ESSENTIALLY (R), ESSENTIALLY (H₁) AND ESSENTIALLY SPECTRALOID OPERATORS ON HILBERT SPACE

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Sufficient conditions for an operator to be essentially (R), essentially (H₁) and essentially spectraloid are obtained. It is shown that R is a proper subset of e(R), e(G₁) is a proper subset of e(H₁), but classes e(R) and e(G₁), e(H₁) and H₁, e(S) and S are non-comparable.

§1. Let \( B(H) \) denote the set of all bounded linear transformations from Hilbert space \( H \) into \( H \). Let \( \sigma(T), \pi_{00}(T), \bar{\sigma}(T), \text{ con } \sigma(T), \bar{W}(T), r(T) \) and \( |W(T)| \) respectively denote the spectrum, the set of all isolated points in \( \sigma(T) \) that are eigen values of finite multiplicity, the hen-spectrum [complement of the unbounded component of the complement of \( \sigma(T) \) (Fujii 1971, 1973)] the convex hull of the spectrum, the closure of the numerical range, the spectral radius and the numerical radius of an operator \( T \). An operator

\[
T \in R \text{ if } \| (T - zI)^{-1} \| = 1/d(z, W(T)), z \notin \bar{W}(T) \quad \text{(Luecke 1972b)}
\]

and

\[
T \in H₁ \text{ if } \| (T - zI)^{-1} \| = 1/d(z, \bar{\sigma}(T)), z \notin \bar{\sigma}(T) \quad \text{(Fujii 1971, 1973)}
\]

Let \( \pi \) be the quotient map from \( B(H) \) onto the Calkin algebra \( B(H)/K \), where \( K \) denotes the set of all compact operators in \( B(H) \). An operator \( T \in B(H) \) is essentially \( R, H₁ \) or a spectraloid according as \( \pi(T) \) is an element of \( R, H₁ \) or spectraloid. We denote each of these sets by \( e(R), e(H₁) \) and \( e(S) \) respectively. Let \( \sigma_e(T), \sigma_e(T), W_e(T), \bar{\sigma}_e(T), r_e(T) \) and \( |W_e(T)| \) denote the essential spectrum, the left essential spectrum, the essentially numerical range (Fillmore et al. 1972), the essential hen-spectrum, the essential spectral radius and the essential numerical radius of an operator \( T \).

Luecke (1975) proved the following basic results:

Theorem A — If \( T = A \oplus B \) on \( H \oplus H \), then

(i) \( \sigma_e(T) = \sigma_e(A) \cup \sigma_e(B) \)

(ii) \( W_e(T) = \text{ con } (W_e(A) \cup W_e(B)) \)

(iii) \( \| \pi(T) \| = \max \{ \| \pi(A) \|, \| \pi(B) \| \} \).
Using these properties sufficient conditions for an operator to be essentially $G_1$ and essentially convexoid are obtained. It is further shown by Luecke (1975) that $e(G_1)$ and $G_1$; $e(C)$ and $C$, are non-comparable, where $e(G_1)$ and $e(C)$ denote the classes of essentially $G_1$ and essentially convexoid operators.

In this note, we obtain sufficient conditions for an operator to be essentially $R$, essentially $H_1$ and essentially spectraloid. It is remarkable to note that class $R$ is a proper subset of $e(R)$, while classes $e(R)$ and $e(G_1)$, $e(H_1)$ and $H_1$, $e(S)$ and $S$ are non-comparable.

§2. It is known (Acharya 1980) that if $T = A \oplus B$ be defined on $H \oplus H$, then

(i) $B \in R$ with $\bar{W}(A) \subseteq \bar{W}(B)$ implies that $T \in R$

(ii) $B \in H_1$ with $\bar{W}(A) \subseteq \bar{\sigma}(B)$ implies that $T \in H_1$.

We have the sufficient conditions for an operator to be in $e(R)$ and $e(H_1)$ as follows:

**Theorem 1** — If $T = A \oplus B$ on $H \oplus H$, where $B$ is essentially $R$ with $W_e(A) \subseteq W_e(B)$ then $T$ is essentially $R$.

**Theorem 2** — If $T = A \oplus B$ on $H \oplus H$, where $B$ is essentially $H_1$ with $W_e(A) \subseteq \widetilde{\sigma}(B)$ then $T$ is essentially $H_1$.

Proofs for both the Theorems can be constructed on the same lines as in Luecke (1975, Theorem 3). For completeness we give the proof for Theorem 2 as follows:

**Proof:** Here $\widetilde{\sigma}(T) = \widetilde{\sigma}(A) \cup \widetilde{\sigma}(B) = \widetilde{\sigma}(B)$.

For $z \notin \widetilde{\sigma}(B)$, $\| (\pi(A) - zI)^{-1} \| \leq 1/d(z, W_e(A))$

$\leq 1/d(z, \widetilde{\sigma}(B))$.

