DUAL INTEGRAL EQUATIONS INVOLVING THE GENERALIZED FOX FUNCTION AS THE KERNEL

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Fractional integral operators are used to obtain a formal solution of dual integral equations having a special case of the multivariable H-function of Srivastava and Panda (1976) as kernel by reducing them to the ones with a common kernel. The results of Parashar and Goyal (1973), Jain and Goyal (1976), Goyal and Aggarwala (1974) and many others, follow as particular cases of the present investigation.

1. FORMULATION OF PROBLEM

In this paper we propose to obtain the formal solution of the following dual integral equations

\[
\int_0^\infty \cdots \int_0^\infty H^*(x_1u_1, \ldots, x_nu_n) f(u_1, \ldots, u_n) \, du_1 \cdots du_n
\]

\[= \phi'(x_1, \ldots, x_n); 0 < x_1, \ldots, x_n < 1 \quad \cdots (1.1)\]

\[
\int_0^\infty \cdots \int_0^\infty H_1^* (x_1u_1, \ldots, x_nu_n) f(u_1, \ldots, u_n) \, du_1 \cdots du_n
\]

\[= \psi'(x_1, \ldots, x_n); x_1, \ldots, x_n > 1 \quad \cdots (1.2)\]

where \(\phi'(x_1, \ldots, x_n)\), \(\psi'(x_1, \ldots, x_n)\) are given and \(f(x_1, \ldots, x_n)\) is required to be determined. Also we assume that \(H_1^* (u_1x_1, \ldots, u_nx_n)\) of (1.2) is of the same type as \(H^* (u_1x_1, \ldots, u_nx_n)\) with \(c_j^k\) replaced by \(e_j^k\) for \(j = 1, \ldots, p_k\); \(\forall k \in (1, \ldots, n)\) and \(d_j^k\) replaced by \(f_j^k\) for \(j = 1, \ldots, q_k\); \(\forall k \in (1, \ldots, n)\). We also assume that \(H_1^*\) satisfies all the conditions given for \(H^*\) and have common contours with it.

The formal solution so obtained is

\[
f(x_1, \ldots, x_n) = \int_0^\infty \cdots \int_0^\infty H^{**}(x_1u_1, \ldots, x_nu_n) t(u_1, \ldots, u_n) \, du_1 \cdots du_n
\]

\[\cdots (1.3)\]
where
\[
I(x_1, \ldots, x_n) = T_{l_{k+1}}^{*k} \left[ T_{l_{k+2}}^{*k} \cdots T_{l_{q_k}}^{*k} T_1^{k} \cdots T_{m_k}^{k} \{ \varphi'(x_1, \ldots, x_n) \} \right]
\]
\[
0 < x_1, \ldots, x_n < 1 \quad \ldots (1.4)
\]
\[
= R_{m_{k+1}}^{*k} \left[ R_{m_{k+2}}^{*k} \cdots R_{p_k}^{*k} R_1^{k} \cdots R_{l_{k}}^{k} \{ \psi'(x_1, \ldots, x_n) \} \right]
\]
\[
x_1, \ldots, x_n > 1 \quad \ldots (1.5)
\]

also
\[
H^{**}(x_1, \ldots, x_n) = H_{p_0, p_1 + q_1, \ldots, p_{n+q_n}}^{p_{n-1}, p_{n-2}, \ldots, p_{n-1}, q_n} \left[ \left[ (1 - a_{m+1, p} - \sum_{k=1}^{n} a_{m+1, p}, a_{m+1, p}, \ldots, a_{m+1, p}) \right] \right]
\]
\[
\begin{align*}
&\{(1 - c_{m_1 + 1, p_1}^{1}, c_{m_1 + 1, p_1}^{1}, \gamma_{m_1 + 1, p_1}^{1}), (1 - e_{m_1}^{1}, e_{m_1}^{1}, \gamma_{m_1}^{1})\}; \ldots; \\
&\{(1 - c_{m_1 + 1, p_1}^{n}, c_{m_1 + 1, p_1}^{n}, \gamma_{m_1 + 1, p_1}^{n}), (1 - e_{m_1}^{n}, e_{m_1}^{n}, \gamma_{m_1}^{n})\}; \\
&\{(1 - f_{i_{1} + 1, q_{1}}^{1}, f_{i_{1} + 1, q_{1}}^{1}, \delta_{i_{1} + 1, q_{1}}^{1}), (1 - d_{i_{1}}^{1}, d_{i_{1}}^{1}, \delta_{i_{1}}^{1})\}; \ldots; \\
&\{(1 - f_{i_{1} + 1, q_{1}}^{n}, f_{i_{1} + 1, q_{1}}^{n}, \delta_{i_{1} + 1, q_{1}}^{n}), (1 - d_{i_{1}}^{n}, d_{i_{1}}^{n}, \delta_{i_{1}}^{n})\} \right] \\
\end{align*}
\]
\[
\quad \ldots (1.6)
\]

