ON THE RATE OF CONVERGENCE OF THE HERMITE-FEJÉR PROCESS
ON THE TCHEBYCHEFF MATRIX OF THE SECOND KIND

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In the present paper the estimate for rate of convergence of the sequence
$H_n(f, x)$ is obtained.

§1. Let $f(x)$ be a continuous function defined on the closed interval $[-1, 1]$ and

$$U_{n-2}(x) = \frac{\sin (n - 1) \theta}{\sin \theta}, \quad \cos \theta = x$$

be the $(n - 2)$th Tchebycheff polynomial of second kind. Let

$$x_{kn} = \cos \frac{(k - 1) \pi}{n - 1}, \quad k = 1, \ldots, n$$

be the zeros of $(1 - x^2) U_{n-2}(x)$. Then the Hermite-Fejér interpolation polynomial
$H_n(f, x)$ of degree $\leq 2n - 1$ constructed on the nodes $x_{kn}$ is given by

$$H_n(f, x) = f(1) \left[ 1 + \frac{2n^2 - 4n + 3}{3} (1 - x) \right]\left[\frac{1 + x U_{n-2}(x)}{n - 1}\right]^2$$

$$+ f(-1) \left[ 1 + \frac{2n^2 - 4n + 3}{3} (1 + x) \right]\left[\frac{1 - x U_{n-2}(x)}{n - 1}\right]^2$$

$$+ \sum_{k=2}^{n-1} f(x_{kn}) \left[ 1 + \frac{x_{kn}}{1 - x_{kn}} (x - x_{kn}) \right]\left[\frac{(1 - x^2) U_{n-2}(x)}{(n - 1)(x - x_{kn})}\right]^2.$$

Saxena (1974) has shown that the sequence of these polynomials $H_n(f, x)$ converges
uniformly to the given continuous function $f(x)$ in the interval $[-1, 1]$ as $n$ tends to
infinity. The aim of the present paper is to obtain the estimate for rate of convergence
of the sequence $H_n(f, x)$.

Let us denote by $C_* [-1, 1]$ the class of all those functions for which

$$\omega(f; t) \leq C_1 \omega(t)$$

where $\omega(f; t)$ is the modulus of continuity of $f(x)$, $\omega(t)$ is a certain modulus of
continuity and $C_1$ (later on $C_2, C_3, \ldots$) some positive constant. We shall prove
Theorem 1 — If \( f(x) \in C_{\alpha} [-1, 1] \), then for \( x \in [-1, 1] \), we have

\[
| H_n(f, x) - f(x) | = O(1) \sum_{i=1}^{n} \frac{1}{i^2} \omega \left( \frac{i(1 - x^2)^{1/2}}{n} \frac{U_{n-2}(x)}{n} \right).
\]

...(1,3)

The result appears to be interesting in the sense that \( f(x) \) can be approximated arbitrarily at the nodes (1.1) by \( H_n(f, x) \). Further, if \( f(x) \) belongs to Lipschitz class of order \( \alpha \) i.e. \( \omega(t) \leq t^\alpha \), we get from (1,3)

\[
| H_n(f, x) - f(x) | \leq C_2 \frac{(1 - x^2)^{1/2}}{n^\alpha} \frac{|U_{n-2}(x)|}{n} \quad \text{if} \quad 0 < \alpha < 1
\]

and

\[
| H_n(f, x) - f(x) | \leq C_3 \frac{(1 - x^2)^{1/2}}{n} \frac{|U_{n-2}(x)|}{n} \log n \quad \text{if} \quad \alpha = 1.
\]

Evidently, our theorem is the best possible for \( f(x) \in \text{Lip} \alpha (0 < \alpha < 1) \) when \( x \in [-1, 1] \). We shall also show that our estimation is precise for

\[
f(x) \in \text{Lip} 1(- 1 \leq x \leq 1)
\]

by proving the following:

Theorem 2 — There exists a function \( f(x) \) belonging to \( \text{Lip} 1 \) and a constant \( C_4 \) such that

\[
| H_n(f, 0) - f(0) | \geq C_4 \frac{\log n}{n} ; \quad n = 2, 4, 6, \ldots .
\]

In our proof of Theorem 1, we shall need the following lemmas. We shall henceforth write \( x_k \) for \( x_{kn} \) for the sake of simplicity.