Now $\| (\pi(T) - zI)^{-1} \|$ = max \{ $\| (\pi(A) - zI)^{-1} \|$, $\| (\pi(B) - zI)^{-1} \|$ \}

= max \{ $\| (\pi(A) - zI)^{-1} \|$, $1/d(z, \widetilde{\sigma}(B))$ \}

= $1/d(z, \widetilde{\sigma}(B))$

= $1/d(z, \widetilde{\sigma}(T))$.

Therefore, $T$ is essentially $H_1$. 
If $T = A \oplus B$ on $H \oplus H$ and $B$ is a spectraloid with $|W(A)| \leq r(B)$, then $T$ is spectraloid (Acharya 1980). We have the following:

**Theorem 3** — If $T = A \oplus B$ on $H \oplus H$ and $B$ is essentially spectraloid with $|W_e(A)| \leq r_e(B)$, then $T$ is essentially spectraloid.

**Proof:** Since $B$ is essentially spectraloid,

$$r_e(B) = |W_e(B)|.$$

Now,

$$r_e(T) = \max \{r_e(A), r_e(B)\} = r_e(B)$$

and

$$|W_e(T)| = \max \{|W_e(A)|, |W_e(B)|\} = |W_e(B)| = r_e(B).$$

Thus, $r_e(T) = |W_e(T)|$. Hence $T$ is essentially spectraloid.

§3. According to Putnam (1968)

$$\partial \sigma(T) \subseteq \sigma_{le}(T) \cup \pi_{oo}(T)$$

where $\partial M$ denotes the boundary of a set $M$. It is known that $T \in R$ if and only if $\partial W(T) \subseteq \sigma(T)$ (Luecke 1972b). It is not difficult to observe that $T \in e(R)$ if and only if $\partial W_e(T) \subseteq \sigma_e(T)$. Now we use these results to show the following:

**Theorem 4** — Class $R$ is a proper subset of $e(R)$.

**Proof:** Let $T \in R$. Hence $\partial W(T) \subseteq \sigma(T)$.

Now

$$\partial W(T) \subseteq \partial \sigma(T) \subseteq \sigma_{le}(T) \cup \pi_{oo}(T) \quad \text{(Putnam 1968)}.$$

Further $\partial W(T) \cap \pi_{oo}(T) = \emptyset$. Hence

$$\partial W(T) \subseteq \sigma_{le}(T) \subseteq \sigma_e(T) \subseteq W_e(T) \text{ and } W_e(T)$$

is a convex set which is again a subset of $W_e(T)$. Therefore, $\partial W(T) = \partial W_e(T)$ or $\partial W_e(T) \subseteq \sigma_e(T)$, i.e. $T \in e(R)$.

To show that this inclusion is proper, consider \[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\oplus 0 \text{ on } H = M \oplus M^\perp,
\]

where dimension of $M$ is two. Here $\pi(T) = 0$ so that $T$ is essentially $R$, but $T \notin R$.

Using the technique of Luecke (1975, Theorem 10) we give a non-trivial example to show the following:

**Theorem 5** — $e(G_3)$ is a proper subset of $e(H_3)$.

**Proof:** Consider $T = A \oplus N$ on $(M_1 \oplus M_2) \oplus M_3$, (each $M_i$ has infinite dimensions) with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $N$, a normal operator with $\sigma(N) = C$, where $C$ is unit circle in the complex plane.
Now
\[ \| \pi(T - zI)^{-1} \| \]
\[ = \max \{ \| \pi(A - zI)^{-1} \|, \| \pi(N - zI)^{-1} \| \} \]
\[ \geq \| \pi(A - zI)^{-1} \| \]
\[ = \| (\pi(A) - z)^{-1} \| \]
\[ = \left\| \left( \begin{array}{cc} -1/z & 1/z^2 \\ 0 & -1/z \end{array} \right) \right\| \]
\[ = \left\| \left( \begin{array}{cc} -1/z & 1/z^2 \\ 0 & -1/z \end{array} \right) \right\| \]
\[ \geq \frac{1}{|z|^2} \]

and \( \sigma_c(T) = \{0\} \cup C \). Choose \( z = 1/10 \notin \sigma_c(T) \).

Then \( 1/d(z, \sigma_c(T)) = 10 \). But \( \| \pi(T - zI)^{-1} \| > 100 \). Hence \( T \notin e(G_1) \). However \( \partial W_c(T) \subseteq \sigma_c(T) \) implies that \( T \in e(R) \subseteq e(H_1) \).

**Corollary 1** — Classes \( e(G_1) \) and \( e(R) \) are non-comparable.

With the help of technique given in examples in Theorems 6 and 7 of Luecke (1975) it is easy to see that:

**Corollary 2** — Classes \( H_1 \) and \( e(H_1) \) are non-comparable.

**Corollary 3** — Classes \( S \) and \( e(S) \) are non-comparable.

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**REFERENCES**