It is clear from the nature of the solution that it can be written by inspection from eqns. (1.1) and (1.2)
\[
f(x_1, \ldots, x_n) = \int_0^1 \int_0^1 H^{**}(u_1 x_1, \ldots, u_n x_n) T_{l_{k+1}}^{*k} \left[ T_{l_{k+2}}^{*k} \cdots T_{l_{q_k}}^{*k} T_1^{k} \cdots T_{m_k}^{k} \{ \varphi'(u_1, \ldots, u_n) \} \right] du_1 \ldots du_n
\]
\[
+ \int_0^\infty \int_1^\infty H^{**}(x_1 u_1, \ldots, x_n u_n) R_{m_{k+1}}^{*k} \left[ R_{m_{k+2}}^{*k} \cdots R_{p_k}^{*k} R_1^{k} \cdots R_{l_{k}}^{k} \{ \psi'(u_1, \ldots, u_n) \} \right] du_1 \ldots du_n. \quad \ldots (1.7)
\]

Here the $H^*(x_1 u_1, \ldots, x_n u_n)$ is a particular case of the multivariable $H$-function of Srivastava and Panda [1976, eqns. (1.3) – (1.7), p. 130] in the contracted notation (slightly different)

\[
H_{m_0, (m_1, 1); \ldots, (m_{n+1})}^{n, (m_0, q_1); \ldots, (p_{n+1})} \begin{pmatrix}
\begin{bmatrix}
\left[ x_1; a_1^{i_1}, \ldots, a_n^{i_n} \right], \\
\vdots \\
\left[ (c_1^{1}, \gamma_1^{1}); \ldots, (c_n^{n}, \gamma_n^{n}) \right], \\
\left[ (d_1^{1}, \delta_1^{1}); \ldots, (d_n^{n}, \delta_n^{n}) \right]
\end{pmatrix}
\end{pmatrix}
\]

(equation continued on p. 1178)
\begin{align*}
&= (1/2\pi i)^n \int \cdots \int \psi(\sum_{k=1}^{n} s_k) \prod_{k=1}^{n} \{\phi_k(s_k) x_k^{-s_k} ds_k\} \quad \cdots (1.8) \\
\end{align*}

where

\begin{align*}
\phi_k(s_k) &= \prod_{i=1}^{m_k} \Gamma(1 - c^k_j + \gamma^k_j s_k) \prod_{j=1}^{l_k} \Gamma(d^k_j - \delta^k_j s_k) \\
&\times [\prod_{j=1+m_k}^{P_k} \Gamma(c^k_j - \gamma^k_j s_k) \prod_{j=1+l_k}^{q_k} \Gamma(1 - d^k_j + \delta^k_j s_k)]^{-1} \quad (1.9) \\
\psi(\sum_{k=1}^{n} s_k) &= \prod_{j=1}^{m} \Gamma(a_j + \sum_{k=1}^{n} \alpha^j_k s_k) \prod_{j=1+m}^{p} \Gamma(1 - a_j - \sum_{k=1}^{n} \alpha^j_k s_k)]^{-1} \quad (1.10) \\
\end{align*}

The integers \( m, p, (m_k), (l_k), (p_k), (q_k) \) \( \forall \ k \in (1, \ldots, n) \) satisfy the inequalities