**Lemma 1** — Let \( x_i \) denote the nearest root to \( x \), then

\[
\left| \frac{(1 - x^2) U_{n-2}(x)}{(n - 1)(x - x_k)} \right| \leq \begin{cases} \frac{2}{i} & \text{if} \quad k \neq j, \quad k = j \pm 1 \\ 2 & \text{if} \quad k = j \quad \text{or} \quad j - 1 \end{cases}
\]

**Proof:**

\[
\left| \frac{(1 - x^2) U_{n-2}(x)}{(n - 1)(x - x_k)} \right| \leq \frac{2}{(n - 1)} \left| \frac{\sin n \left( \theta - \theta_k \right)}{\sin \left( \frac{n \theta}{2} \right)} \right|
\]

\[
= \frac{2}{i} \quad \text{if} \quad k \neq j, \quad k = j \pm 1
\]

\[
= 2 \quad \text{if} \quad k = j \quad \text{or} \quad j - 1
\]

**Lemma 2** — \( \sum_{k=2}^{n-1} \frac{1}{1 - x_k^2} = \sum_{k=2}^{n-1} \frac{1}{1 - x_k} = \sum_{k=2}^{n-1} \frac{1}{1 + x_k} = \frac{n(n - 2)}{3} \).
For the proof of Lemma 2 refer to Saxena (1974). We shall also use the following property of modulus of continuity, viz.

\[ \omega(f, \lambda \delta) \leq (\lambda + 1) \omega(f, \delta); \lambda \geq 0. \]  \( \text{(1.4)} \)

**Proof of Theorem 1** — Now making suitable arrangements in the representation (1.2), after using Lemma 2, and then owing to the uniqueness of the polynomial, we obtain

\[
H_n(f, x) - f(x) = [f(1) - f(x)] \left(2 - x\right) \left[\frac{(1 + x) \ U_{n-2}(x)}{2(n - 1)}\right]^2
\]

\[
+ [f(-1) - f(x)] \left(2 + x\right) \left[\frac{(1 - x) \ U_{n-2}(x)}{2(n - 1)}\right]^2
\]

\[
+ \left[\frac{(1 + x)(1 - x^2) \ U_{n-2}^2(x)}{2(n - 1)^2}\right] \sum_{k=2}^{n-1} \left[\frac{f(1) - f(x_k)}{1 - x_k}\right]
\]

\[
+ \frac{(1 - x)(1 - x^2) \ U_{n-2}^2(x)}{2(n - 1)^2} \sum_{k=2}^{n-1} \left[\frac{f(-1) - f(x_k)}{1 + x_k}\right]
\]

\[
+ \sum_{k=2}^{n-1} \left[f(x_k) - f(x)\right] \left[\frac{(1 - x^2) \ U_{n-2}(x)}{(n - 1)(x - x_k)}\right]^2
\]

\[
+ \sum_{k=1}^{n-1} \left[\frac{f(x_k) - f(x)}{x - x_k}\right] \left[\frac{x(1 - x^2) \ U_{n-2}^2(x)}{(n - 1)^2}\right]
\]

\[= \sum_{k=1}^{6} S_k(x). \quad \text{(2.1)} \]

First we shall estimate \( S_6(x) \). Thus, using Lemma 1, Property (1.4) and the fact that \(| U_{n-2}(x) | \leq n - 1, |(1 - x^2)^{1/2} U_{n-2}(x)| \leq 1\), we have for \( n > 1 \)

\[
| S_6(x) | \leq \sum_{k=j \pm 1, k \neq j, j-1} \omega \left[\frac{i(1 - x^2)^{1/2} \ U_{n-2}(x)}{n - 1}\right]
\]

\[
\times \left[1 + \frac{(n - 1) \ | x - x_k |}{i(1 - x^2)^{1/2} \ U_{n-1}(x)}\right] \left[\frac{(1 - x^2) \ U_{n-2}(x)}{(n - 1)(x - x_k)}\right]^2
\]

\[
+ \omega \left[\frac{(1 - x^2)^{1/2} \ U_{n-2}(x)}{(n - 1)}\right] \left[1 + \frac{(n - 1) \ | x_j - x |}{(1 - x^2)^{1/2} \ U_{n-2}(x)}\right] \times
\]

