NEW CRITERIA FOR $p$-VALENCE

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In this paper we consider the classes $T_{n+p-1}(\alpha)$ of functions

$$f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \ldots,$$

regular in the unit disc $D$ and satisfying

$$\text{Re} \left( \frac{(D^{n+p-1}f)'}{p z^{p-1}} \right) > \alpha, \ 0 \leq \alpha < 1, \ z \in D$$

where $D^{n+p-1}f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z)$. It is proved that

$$T_{n+p}(\alpha) \subset T_{n+p-1}(\alpha).$$

Since $T_0(\alpha)$ is the class of functions $f(z)$, with $\text{Re} \left( \frac{f'(z)}{pz^{p-1}} \right) > \alpha$ all functions in $T_{n+p-1}(\alpha)$ are $p$-valent.

1. Introduction

Let $A(p)$ denote the class of functions

$$f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \ldots, \quad (1.1)$$

$p$ a positive integer which are regular in the unit disc $D = \{z : |z| < 1\}$. Let

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}z^{p+n},$$

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n}z^{p+n}$$

belong to $A(p)$. We denote the Hadamard product or the convolution of $f$ and $g$ by

$$(f \ast g)(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}b_{p+n}z^{p+n}.$$

In this paper we shall prove that a function $f \in A(p)$ and satisfy one of the conditions

$$\text{Re} \left( \frac{(D^{n+p-1}f(z))'}{p z^{p-1}} \right) > \alpha, \ 0 \leq \alpha < 1, \ z \in D \quad (1.2)$$
\[ D^{n+p-1}f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z) \]

is \( p \)-valent in \( E \). We denote by \( T_{n+p-1}(\alpha) \) the classes of functions \( f(z) \in A(p) \) and satisfying (1.2). It is easy to see that

\[ z(D^{n+p-1}f(z))' = (n + p)D^{n+p}f(z) - nD^{n+p-1}f(z). \]

Using (1.4) condition (1.2) can be re-written in the form

\[ \text{Re} \left\{ (n + p) \frac{D^{n+p}f(z)}{pz^p} - n \frac{D^{n+p-1}f(z)}{pz^p} \right\} > \alpha, \quad z \in E. \]

We shall show that

\[ T_{n+p}(\alpha) \subset T_{n+p-1}(\alpha). \]

Since \( T_0(\alpha) \) is the class of functions which satisfy the condition

\[ \text{Re} \left\{ \frac{f'(z)}{pz^{\alpha-1}} \right\} > \alpha \geq 0 \]

and we know from (Umezawa 1957) that such functions are \( p \)-valent, the \( p \)-valence of functions in \( T_{n+p-1}(\alpha) \) follows from (1.6).

By putting \( p = 1 \), we shall get the criteria for univalence.

2. The Classes \( T_{n+p-1}(\alpha) \)

**Theorem 1** — \( T_{n+p}(\alpha) \subset T_{n+p-1}(\alpha), \ 0 \leq \alpha < 1, \ n \) is any integer greater than \( p \).

We need the following lemma due to Jack (1971).

**Lemma 1** — Let \( w(z) \) be non-constant and regular in \( |z| < 1, \ w(0) = 0 \). Then if \( |w(z)| \) attains its maximum value on the circle \( |z| = r < 1 \), at \( z_0 \), we can write

\[ z_0 w'(z_0) = kw(z_0) \]

where \( k \) is a real number greater than or equal to one.

**Proof of Theorem 1** — Let \( f(z) \in T_{n+p}(\alpha) \). Define a regular function \( w(z) \) in \( E \) such that \( w(0) = 0, \ w(z) \neq -1 \) by

\[ (n + p) D^{n+p}f(z) - nD^{n+p-1}f(z) = pz^p \frac{1 + (2\alpha - 1) w(z)}{1 + w(z)}. \]

Differentiating (2.1), we get

\[ \frac{(D^{n+p}f(z))'}{pz^{\alpha-1}} = \frac{1 + (2\alpha - 1) w(z)}{1 + w(z)} - \frac{2(1 - \alpha) zw'(z)}{n + p (1 + w(z))^2} \]
We claim that $|w(z)| < 1$ for all $z \in E$. For, otherwise, by Lemma 1, there exists a $z_0$, $|z_0| < 1$ such that

$$z_0 w'(z_0) = kw(z_0)$$

...(2.3)

with $|w(z_0)| = 1$ and $k \geq 1$.

