \( g_{\alpha_k} \) \( k = 0, 1, 2, 3 \) satisfying (3), then define the function
\[
F(x) = \begin{vmatrix}
V_{i-1} \\
V_i \\
V_{i-1} \\
V(x)
\end{vmatrix} \quad \text{for} \quad x \in [x_0, x_n].
\]

Then \( F(x) \) is a generalized polynomial with at least two terms. Further
\[
\begin{vmatrix}
V_{i-1} \\
V_i \\
V_{i-1}' \\
V_i'
\end{vmatrix} = 0
\]

implies that \( F(x) \) has two double zeros \( x_{i-1} \) and \( x_i \), which contradicts the property \( D \) of the sequence \( \{g_{\alpha}\}_{\alpha=0}^{\infty} \).

In case there exists a function, say \( g_{\alpha_3} \), satisfying (3), then by (1)
\[
g'_{\alpha_3}(x_i) \neq 0.
\]

Define
\[
H(x) = \begin{vmatrix}
V_{i-1} \\
V_i \\
V(x) \\
V_i'
\end{vmatrix} \quad \text{for} \quad x \in [x_0, x_n].
\]
Again, that
\[
\begin{vmatrix}
  V_{i-1} \\
  V_i \\
  V'_i \\
  V''_i \\
\end{vmatrix} = 0
\]
leads to the same contradiction as above. Thus the lemma is proved.

**Corollary 2.2** — If condition (1) of Lemma 2.1 holds and neither \( x_{i-1} \) nor \( x_i \) is a triple zero for
\[
g_{\alpha_k}, \ k = 0, 1, 2, 3.
\]

Then
\[
\begin{vmatrix}
  V_{i-1} \\
  V_i \\
  V'_i \\
  V''_i \\
\end{vmatrix} \neq 0 \quad \text{and} \quad \begin{vmatrix}
  V_{i-1} \\
  V_i \\
  V'_i \\
  V''_i \\
\end{vmatrix} \neq 0.
\]

**Proof:** If there exists none of \( g_{\alpha_k}, \ k = 0, 1, 2, 3; \) such that
\[
g_{\alpha_k}(x_{i-1}) = 0, \ g_{\alpha_k}(x_i) = 0 \quad \text{and} \quad g'_{\alpha_k}(x_i) = 0.
\]

Then
\[
\begin{vmatrix}
  V_{i-1} \\
  V_i \\
  V'_i \\
  V''_i \\
\end{vmatrix} = 0
\]
implies that the following generalized polynomial
\[
F(x) = \begin{vmatrix}
  V_{i-1} \\
  V_i \\
  V'_i \\
  V(x) \\
\end{vmatrix}
\]
has a zero \( x_{i-1} \) and a triple zero \( x_i \) which contradicts the property \( \mathcal{D} \) of the sequence \( \{g_{a_i}\}_{a=0}^\infty \).

Suppose there exists a function, say \( g_{a_3} \), such that (5) holds. By (4)

\[
g_{a_3}'(x_i) \neq 0.
\]

Then

\[
\begin{vmatrix}
V_{i-1} \\
V_i \\
V'_i \\
V''_i
\end{vmatrix} = 0
\]

implies that the following generalized polynomial

\[
H(x) = \begin{vmatrix}
V_{i-1} \\
V_i \\
V(x) \\
V''_i
\end{vmatrix}
\]

has a zero \( x_{i-1} \) and a triple zero \( x_i \) which contradicts the property \( \mathcal{D} \) of the sequence \( \{g_{a_i}\}_{a=0}^\infty \).

That

\[
\begin{vmatrix}
V_{i-1} \\
V_i \\
V''_{i-1} \\
V''_{i-1}
\end{vmatrix} \neq 0
\]

may be proved in a similar way.