\( 0 \leq m \leq p; \ 0 \leq m_k \leq p_k; \ 0 \leq l_k \leq q_k; \ q_k \geq 0, \ \forall \ k \in (1, \ldots, n) \). The values \( x_k = 0, \ \forall \ k \in (1, \ldots, n) \) are excluded. An empty product is interpreted as unity. The contour \( L_k \) is in the \( s_k \) plane for every \( k \in (1, \ldots, n) \) and runs from \(-i\infty\) to \(+i\infty\) with loops if necessary, to ensure that the poles of \( \Gamma(a_j + \sum_{k=1}^{n} \alpha^j_k s_k) \) for \( j = 1, \ldots, m \) and \( \Gamma(1 - c^k_j + \gamma^k_j s_k) \) for \( j = 1, \ldots, m_k \) lie to the left of the contour \( L_k \) and that of \( \Gamma(d^k_j - \delta^k_j s_k) \) for \( j = 1, \ldots, l_k \) lie to the right of the contour \( L_k \). Also \( |\arg x_k| < \lambda^* \pi, \lambda^* > 0 \) where

\[
\lambda^* = \frac{1}{2} \left[ \sum_{j=1}^{l_k} |\delta^k_j| + \sum_{j=1+m}^{p} |\alpha^j_k| - \sum_{j=1+m_k}^{p_k} |\gamma^k_j| \right] \\
+ \frac{1}{2} \left[ \sum_{j=1}^{p_k} |\alpha^j_k| + \sum_{j=1}^{q_k} |\gamma^k_j| + \sum_{j=1}^{l_k} |\delta^k_j| \right] \\
\]

\((\alpha^j_k), (\gamma^k_j), (\delta^k_j), \ \forall \ k \in (1, \ldots, n) \) are all positive.

Using the multidimensional Mellin transform of \( H(x_1, \ldots, x_n) \) given by

Srivastava and Panda (1978, Lemmas 1 and 2, p. 125), constructing the Parseval theorem

for \( n \) variables, viz.

If \( M \{y(u_1, \ldots, u_n)\} = Y(s_1, \ldots, s_n) \)

and

\[
M \{f(x_1u_1, \ldots, x_nu_n)\} = \prod_{k=1}^{n} (x_k^{-s_k}) F(s_1, \ldots, s_n) \\
\]
then
\[
\int_0^\infty \ldots \int_0^\infty y(x_1u_1, \ldots, x_nu_n) f(u_1, \ldots, u_n) \, du_1 \ldots du_n
\]
\[
= (1/2\pi i)^n \int_{-i\infty}^{+i\infty} \ldots \int_{-i\infty}^{+i\infty} Y(s_1, \ldots, s_n)
\times F(1 - s_1, \ldots, 1 - s_n) \prod_{k=1}^n (x_k^{-s_k} \, ds_k).
\]...
(1.11)

On applying (1.11) to (1.1) and (1.2) we obtain
\[
(1/2\pi i)^n \int_{L_1}^{L_n} \ldots \int_{L_1}^{L_n} \psi\left( \sum_{k=1}^n s_k \right) F(1 - s_1, \ldots, 1 - s_n) \prod_{k=1}^n \{ \phi_k(s_k) x_k^{-s_k} \, ds_k \}
\]
\[
= \psi'(x_1, \ldots, x_n); \text{ where } 0 < x_k < 1, \forall \ k \in (1, \ldots, n)
\]
...
(1.12)

and
\[
(1/2\pi i)^n \int_{L_1}^{L_n} \ldots \int_{L_1}^{L_n} \psi\left( \sum_{k=1}^n s_k \right) F(1 - s_1, \ldots, 1 - s_n) \prod_{k=1}^n \{ \phi_k^*(s_k) x_k^{-s_k} \, ds_k \}
\]
\[
= \psi'(x_1, \ldots, x_n); \ x_k > 1, \forall \ k \in (1, \ldots, n)
\]
...
(1.13)

and \(\phi_k^*(s_k)\) represents \(\phi_k(s_k)\) with \(c_j^k\) replaced by \(e_j^k\) for \(j = 1, \ldots, p_k; \forall \ k \in (1, \ldots, n)\)
and \(d_j^k\) replaced by \(f_j^k, \ j = 1, \ldots, q_k; \forall \ k \in (1, \ldots, n)\).