(2.2) in conjunction with (2.3) gives

$$\frac{(D^{n+p}f(z_0))'}{pz_0^{n-1}} = \frac{1 + (2\alpha - 1)w(z_0)}{1 + w(z_0)} - \frac{2(1 - \alpha)}{n + p} \cdot \frac{kw(z_0)}{(1 + w(z_0))^2}. \quad \text{(2.4)}$$

Since $\text{Re} \frac{1 + (2\alpha - 1)w(z_0)}{1 + w(z_0)} = \alpha$, $k \geq 1$ and $\frac{w(z_0)}{(1 + w(z_0))^2}$ is real and positive, we see that $\text{Re} \frac{(D^{n+p}f(z_0))'}{pz_0^{n-1}} < \alpha$. This contradicts the hypothesis that $f(z) \in T_{n+p}(\alpha)$.

Hence $|w(z)| < 1$, $z \in E$ and it follows from (2.1) that $f(z) \in T_{n+p-1}(\alpha)$.

**Theorem 2** — Let $c$ be any real number greater than $p$. If $f(z) \in T_{n+p-1}(\alpha)$, then

$$F(z) = \frac{c + p}{z^c} \int_0^z t^{c-1}f(t) \, dt \in T_{n+p-1}(\alpha), \text{ for } c + p > 0.$$  

**Proof**: One can easily verify that the function $F(z)$ satisfies

$$z(D^{n+p-1}F(z))' = (p + c) D^{n+p-1}f(z) - c D^{n+p-1}F(z). \quad \text{(2.5)}$$

Define a regular function $w(z)$ in $E$ by

$$\frac{(D^{n+p-1}F(z))'}{z^{n-1}} = p \cdot \frac{1 + (2\alpha - 1)w(z)}{1 + w(z)}. \quad \text{(2.6)}$$

Obviously $w(0) = 0$, $w(z) \neq -1$ for $z \in E$.

Using (1.4), (2.6) can be re-written as

$$(n + p) D^{n+p}F(z) - n D^{n+p-1}F(z) = p z^{p} \cdot \frac{1 + (2\alpha - 1)w(z)}{1 + w(z)}. \quad \text{(2.7)}$$

Differentiating (2.7) and using (2.5), we get

$$\frac{(D^{n+p-1}f(z))'}{pz^{n-1}} = \frac{1 + (2\alpha - 1)w(z)}{1 + w(z)} - \frac{2(1 - \alpha)}{p + c} \cdot \frac{zw'(z)}{(1 + w(z))^2}. \quad \text{(2.8)}$$

Now proceeding as in Theorem 1, we can show that $F(z) \in T_{n+p-1}(\alpha)$.

**Theorem 3** — Let $f(z) \in A(p)$ and satisfy the condition

$$\text{Re} \frac{(D^{n+p-1}f(z))'}{pz^{n-1}} > \alpha - \frac{(1 - \alpha)}{2(p + c)} \cdot \frac{zw'(z)}{(1 + w(z))^2}, \text{ for } c + p > 0.$$
Then the function

\[ F(z) = \frac{p + c}{z^{p}} \int_{0}^{z} t^{z-1} f(t) \, dt \in T_{n+p-1}(z). \]

The proof of this theorem is similar to that of Theorem 2 and so we omit it.

**Corollary 3(a)** — By putting \( n + p = c = 1 \) and \( \alpha = 0 \) in Theorem 3, it follows that if \( f(z) \in A(p) \) and satisfies the condition

\[ \text{Re} \left( \frac{f'(z)}{pz^{p-1}} \right) > -\frac{1}{2(p + 1)} \]

then

\[ \text{Re} \left( \frac{F'(z)}{pz^{p-1}} \right) > 0 \]

and hence \( F(z) \) is \( p \)-valent in \( E \).

