ENUMERATION OF INCONGRUENT CYCLIC \( k \)-GONS

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If the circumference of a circle with an arbitrarily fixed radius is divided into \( n \) equal parts, we find in this paper, a formula for the number \( R(n, k) \) of mutually incongruent convex \( k \)-gons that can be obtained by joining \( k \) of the \( n \) points of division. The problem was first raised by Richard H. Reis. The prologue gives an account of his contributions to the solution of the problem.

1. Prologue

1.1. Men of letters are known for their apathy towards Mathematics in general and computational work in particular and they openly confess this with a sense of pride. I was, therefore, not a little surprised when I received a letter, dated April 13, 1978, from Richard H. Reis, Professor of English at the Southeastern Massachusetts University, N. Dartmouth (U.S.A.), posing a problem in Partition Theory. The problem was essentially this:

Given a regular \( k \)-gon with positive integers, summing to a given number \( n \), written at the vertices, Reis was interested in finding \( R(n, k) \)—the number of such polygons when reflections and rotations were considered redundant.

Reis had been working on the problem for about one year and had already obtained formulae for \( R(n, k) \) for values of \( k \ll 5 \), and the asymptotic result true for any fixed \( k \) and large \( n \).

Remembering that Mr. Martin Gardner had discussed the necklace-stringing problem in an article in Scientific American some years earlier, Reis wrote to Gardner to find if he knew someone who could tell him if his problem was new. Mr. Gardner identified his problem as one in Partition Theory (of which Reis had never heard) and referred him to Professor George E. Andrews, who then referred him to me.

In the middle 30's (if I remember aright), I had come across a similar problem wherein \( k \) coins of which \( k_1 \) were of one kind, \( k_2 \) of a second kind, \( k_3 \) of a third kind and so on, were to be arranged round a circle at equal distances. Here it was left vague if rotations alone or both rotations and reflections were to be considered redundant.
Finding the problem too difficult to solve in all its generality, I had put it aside and forgotten all about it till I received this letter from Reis. The letter revived my interest in the problem and I decided to attack it seriously. But at the time I was busy giving finishing touches to my book "Selected Topics in Number Theory" which had been accepted for publication by the Abacus Press. I was, therefore, forced to postpone the study of the problem posed by Reis till I had submitted the typescript of my book to the publishers. In the mean time Reis continued his work and succeeded in obtaining some results with his crude empirical methods. I was highly impressed by his zeal and insight. But all that I was doing during this period was writing encouraging letters to him without devoting any time to the problem myself. My letters to Reis did at least one thing; that is they goaded him on and more and more success came to him. It was sometime in August that he sent me a table of values of $R(n, k)$ for $k \leq 12$ and $n$ going up to $k + 30$ for $k \leq 6$ but not beyond $k + 17$ in other cases. The most remarkable relation to which he was led by a study of his table, was that

$$R(n, k) = R(n, n - k), \ 1 \leq k < n.$$ 

Like the rest of the material, I put this aside also.

On October 20, 1978 I was able to dispatch the final typescript of my book to the publishers and started looking at the problem of Reis with all seriousness. I requested Reis to send me all the results he had obtained. While I told him about the method I was going to use, I thought it might be best for me to study the problem independently, for then we could compare our results. Finding a few days later that my results agreed with those in the table of Reis, I looked into his papers to see what method he had used. I was surprised to find that we had both used the same procedure.

It was for the first time in November 1978 that Reis was able to give some really good general results. These gave $R(n, k)$ for $(n, k) = 1$ or 2 and also when $k$ was an odd prime and $n$ a multiple of $k$. But his formulae were expressed in a very complicated form. It did not take me long to give them an elegant shape. I decided to let Reis continue in his own independent way, while I went ahead in mine. Every letter from him from this time on brought some new results. By the middle of January, 1979, I had found the exact formula for $R(n, k)$ for all $n$ and $k$ and Reis had covered the same ground almost if not exactly.