2. Reduction of (1.12) and (1.13) into One with a Common Kernel

The integral operators are used to transform (1.12) and (1.13) into two others with a common kernel. Thus we transform
\[
\prod_{k=1}^n \left[ \frac{m_k}{\prod_{j=1}^{l_k}} \frac{\Gamma(1 - c_j^k + \gamma_j^k s_k)}{\Gamma(1 - c_j^k + \gamma_j^k s_k)} \right] \text{ of (1.12)}
\]
into
\[
\prod_{k=0}^n \left[ \frac{m_k}{\prod_{j=1}^{l_k}} \frac{\Gamma(1 - e_j^k + \gamma_j^k s_k)}{\Gamma(1 - f_j^k + \delta_j^k s_k)} \right] \text{ of (1.13)}
\]
\[
\prod_{k=1}^n \left[ \frac{l_k}{\prod_{j=1}^{p_k}} \frac{\Gamma(f_j^k - \delta_j^k s_k)}{\Gamma(e_j^k - \gamma_j^k s_k)} \right] \text{ of (1.13)}
\]
into
\[ \prod_{k=1}^{n} \frac{\prod_{j=1}^{l_k} \Gamma(d_j^k - s_j^k)}{\prod_{j=1+m_k}^{p_k} \Gamma(c_j^k - \gamma_j^k s_k)} \] of (1.12).

In making these transformations we use the fractional integral operators \( T \) and \( R \) defined by Erdélyi (1950–51), viz.
\[
T [x, \beta; \gamma : w(x)] = \left( \frac{\Gamma(\alpha)}{\Gamma(\alpha + \beta - 1)} \right) x^{-\alpha + \beta - 1} \int_{0}^{x} (x^\gamma - v^\gamma)^{\alpha - 1} v^\beta w(v) \, dv
\]
\[
R [x, \beta; \gamma : w(x)] = \left( \frac{\Gamma(\alpha)}{\Gamma(\alpha + \beta - 1)} \right) \int_{0}^{x} (v^\gamma - x^\gamma)^{\alpha - 1} v^{-\beta - \alpha + \gamma - 1} w(v) \, dv
\]
provided \( w(x) \in L_\nu(0, \infty), p' \geq 1; \alpha > 0; \beta > (1 - p')/p' \). If in addition \( w(x) \) can be differentiated sufficiently often, \( T \) and \( R \) exist for negative as well as positive (see also Fox 1965). In this case for brevity we write
\[
T [e_j^k - c_j^k, (1 - e_j^k) (- \gamma_j^k)^{-1} - 1; (- \gamma_j^k)^{-1} : w(x)] = T_{j}^{\star} [w(x)]
\]
\[
T [d_j^k - f_j^k, (1 - d_j^k) (- \delta_j^k)^{-1} - 1; (- \delta_j^k)^{-1} : w(x)] = T_{j}^{\star} [w(x)]
\]
\[
R [f_j^k - d_j^k, d_j^k (- \delta_j^k)^{-1}; (- \delta_j^k)^{-1} : w(x)] = R_{j}^{\star} [w(x)]
\]
\[
R [c_j^k - e_j^k, e_j^k (- \gamma_j^k)^{-1}; (- \gamma_j^k)^{-1} : w(x)] = R_{j}^{\star} [w(x)].
\]

Now, replace \( x_1 \) by \( v_1 \), multiply by \( (v_1)^{\theta} (x_1^\theta - v_1^\theta)^{\theta_1} \) where \( A = (1 - e_{m_1}^1) \theta - 1 \), \( \theta = (- \gamma_{m_2}^1)^{-1} \) and \( \theta_1 = e_{m_1}^1 - c_{m_1}^1 - 1 \), integrate with respect to \( v_1 \) under the limit \( 0 < v_1 < x_1, 0 < x_1 < 1 \) and apply the well-known beta function formula to obtain
\[
T_{m_1}^{1} [\phi(x_1, x_2, ..., x)] = \frac{x_1^{Z}}{\theta \Gamma(e_{m_1}^1 - c_{m_1}^1)} \int_{0}^{x_1} (v_1)^{\theta} (x_1^\theta - v_1^\theta)^{\theta_1} \phi'(v_1, x_2, ..., x_n) \, dv_1
\]
(here \( Z = \theta c_{m_2}^1 \)).

