ON COMPOSITION OF INTEGRAL OPERATORS WITH FOURIER TYPE KERNELS

V. M. Bhise and Madhavi Dighe

Department of Mathematics, G. S. Institute of Technology and Science, Indore 452003, M.P.

(Received 22 March 1979; after revision 18 December 1979)

In this paper an integral operator with a Fourier type kernel is expressed as the composition of two operators, one of which is a fractional integral operator and the other an operator with a Fourier type kernel. The composition is also estimated as a bounded operator from $L^p(R^+)$ to $L^q(R^+)$. 

§1. Composition of different operators has been studied by Okikiolu (1966) while Srivastava and Buschman (1973) have studied the composition of fractional integral operators. In this paper we study the composition of an operator with the Fourier type kernel and a fractional integral operator. Special cases have been obtained which lead to known results by Okikiolu (1966). In the next section, we define the operators, find their composition and give their special cases. In section 3 we establish a theorem for the estimation of the composition of the operators, by treating the composition as a Fourier type integral.

§2. An integral operator is defined by Okikiolu (1971, p. 334) as a Fourier type integral or an operator having a Fourier type kernel, on suitable spaces of measurable functions on $R$, if

$$Tf(x) = x^n \int_R t^n k(xt) f(t) \, dt$$

where $k(t)$ is a suitable kernel. Okikiolu (1966) has obtained the composition of a fractional integral operator and Fourier type integral operators having the sine and cosine kernels. We have generalized these results by considering the $H$-function in the kernels of the Fourier type integral operators. We have the known integral (Anandani 1968)

$$H_{p+1,q+1}^{m+n+1} \left[ c(x^{2 \ell} \beta)^k \left\{ (1 + \frac{1}{2} a - \frac{1}{2} k^{-1}, 1), \{(a_p + \frac{1}{2} a A_p, A_p)\} \right\} \right]$$

$$= \frac{2k}{\Gamma(a + \frac{1}{2})} c^{a/2} x^{a-k-1} \int_0^x (x^{2k} - y^{2k})(2a-1)/2$$

$$\times H_{p,q}^{m,n} \left[ c(t^{2 \ell} \beta)^k \right] \left\{ (a_p, A_p) \right\} \left\{ (b_q, B_q) \right\} \, dy$$

...(2.1)
where \( \{(a_p, A_p)\} \) stands for the set of parameters \((a_1, A_1), (a_2, A_2), \ldots (a_p, A_p)\); \( a + \frac{1}{2} > 0; c, k > 0; m, n, p, q \) are integers satisfying \( 0 \leq m \leq q, 0 \leq n \leq p; A_s \) and \( B_t \) are all positive.

\[
\phi = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \sum_{j=n+1}^{p} A_j - \sum_{j=m+1}^{q} B_j > 0, \quad | \arg c(t^2y^2)^k | < \frac{1}{2} \phi \pi,
\]

\[
\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0, \quad 1/2k > - \min \text{Re}(b_h/B_h), 1 \leq h \leq m.
\]

Multiplying eqn. (2.1) by \( t^v f(t) \), integrating with respect to \( t \) and changing the order of integration in the right hand side, it follows from Fubini theorem that for \( t^{v+ka} f(t) \) in \( L^1(R^+) \),

\[
\int_{0}^{\infty} t^{v} H_{p+1,q+1}^{m,n+1} \left[ c(x^2t^2)^k \left| (1 + \frac{1}{2} a - \frac{1}{2} k^{-1}, 1) \{(a_p + \frac{1}{2} aA_p, A_p)\} \right| \right] f(t) dt
\]

\[
= \frac{2kc^{a/2}}{\Gamma(a + \frac{1}{2})} \frac{x^{-kz+k-1}}{x^{2k - y^{2k}(2a-1)/2}} dy \int_{0}^{\infty} t^{v+ka} f(t)
\]

\[
\times H_{p,q}^{m,n} \left[ c(x^2t^2)^k \left| \{(a_p, A_p)\} \right| \right] dt. \quad \ldots(2.2)
\]

Now, denoting the fractional integral operator \( R_k \) by

\[
R_k f(x) = x^{v+\sigma-kz+k-1} \int_{0}^{x} (x^{2k} - y^{2k}(2a-1)/2) y^{-\sigma-v-k} f(y) dy \quad \ldots(2.3)
\]

and defining the Fourier type integral operators involving the \( H \)-functions as

\[
T_{H} f(x) = \frac{\Gamma(a + \frac{1}{2})}{2kc^{a/2}} x^{v+\sigma} \int_{0}^{\infty} t^{v} f(t)
\]

\[
\times H_{p+1,q+1}^{m,n+1} \left[ c(x^2t^2)^k \left| (1 + \frac{1}{2} a - \frac{1}{2} k^{-1}, 1) \{(a_p + \frac{1}{2} aA_p, A_p)\} \right| \right] dt. \quad \ldots(2.4)
\]

\[
H_{v}^{*} f(x) = x^{v+\sigma} \int_{0}^{\infty} t^{v} f(t) H_{p,q}^{m,n} \left[ c(x^2t^2)^k \left| \{(a_p, A_p)\} \right| \right] dt \quad \ldots(2.5)
\]

we have from (2.2)

\[
T_{H} f(x) = R_k H_{v}^{*+ka} f(x). \quad \ldots(2.6)
\]
Thus the operator $T_H$ can be expressed as the composition of the operators $R_k$ and $H^{e+ka}_\sigma$.

Special cases

(i) If all the $A_j$ and $B_j$ in (2.4) and (2.5) are equal to 1, we get from (2.6) the composition of operators involving the $G$-functions.