**Theorem 2.3** — For given \( 0 < x_{i-1} < x_i \leq R, \ a_0, \ a_1, \ a_2, \ a_3 \) and constants \( f_{i-1}, \ f_i, \ f'_{i-1} \) and \( f''_i \), if \( x_{i-1}, x_i \) and \( g_{a_k}, \ k = 0, 1, 2, 3 \); satisfy (1) and (4), then there exists a unique generalized polynomial

\[
F(x) = \sum_{k=0}^{3} C_k g_{a_k}(x)
\]
such that

\[ F(x_{i-1}) = f_{i-1}, \ F(x_i) = f_i, \ F'(x_{i-1}) = f'_{i-1} \text{ and } F'(x_i) = f'_i. \quad ... (6) \]

**Proof:** Let

\[
\Delta_{i-1} = \begin{vmatrix} V_{i-1} \\ V_i \\ V'_{i-1} \\ V'_i \end{vmatrix}.
\]

Define

\[
F(x) = \begin{vmatrix} V(x) \\ V_i \\ V'_{i-1} \\ V'_i \end{vmatrix} \frac{f_{i-1}}{\Delta_{i-1}} + \begin{vmatrix} V_{i-1} \\ V(x) \\ V'_{i-1} \\ V'_i \end{vmatrix} \frac{f_i}{\Delta_{i-1}} + \begin{vmatrix} V_{i-1} \\ V_i \\ V'_{i-1} \\ V'_i \end{vmatrix} \frac{f'_{i-1}}{\Delta_{i-1}} + \begin{vmatrix} V_{i-1} \\ V_i \\ V'_{i-1} \\ V'_i \end{vmatrix} \frac{f'_i}{\Delta_{i-1}}.
\]

By Lemma 2.1

\[ \Delta_{i-1} \neq 0. \]

Clearly, \( F(x) \) is a generalized polynomial of the form \( \Sigma_{k=0}^3 C_k g_{a_k}(x) \) which satisfies (6). The uniqueness follows from the fact that

\[ \Delta_{i-1} \neq 0. \]

**Theorem 2.4** — Let \( 0 < x_0 < ... < x_n \leq R \), let \( a_0, a_1, a_2, a_3 \) be given non-negative integers and \( f \in C'_{[x_0, x_n]} \). Suppose further that \( x_{i-1}, x_i, i = 1, 2, ..., n \); and \( g_{a_k}, k = 0, 1, 2, 3 \); satisfy (1) and (4) and any generalized polynomials \( g, h \) of the form \( \Sigma_{k=0}^3 C_k g_{a_k}(x) \) satisfy the following conditions:

(iii) If \( g(a) = 0, g'(a) = 0 \) and \( g(b) = 0 \), then there exists an odd number of inflection points between \( a \) and \( b \).

(iv) If \( g(a) = 0, g'(a) = 0, g(b) = 0 \) and

\[
h(a) = 0, h(b) = 0, h'(b) = 0
\]

then

\[
\left| \frac{g'(b)}{g'(b)} \right| \geq \left| \frac{h'(b)}{h'(a)} \right|.
\]
Then there exists a unique function \( F \in \mathcal{C}^2_{[x_0, x_n]} \) such that

\[
F(x_i) = f(x_i) \quad i = 0, 1, \ldots, n, \quad F'(x_0) = f'(x_0), \quad F'(x_n) = f'(x_n);
\]

and in each interval \([x_{i-1}, x_i] \), \( i = 1, 2, \ldots, n \), \( F \) agrees with a generalized polynomial containing at most four terms \( g_{a_0}, g_{a_1}, g_{a_2} \) and \( g_{a_3} \).

**Remarks:** (1) By the property (9) of the sequence \( \{g_x\}_{x=0}^\infty \), the generalized polynomial of four terms can have at most three zeros. Condition (iii) implies that for any double zero \( a \) and single zero \( b \) of such a generalized polynomial \( g \) the following holds:

\[
g''(a) \cdot g''(b) \leq 0 \quad \text{and} \quad g''(a) \neq 0.
\]

(2) The graphs of the generalized polynomials

\[
\frac{g(x)}{g'(b)} \quad \text{and} \quad \frac{h(x)}{h'(a)}
\]

in (iv) are as indicated in Fig. 1 and Fig. 2, respectively.