I am sure, if Reis had some knowledge of Partition Theory, his insight would have enabled him to solve the problem without the least help from anyone. Simply because Euler's phi function had appeared in the formula for the number of necklaces with a given number of beads chosen from beads of two different colours without restriction, it was not enough reason to predict that it must appear in the
solution of his problem. It could only be due to his insight that he could insist that it will and it did.

More than half the credit for solving the problem must go to Reis. My own contribution is the geometrical way of representing a decomposition of \( n \) into \( k \) parts and providing the proofs of the results we obtained independently.

1.2. Reis used the method of finite differences. To show how it worked, I consider the case \( k = 4 \).

For \( m \geq 1 \), the table computed by Reis gives

\[
\begin{array}{cccccc}
  m = 1 & 2 & 3 & 4 & 5 & 6 \\
R(4m, 4) = 1 & 8 & 29 & 72 & 145 & 256 \\
\end{array}
\]

Taking the differences as usual, we get

\[
\begin{array}{cccccc}
  1 & 8 & 29 & 72 & 145 & 256 \\
7 & 21 & 43 & 73 & 111 \\
14 & 22 & 30 & 38 \\
8 & 8 & 8 \\
\end{array}
\]

Hence

\[
R(4m, 4) = 1 + 7(m - 1; 1) + 14(m - 1; 2) + 8(m - 1; 3).
\]

Here and in what follows, we write \( (j; r) \) for \( \binom{j}{r} \).

Similarly, we have

\[
\begin{align*}
R(4m + 1, 4) &= 1 + 9(m - 1; 1) + 16(m - 1; 2) + 8(m - 1; 3); \\
R(4m + 2, 4) &= 3 + 13(m - 1; 1) + 18(m - 1; 2) + 8(m - 1; 3); \\
R(4m + 3, 4) &= 4 + 16(m - 1; 1) + 20(m - 1; 2) + 8(m - 1; 3).
\end{align*}
\]

It will be seen that in each of the above cases, \( R \) is a cubic in \( m \) and, therefore, in \( n \) also. In general \( R(n, k) \) is a polynomial in \( n \) of degree \( (k - 1) \), not necessarily with integral coefficients.

The only drawback in this method is that quite a large number of values of \( R(n, k) \) are needed before the formulae can be obtained and then for each \( k \) as many as \( k \) distinct formulae are necessary.

1.3. A few extracts from the letters I received from Reis, will be of interest to the reader.

April 13, 1978:

Prof. Andrews informs me that he has not encountered this problem before, but he thinks that you may have studied it, if anyone has.
June 19, 1978:

I believe that the general problem can be entirely solved.

June 30, 1978:

Besides, I am not a Mathematician and do not know how to program a computer; such results as I have obtained have been produced with pencil, paper, and a small pocket calculator.

If I tried to write an article about my results so far, without the help of a skilled mathematician, I'd no doubt do it clumsily and it would not get printed.

Please let me know if you would be interested in helping me put my first draft into publishable form ...

December 26, 1978:

I suspect that Euler's phi function will be involved somewhere.

I have never been able to understand the difference between partitions and distributions anyhow.

Yes, I suppose it is rather surprising for somebody trained in literary criticism to have some degree of mathematical talent, or even to be interested in Mathematics at all. My colleagues in our English department think I'm a Martian or something! On the other hand, my own explanation of my unusual combination of interests is that poetry, music, mathematics and chess (I am also interested in music and chess, by the way) share two features: all have pattern, and all have beauty. So perhaps I'm not so odd after all!

January 9, 1979:

The conjecture (that my method of finding values of $R(n,k)$ would somehow or other turn out to involve Euler's phi function) now seems a safe bet, don't you think?

January 15, 1979:

I have at last completed the set of algorithms whereby $R(n,k)$ can be found for any combination of $k$ and $n$, including those with which I had been having trouble. As I had conjectured, Euler's phi function is involved, ...