On transforming it successively by the application of the operator \( T_{j}^{\star} \) for \( j = m_k - 1, m_k - 2, ..., 3, 2, 1 \) for \( k = 1 \) and \( j = m_k, m_k - 1, ..., 3, 2, 1 \) for \( k = 2, 3, ..., n \) and then applying the operator \( T_{j}^{\star} \) (successively) for \( j = q_k, q_k - 1, q_k - 2, ..., l_k + 3, \)
\( l_k + 2, l_k + 1; \forall k \in (1, ..., n) \) we finally get the right-hand side of \( t(x_1, ..., x_n) \) given in (1.4).

Similarly for making the second transformation after replacement as in first transformation multiply by

\[(v_1)^B (v_1^\theta - x_1^\theta) \theta^*\]

where \( B = (1 - f^1_{i_1}) \theta^* - 1, \theta^* = (-\delta^1_{i_1})^{-1} \)

and \( \theta^*_1 = f^1_{i_1} - d^1_{i_1} - 1. \)

Integrate with respect to \( v_1 \) from \( x_1 \) to \( \infty (x_1 > 1) \) and apply the well-known beta function formula to obtain

\[ R^1_{i_1} [\psi(x_1, ..., x_n)] \]

\[ = \frac{x_1^Y}{\theta^* \Gamma(f^1_{i_1} - d^1_{i_1})} \int_{x_1}^{\infty} (v_1)^B (v_1^\theta - x_1^\theta) \theta^*_1 \psi(v_1, x_2, ..., x_n) dv_1 \]

(Here \( Y = \theta^* d^1_{i_1} \)).

On transforming the above successively by the application of the operators \( R^k_j \) for \( j = l_k - 1, l_k - 2, ..., 2, 1 \) for \( k = 1 \) and \( j = l_k, l_k - 1, ..., 2, 1 \) for \( k = 2, 3, 4, ..., n \) and applying \( R^k_j \) successively for \( j = p_k, p_k - 1, ..., m_k + 2, m_k + 1, \forall k \in (1, ..., n) \) we finally get \( t(x_1, ..., x_n) \) given in (1.5). Thus the equations can be rewritten in the form

\[ t(x_1, ..., x_n) = (1/2\pi i)^n \int_{(L_n)} \psi \left( \sum_{k=1}^{n} s_k \right) F(1 - s_1, ..., 1 - s_n) \]

\[ \prod_{k=1}^{n} \left[ \prod_{j=1}^{m_k} \Gamma(1 - e_j^k + \gamma_j^k s_k) \prod_{j=1}^{l_k} (d_j^k - \delta_j^k s_k) x_k^{-s_k} ds_k \right] \]

\[ \prod_{k=1}^{n} \left[ \prod_{j=1}^{p_k} \Gamma(e_j^k - \gamma_j^k s_k) \prod_{j=1}^{q_k} \Gamma(1 - f_j^k + \delta_j^k s_k) \right] \]

The above equation is the reduction of (1.12) and (1.13) into the one with a common kernel. On treating the kernel of the above equation as an unsymmetrical Fourier kernel, and following the procedure adopted by Fox (1965), we easily arrive at the solution given in (1.3).
3. PARTICULAR CASES

(i) If $a_{4m} = \beta_{4m} = \gamma_{p_{4},n} = \delta_{q_{4},n} = 0$, $x_{1} = x$, $x_{2} = y$, $x_{3} = z$, $p_{4,n} = q_{4,n} = 0$
and take the limit as $x_{4,n}$ tends to zero, we obtain the dual integral equations and
the solutions given by Jain and Goyal (1976) which in turn contain the results of
Parashar and Goyal (1973), Saxena (1967) as corollaries.

(ii) If $\alpha$'s, $\beta$'s, $\gamma$'s, $\delta$'s are all unity $\forall \ k \in (1, ..., n)$, the main result of Goyal
and Aggarwala (1974) becomes a corollary of the present investigation.

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