(equation continued on p. 1332)
\[ \times \left[ \frac{(1 - x^2)}{(n - 1)(x - x_i)} \right]^2 + \omega \left[ \frac{(1 - x^2)^{1/2}}{n - 1} \right] \]
\[ \times \left[ 1 + \frac{(n - 1)(x_{i-1} - x)}{(1 - x^2)^{1/2}} \right] \left[ \frac{(1 - x^2)U_{n-2}(x)}{(n - 1)(x - x_{i-1})} \right]^2 \]
\[ \leq 6 \sum_{i=1}^{n} \frac{1}{i^2} \omega \left[ \frac{i(1 - x^2)^{1/2}}{n - 1} \right]. \quad \ldots(2.2) \]

Similarly,
\[ |S_k(x)| < 3\omega \left[ \frac{(1 - x^2)^{1/2}}{n - 1} \right]. \quad \ldots(2.3) \]

Again using Lemma 2 and proceeding as in estimate (2.2), we have
\[ |S_k(x)| \leq \frac{8}{3} \omega \left[ \frac{(1 - x^2)^{1/2}}{n - 1} \right]; \quad \text{if } k = 3, 4. \quad \ldots(2.4) \]

For \( k = 1, 2 \) it is easy to show
\[ |S_k(x)| \leq \frac{8}{3} \omega \left[ \frac{(1 - x^2)^{1/2}}{n - 1} \right]. \quad \ldots(2.5) \]

Thus taking into consideration (2.1) and estimates (2.2) – (2.5) we get,
\[ |H_n(f, x) - f(x)| < 17 \sum_{i=1}^{n} \frac{1}{i^2} \omega \left[ \frac{i(1 - x^2)^{1/2}}{n - 1} \right]. \]

This completes the proof of Theorem 1.

**Proof of Theorem 2** — Let \( f(x) = |x| \) and \( n = 2p - 2 \) (\( p \geq 2 \)); then from (2.1) we have
\[ H_n(f, 0) - f(0) = \frac{U_{n-2}^2(0)}{(n - 1)^2} + 2 \sum_{k=2}^{p-1} \frac{1}{(1 + x_k)(n - 1)^2} \]
\[ + \frac{2U_{n-2}^2(0)}{(n - 1)^2} \sum_{k=1}^{p-1} \frac{1}{x_k} \]
\[ \geq \frac{2}{(n - 1)^2} \sum_{k=1}^{p-1} \frac{1}{x_k} \]

and the theorem follows by a similar argument as in Vertesi (1971).
Remark : From Theorem 1 we can easily deduce the following three forms also:

(a) \[ |H_n(f, x) - f(x)| = O(1) \omega \left[ \frac{(1 - x^2)^{1/2} \ |U_{n-2}(x)|}{n \log n} \right] \]

(b) \[ |H_n(f, x) - f(x)| = O(1) \sum_{i=1}^{n} \frac{1}{n} \omega \left[ \frac{(1 - x^2)^{1/2} \ |U_{n-2}(x)|}{i} \right] \]

(c) \[ |H_n(f, x) - f(x)| = O(1) \omega \left[ \frac{(1 - x^2)^{1/2} \ |U_{n-2}(x)|}{n^{1-\epsilon}} \right] \]

\[ \epsilon > 0, \ \text{\epsilon being small and fixed.} \]

These forms are comparable with the corresponding forms given by Moldvan (1954), Bojanic (1972) and Telyakovskii (1966) or Gopengauz (1967).

References