February 3, 1979:

For me, this (to exchange ideas in correspondence) has been a delightful and rewarding experience, in which I have learned a good deal about combinatorial mathematics that I didn't know before. And of course I warmly appreciate the kindness of your remarks about my mathematical talents, such as they are. I'm
actually thinking of dreaming up a new problem, in order to have a reason for corresponding with you further!"

Needless to say that if I have discovered Reis, I am proud of my discovery.

In the following pages is given an account of how the final answer to the problem of Reis was obtained. Now that the expression for $R(n, k)$ is known, shorter proofs of the result should be possible.

2. **Introduction**

2.1. **Notations**

In what follows $x$ denotes an arbitrary real number; other small letters denote positive integers unless stated otherwise.

$(g, h)$ denotes as usual the g.c.d. of $g$ and $h$.

As already stated, we write $(g; h)$ for \( \begin{pmatrix} g \\ h \end{pmatrix} \).

$\phi(m)$ denotes Euler’s totient function and is the number of positive integers $\leq m$ which are prime to $m$.

$[x]$ denotes the largest integer $\leq x$.

2.2. **Partitions and Decompositions**

When a natural number $n$ is expressed as a sum of one or more natural numbers and the order in which the summands are written is irrelevant, we get what we call a partition of $n$. When the order in which the summands are written is relevant, we have a decomposition of $n$. The total number of partitions of $n$ is denoted by $p(n)$, while $p(n, k)$ denotes the number of partitions of $n$ into exactly $k$ summands.

In writing the parts in a partition, we write them in ascending order of magnitude and, when there is no cause for confusion, we omit the plus signs also. Thus the seven partitions of 5 are

11111, 1112, 122, 113, 23, 14, 5

and we have $p(5) = 7$.

On the other hand, the partitions of 10 into six parts are

111115, 111124, 111133, 111223, 112222

so that $p(10, 6) = 5$.

The decompositions of $n$ into $k$ parts are provided by the solutions of the Diophantine equation
\[ u_1 + u_2 + \ldots + u_k = n \] ...

in positive integers \( u \).

It is well known that (1) has exactly \((n - 1; k - 1)\) such solutions. Hence \( n \) has exactly \((n - 1; k - 1)\) decompositions into \( k \) parts. Thus the ten decompositions of 6 into four parts are:

1113, 1131, 1311, 3111;
1122, 1212, 1221, 2112, 2121, 2211.

The frequency of a part in a partition is the number of times the part appears in the partition.

A partition in which

\[ a_1 \text{ occurs as a part } h_1 \text{ times;} \]
\[ a_2 \text{ occurs as a part } h_2 \text{ times;} \]
.................................
\[ a_i \text{ occurs as a part } h_i \text{ times;} \]

\( a \)'s all distinct, is said to be of the type

\[(h_1 \ h_2 \ldots \ h_i).\] ...

Here, the order in which the \( h \)'s are written is immaterial. Thus

1113344666 is a partition of 35 and it is of the type (2 2 3 3). The number of decompositions to which a partition of the type \((h_1 \ h_2 \ldots \ h_i)\) leads is given by

\[ \frac{(h_1 + h_2 + \ldots + h_i)!}{h_1! \ h_2! \ldots \ h_i!}. \] ...

In this paper, we will usually need to write out decompositions arising from a given partition and starting with a given element. Thus, the decompositions arising from the partition 111223 and starting with 2 are twenty in number, while there are thirty which start with 1 and only ten which start with 3. Since our interest will be in having to record the minimum number of decompositions, it will be best if we choose as our starting element one of those the frequency of which is the least.

It is noteworthy that the number of decompositions arising from a given partition, depends only on the frequencies of the parts and not on the size of those parts.

2.3 Graphical Representation of a Decomposition

Take a circle with an arbitrarily fixed radius. Divide the circumference into \( n \) equal parts. Call each part a step. Then any decomposition
of \( n \), can be represented graphically as follows:

Select one of the points of division as the starting point. From here move \( c_1 \) steps in the counter-clockwise direction and there put a mark. Then move ahead \( c_2 \) steps and again put a mark. Continue in this manner till you have finally moved \( c_k \) steps and put a mark. This will bring you to the starting point.