**Fig. 1.**

**Fig. 2.**

Condition (iv) implies that the curvature of \( \frac{g(x)}{g'(b)} \) at \( b \) is greater than that of \( \frac{h(x)}{h'(a)} \) at \( b \).

**Proof of Theorem 2.4** — Define

\[
F(x) = \begin{vmatrix}
V(x) & V_{i-1} \\
V_i & V'_{i-1} \\
V'_{i} & \frac{f_i}{\Delta_{i-1}} + \frac{f_{i-1}}{\Delta_i}
\end{vmatrix} + \begin{vmatrix}
V_{i-1} \\
\frac{U_{i-1}}{\Delta_{i-1}} \\
\frac{U_i}{\Delta_i}
\end{vmatrix} + \begin{vmatrix}
V_i \quad V'(x) \\
V'_{i} \quad V'_{i-1} \\
\frac{V'_i}{V(x)} \quad V'(x)
\end{vmatrix}
\]

for \( x \in [x_{i-1}, x_i] \) \( \ldots(7) \)

where

\[
\Delta_{i-1} = \begin{vmatrix}
V_{i-1} \\
V_i \\
V'_{i-1} \\
V'_i
\end{vmatrix} \quad [i = 1, 2, \ldots, n]
\]
and \( f_i = f(x_i), \ i = 0, 1, \ldots, n; \ U_0 = f'(x_0), \ U_n = f'(x_n) \) with \((n - 1)\) quantities \( U_i \) \((i = 1, 2, \ldots, (n - 1))\) to be determined.

Upon equating \( F''(x_i) \) as calculated on \([x_{i-1}, x_i]\) and \([x_i, x_{i+1}]\), we have the following \((n - 1)\) linear equations:

\[
\begin{vmatrix}
V_i^* \\
V_i \\
V_{i-1} \\
V_i^* \\
\end{vmatrix}
= \begin{vmatrix}
V_{i+1}^* \\
V_i \\
V_i^* \\
V_{i+1} \\
\end{vmatrix}
\begin{vmatrix}
f_{i-1} \\
\frac{f_i}{\Delta_i} \\
\frac{U_{i-1}}{\Delta_{i-1}} \\
U_i \\
\end{vmatrix}
\begin{vmatrix}
V_{i-1} \\
V_i \\
V_i^* \\
V_{i-1} \\
\end{vmatrix}
\begin{vmatrix}
V_{i+1} \\
V_i^* \\
\frac{U_i}{\Delta_i} \\
V_i^* \\
\end{vmatrix}
\begin{vmatrix}
V_i \\
V_{i+1} \\
V_i^* \\
V_i \\
\end{vmatrix}
\begin{vmatrix}
V_i \\
V_{i+1} \\
V_i \\
V_i^* \\
\end{vmatrix}
\]

\( i = 1, 2, \ldots, (n - 1). \) \( \ldots(8) \)

The matrix of the system \((8)\) is tridiagonal with elements

\[
a_{i,j-1} = \frac{V_{j-1}}{\Delta_{i-1}}, \quad j = 2, 3, \ldots, (n-1)
\]

\[
a_{i,j} = \begin{vmatrix}
V_{j-1} \\
V_j \\
V_j^* \\
V_{j-1} \\
\end{vmatrix}
\begin{vmatrix}
V_j \\
V_{j+1} \\
V_j^* \\
V_j \\
\end{vmatrix}
\]