(ii) Further in (i), $k = 1$, $c = \frac{1}{2}$, $m = 1$, $n = 0$, $p = 0$, $q = 2$, $b_1 = 0$, $b_2 = \frac{1}{2}$, we get

$$T_H f(x) = RC^{e+\alpha}_\sigma f(x). \quad \ldots(2.7)$$

which is the result given by Okikiolu (1966).

(iii) Again, by setting in (i) $k = 1$, $c = \frac{1}{2}$, $m = 1$, $n = 0$, $p = 0$, $q = 2$, $b_1 = \frac{1}{2}$, $b_2 = 0$, we get

$$T_H f(x) = RS^{e+\alpha}_\sigma f(x) \quad \ldots(2.8)$$

which is another known result given Okikiolu (1966).

§3. In this section we establish the following theorem for the estimation of the Fourier type integral operator $T_H$.

Theorem — Let $f \in L^p(R^+)$, $p > 1$, and let $q > 1$, $1/q = 1 - (1/p) - \sigma > 0$, $0 < \sigma + (2/p) - 1 < 1/p$, then $T_H f(x)$ given by (2.4) can be extended as a bounded operator from $L^p(R^+)$ to $L^q(R^+)$ and $\| T_H \|_{p \to q} \leq \| T_H \|_{p \to q}$, where

$$V_p f(x) = x^{-2/p} f(x^{-1}) \text{ if } a + \frac{1}{2} > 0 \quad \ldots(3.1)$$

and

$$-2k \min_{1 \leq h \leq m} \Re (b_h/B_h) < 1 + v + ka - (1/p) \leq 2k \left[ 1 - \max_{1 \leq j \leq n} \Re \{a_j/A_j, (2k - 1)/2k\} \right]. \quad \ldots(3.2)$$

Proof: The Fourier type integral operator (2.4) can be written as

$$T_H f(x) = x^\alpha \int_0^\infty g(xt) f(t) \, dt$$

where

$$g(xt) = \frac{\Gamma(a + \frac{1}{2})}{2k c^{a/2}} (xt)^v \times H^{m,n+1}_{p+1,q+1} \left[ c(x^2 t^2)^k \begin{pmatrix} (1 + \frac{1}{2} a - \frac{1}{2} k^{-1}, 1), \{(a_p + \frac{1}{2} aA_p, A_p)\} \\
(b_q + \frac{1}{2} AB_q, B_q), (\frac{1}{2} - \frac{1}{2} a - \frac{1}{2} k^{-1}, 1) \end{pmatrix} \right]. \quad \ldots(3.3)$$
Using Theorem 4.2.9 of Okikiolu (1971, p. 197)
\[ T_H V_p f(x) = x^\mu \int_0^\infty t^{(2-2p)/p} g(x/t) f(t) \, dt \]
where \( V_p \) is given by (3.1).
Hence
\[ T_H V_p f(x) = x^{\mu-1} \int_0^\infty g(x/t) f(t) \, dt \]
where:
\[ \mu = \sigma + 2p^{-1} - 1 \quad \text{and} \quad g(x/t) = (x/t)^{(2p-2)/p} g(x/t). \] ...(3.4)
Assuming \( f \in L^p(\mathbb{R}^+) \), \( p \gg 1, q \gg 1, 1/q = (1/p) - \mu = 1 - (1/p) - \sigma, 0 \leqslant \mu \leqslant 1/p, \)
and using Theorem 4.2.4 of Okikiolu (1971, p. 197) we obtain
\[ \| T_H V_p f \|_q \leq k_1^{1-p} \| f \|_p \]
where
\[ k_1 = \int_0^\infty t^{(\mu-2+1/(1/p))/1-\mu} | g_1(t) |^{1/(1-\mu)} \, dt \]
\[ = \left[ \frac{\Gamma(a + \frac{1}{2})}{2kc^{a/2}} \right]^{1/(1-\mu)} \int_0^\infty t^{(\nu+p-1)/(1-\mu)} \]
\[ \times \left\{ H_{\nu+1,q+1}^{m,n+1} \left[ c t^{2k} \left| (1 + \frac{1}{2} a - \frac{1}{2} k^{-1}, 1), \{(a_p + \frac{1}{2} a A_p, A_p)\} \right|^{|1/(1-\mu)} \right] \right\} \]
by using (3.3), (3.4). Also \( k_1 \) is finite if
\[ -2k \min_{1 \leq k \leq m} \text{Re} (b_h/B_h) < 1 + \nu + ka - (1/p) \]
\[ < 2k [1 - \max_{1 \leq j \leq n} \text{Re} \{a_j/A_j, (2k - 1)/2k\}]. \]
Hence \( T_H V_p \) is a bounded operator from \( L^p(\mathbb{R}^+) \) to \( L^q(\mathbb{R}^+) \). Therefore by Theorem 4.2.9 of Okikiolu (1971, p. 197), \( T_H \) is also a bounded operator from \( L^p(\mathbb{R}^+) \) to \( L^q(\mathbb{R}^+) \)
and \( \| T_H \|_{p \to q} \leq \| T_H V_p \|_{p \to q} \), provided \( T_H f \) and \( T_H V_p f \) have a meaning.

**Special Cases**

In (3.2), setting the parameters as in (2.7) and (2.8), the operators \( T_f \) and \( T_H \)
given by Okikiolu (1966) can be extended from \( L^p(\mathbb{R}^+) \) to \( L^q(\mathbb{R}^+) \) under the conditions
\[ 1/q = 1 - (1/p) - \sigma > 0, a + \frac{1}{2} > 0, 1/p > \nu + a. \]

**Acknowledgement**

The authors are thankful to the referee for some useful suggestions.
References