Join the \( k \) marked points in order by straight lines to get a convex cyclic \( k \)-gon. The sides of this \( k \)-gon can be taken to represent the numbers \( c_1, c_2, \ldots, c_k \); because they are proportional to the angles subtended by the sides at the centre of the circle. The \( k \)-gon, therefore, provides a graphical representation of the given decomposition of \( n \). To make the representation unique, it will be necessary to indicate the starting point by an arrowhead or some such sign.

The following figure represents, for example, the decomposition

\[
22143
\]

of 12.

One may ask

What decompositions does the \( k \)-gon represent if the starting point is not indicated? And what if one is permitted to move in the clockwise direction also?

These questions are easy to answer and are left to the reader.

2.4. *Congruence of \( k \)-gons*

Let us represent the decompositions

\[
1223, 2213, 2312
\]

of 8, by quadrilaterals using equal circles.
It will be seen that the quadrilateral in (2.1) representing the decomposition 1223, can be cut out of the paper and made to fit upon the quadrilateral in (2.3), representing the decomposition 2312, directly, that is by just rotating the paper; but it can be made to fit upon the quadrilateral in (2.2), representing the decomposition 2213, only if we first turn it upside-down and then rotate.

We say that the quadrilaterals in (2.1) and (2.3) are directly congruent, while those in (2.1) and (2.2) are invertedly so. But the three quadrilaterals are mutually congruent anyway.

The definitions can be extended to $k$-gons immediately.

The $k$-gons represented by the decompositions

$$c_1 c_2 \ldots c_k; \ c_2 c_3 \ldots c_k c_1; \ldots; \ c_k c_1 \ldots c_{k-1}$$

...(5)
of \( n \), are all directly congruent among themselves. While any one of these is invertedly congruent to each of the \( k \)-gons represented by the decompositions

\[
\begin{align*}
&c_k \, c_{k-1} \ldots \, c_1; \\
&c_{k-1} \, c_{k-2} \ldots \, c_1 \, c_k; \\
&\ldots \, c_1 \, c_k \ldots \, c_3 \, c_2
\end{align*}
\] ...

(6)

It is not implied here that the decompositions in (5) are all distinct. But if they are distinct in the case of (5), they are so in the case of (6) too. If any decomposition in (5) is identical with some decomposition in (6), then the decompositions in (6) are just a permutation of those in (5). When this happens, every two \( k \)-gons are both directly and invertedly congruent. In fact, each \( k \)-gon has now at least one axis of symmetry. For \( k \) odd, any axis of symmetry runs through one vertex and the middle point of the side opposite to it. When \( k \) is even, any axis of symmetry either runs through two opposite vertices or through the middle points of two opposite sides.

![Fig. 3.1](image1)

![Fig. 3.2](image2)

![Fig. 3.3](image3)
Definition — Two decompositions are said to be equivalent when the \(k\)-gons representing them are congruent (whether directly or invertedly).

2.5. The Number of Symmetrical \(k\)-gons for a given \(n\)

(i) When \(k\) is odd.

Starting from a vertex through which an axis of symmetry passes, we have in this case

\[
c_1 = c_k; \ c_2 = c_{k-1}; \ldots; \ c_h = c_{h+2}; \ c_{h+1} \text{ independent},
\]

where \(h = (k - 1)/2\).