\( j = 1, 2, \ldots, (n-1) \) and

\[
a_{i,j+1} = -\frac{V_j}{\Delta_j}, \quad j = 1, 2, \ldots, (n-2)
\]
Each of the following four generalized polynomials

\[
g_1(x) = \begin{vmatrix} V_{j-1} \\ V_j \\ V'_{j-1} \\ V(x)_{j-1} \end{vmatrix}, \quad j = 1, 2, \ldots, (n-1); \quad g_2(x) = -\begin{vmatrix} V_j \\ V_{j+1} \\ V(x) \\ V'_{j+1} \end{vmatrix}, \quad j = 1, 2, \ldots, (n-1); \]

\[
h_1(x) = \begin{vmatrix} V_{j-1} \\ V_j \\ V(x) \\ V'_{j-1} \end{vmatrix}, \quad j = 2, 3, \ldots, (n-1); \quad h_2(x) = -\begin{vmatrix} V_j \\ V_{j+1} \\ V(x) \\ V'_{j} \end{vmatrix}, \quad j = 1, 2, \ldots, (n-2);
\]

...(10)

has a double and a single zero with the derivative at the single zero either 1 or -1. The graphs of \(g_1, g_2, h_1\) and \(h_2\) at their zeros are shown in (a), (b), (c) and (d), Fig. 3, respectively.

\[\begin{array}{c}
\text{(a)} \\
\text{(b)} \\
\text{(c)} \\
\text{(d)} \\
\end{array}\]

\text{Fig. 3.}

By applying \(\text{(iii)}\) to \(g_1, g_2, h_1\) and \(h_2\), we have

\[
g_1'(x_j) \geq 0 \quad j = 1, 2, \ldots, (n-1); \quad g_2'(x_j) \geq 0 \quad j = 1, 2, \ldots, (n-1)
\]

\[
h_1'(x_j) > 0 \quad j = 2, 3, \ldots, (n-1); \quad h_2'(x_j) > 0 \quad j = 1, 2, \ldots, (n-2)
\]

...(11)

By applying \(\text{(iv)}\) to \(g_2\) and \(h_k(k = 1, 2)\), we have
\[ g_1^*(x_i) = \begin{cases} \frac{g_1^*(x_i)}{g'(x_i)} & \frac{h_1^*(x_i)}{h'_1(x_{j-1})} = h_1^*(x_i), j = 2, 3, \ldots, (n - 1); \\ \frac{g_1^*(x_i)}{g_2^*(x_i)} & \frac{h_1^*(x_i)}{h_2^*(x_{j+1})} = h_1^*(x_i), j = 1, 2, \ldots, (n - 2) \end{cases} \]

That every entry in the matrix of the linear system (8) is non-negative follows from (9), (10) and (11). (12) implies that the matrix is diagonally dominant. Since a tridiagonal, diagonally dominant with non-negative entries is regular (see Rivlin 1969, pp. 107), system (8) has unique solution \{U_1, U_2, \ldots, U_{n-1}\}. With \( U_0 = f'(x_0), U_n = f'(x_n) \) and \( f_i = f(x_i) \) \( i = 0, 1, 2, \ldots, n \) the function (7) is uniquely constructed which satisfies every requirement for Theorem 2.4. Thus the theorem is proved.

(3) Example — Let \( f(x) = \sin x, x_0 = \frac{\pi}{10}, x_1 = \frac{\pi}{2}, x_2 = \pi, g_{\alpha_0}(x) = 1 \),

\[ g_{\alpha_1}(x) = x^{1/4}, g_{\alpha_2}(x) = x^{2/3}, g_{\alpha_3}(x) = x^{4/5}. \]

The generalized polynomial constructed as in Theorem 2.4 is the following (coefficients are rounded off at fifth digit):

\[ F(x) = \begin{cases} 8.11367 - 23.19826x^{1/4} + 48.84553x^{2/3} - 32.85175x^{4/5} & x \in \left[ \frac{\pi}{10}, \frac{\pi}{2} \right] \\ -0.28215 - 5.74643x^{1/4} + 26.47984x^{2/3} - 19.55697x^{4/5} & x \in \left[ \frac{\pi}{2}, \pi \right] \end{cases} \]

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