**Corollary 3(b)** — Taking \( n + p = c = 1 \) and \( \alpha = 1/(2p + 3) \), we see that if

\[ \text{Re} \left( \frac{f'(z)}{pz^{p-1}} \right) > 0, \text{ then } \text{Re} \left( \frac{F'(z)}{pz^{p-1}} \right) > \frac{1}{2p + 3}. \]

**Remark**: By taking \( p = 1 \) in Corollary 3(a) and Corollary 3(b) we get the following results:

(i) \( \text{Re} \, f'(z) \geq -\frac{1}{2} \) implies \( \text{Re} \, F'(z) > 0 \);

(ii) \( \text{Re} \, f'(z) > 0 \) implies \( \text{Re} \, F'(z) > \frac{1}{5} \).

Both these results are extensions of an earlier result due to Libera (1965) viz.; \( \text{Re} \, f'(z) > 0 \) implies \( \text{Re} \, F'(z) > 0 \).

3. **Converse of Theorem 2**

In this section we prove the converse of Theorem 2.

**Theorem 4** — Let \( c + p > 0 \) and \( f(z) \) be defined by

\[ F(z) = \frac{p + c}{z^{p}} \int_{0}^{z} t^{z-1} f(t) \, dt, \ c + p > 0. \]

If \( F(z) \in T_{n+p-1}(z) \), then \( f(z) \in T_{n+p-1}(z) \) in \( |z| < \frac{p + c}{1 + \sqrt{(p + c)^2 + 1}} \). The result is sharp.

**Proof**: Since \( F(z) \in T_{n+p-1}(z) \), we can write

\[ z(D^{n+p-1}F(z)) = pz^{p}[z + (1 - \alpha) u(z)] \quad \ldots (3.1) \]
where \( u(z) \in P \), the class of functions with positive real part in the unit disc \( E \) and normalized by \( u(0) = 1 \). We can re-write (3.1) as

\[
(n + p) \ D^{n+p} F(z) - nD^{n+p-1} F(z) = pz^{p} [\alpha + (1 - \alpha) u(z)]. \quad \quad \quad \quad \quad (3.2)
\]

Differentiating (3.2) and making use of (2.5) we get after a simple computation

\[
\left( \frac{(D^{n+p-1} f(z))'}{pz^{p-1}} - \alpha \right) (1 - \alpha)^{-1} = u(z) + \frac{1}{p + c} \ z u'(z). \quad \quad \quad \quad \quad \quad \quad \quad \quad (3.3)
\]

Using the well-known estimate \( |zu'(z)| \leq \frac{2r}{1 - r^2} \ \text{Re} u(z), \ |z| = r \), (3.3) yields

\[
\text{Re} \left\{ \left( \frac{(D^{n+p-1} f(z))'}{pz^{p-1}} - \alpha \right) (1 - \alpha)^{-1} \right\} \geq \left( 1 - \frac{1}{p + c} \right) \frac{2r}{1 - r^2} \ \text{Re} u(z). \quad \quad \quad \quad \quad (3.4)
\]

The right-hand side of (3.4) is positive if \( r < \frac{p + c}{1 + \sqrt{(p + c)^2 + 1}} \). The result is sharp for the function \( f(z) \) defined by

\[
f(z) = \frac{z^{1-\alpha}}{p + c} \ (z^{\alpha} F(z))'.
\]

where \( F(z) \) is given by

\[
(D^{n+p-1} F(z))' = pz^{p-1} \left( \frac{1 + (2\alpha - 1) z}{1 + z} \right).
\]

**Corollary 4(a)** — By Putting \( n + p = 1 \) and \( \alpha = 0 \), we see that if

\[
\text{Re} \frac{F'(z)}{pz^{p-1}} > 0, \ z \in E, \ \text{then} \ \text{Re} \frac{f'(z)}{pz^{p-1}} > 0 \text{for} \ |z| < \frac{p + c}{1 + \sqrt{(p + c)^2 + 1}}.
\]

By putting \( c = 1 \), we get the result obtained by Goel (1972).

**References**