The related decomposition can be written in the form

\[
c_1 \ c_2 \ldots \ c_h \ c_{h+1} \ c_h \ldots \ c_2 \ c_1.
\] ...(7)

The number of symmetric \(k\)-gons for the given \(n\) will, therefore, be the same as the number of solutions in positive integers of the equation

\[
2c_1 + 2c_2 + \ldots + 2c_h + c_{h+1} = n.
\] ...(8)

If \(n\) is odd, so also must \(c_{h+1}\) be. The equation can, therefore, be written

\[
c_1 + c_2 + \ldots + c_h + (c_{h+1} + 1)/2 = (n + 1)/2.
\] ...(9)

Hence the number of symmetric \(k\)-gons is

\[
\binom{n-1}{2} \binom{k-1}{2}.
\] ...(10)

If \(n\) is even, so also is \(c_{h+1}\), and (8) can be written as

\[
c_1 + c_2 + \ldots + (c_{h+1}/2) = n/2.
\] ...(11)

In this case, therefore, the number of symmetric \(k\)-gons is

\[
\binom{n-2}{2} \binom{k-1}{2}.
\] ...(12)

Results (10) and (12) can be combined into the single result

\[
\left\lfloor \frac{n-1}{2} \right\rfloor \cdot \left\lfloor \frac{k}{2} \right\rfloor.
\] ...(13)

(ii) When \(k\) is even.

(a) When \(n\) is even.

If the \(k\)-gon has an axis of symmetry passing through two opposite vertices, then we have starting from one of these vertices

\[
c_1 = c_k; \ c_2 = c_{k-1}; \ldots; \ c_i = c_{i+1}; \quad \text{with} \quad j = k/2.
\]
The corresponding decomposition is
\[ c_1 \ c_2 \ ... \ c_{i-1} \ c_i \ c_1 \ c_{i-1} \ ... \ c_1. \] \(\text{...(14)}\)

The number of such decompositions is the same as the number of solutions of the Diophantine equation
\[ c_1 + c_2 + ... + c_i = n/2 \] \(\text{...(15)}\)

which is given by
\[ \left( \frac{n}{2} - 1; \frac{k}{2} - 1 \right). \] \(\text{...(16)}\)

Note that the decompositions
\[ c_1 \ c_2 \ ... \ c_i \ c_{i-1} \ ... \ c_2 \ c_1 \quad \text{and} \quad c_1 \ c_2 \ ... \ c_{i-1} \ ... \ c_1 \ c_i \] are equivalent. Also they are distinct unless
\[ [c_1, c_2, ..., c_{i-1}, c_i] = [c_i, c_{i-1}, ..., c_2, c_1]. \] \(\text{...(A)}\)

Hence all the solutions do not give incongruent symmetric \(k\)-gons.

If the \(k\)-gon has an axis of symmetry passing through the middle points of two opposite sides, then starting with one of these sides, we have (we use \(d\)'s to distinguish them from the \(c\)'s used in the foregoing case)
\[ d_2 = d_k; \ d_3 = d_{k-1}; ...; \ d_i = d_{i+2}; \ \text{with} \ j = k/2 \]
and \(d_1\) and \(d_{i+1}\) free.

The corresponding decomposition is
\[ d_1 d_2 \ ... \ d_i d_{i+1} d_1 \ ... \ d_2. \]

The number of such decompositions is given by the number of positive integral solutions of the Diophantine equation
\[ d_1 + 2d_2 + ... + 2d_i + d_{i+1} = n \]
which can be written as
\[ \frac{d_1 + 1}{2} + d_2 + ... + d_i + \frac{d_{i+1} + 1}{2} = \frac{n}{2} + 1 \] \(\text{...(17)}\)
when \(d_1\) and \(d_{i+1}\) are both odd; and as
\[ \frac{d_1}{2} + d_2 + ... + d_i + \frac{d_{i+1}}{2} = \frac{n}{2} \] \(\text{...(18)}\)
when \(d_1\) and \(d_{i+1}\) are both even.

Evidently (17) leads to \(\left( \frac{n}{2}; \ \frac{k}{2} \right)\) and (18) to \(\left( \frac{n}{2} - 1; \ \frac{k}{2} \right)\) decompositions.
Note that in this case the decompositions
\[ d_1 d_2 \ldots d_i d_{i+1} d_i \ldots d_2 \quad \text{and} \quad d_{i+1} d_i \ldots d_2 d_1 d_2 \ldots d_i \]
are equivalent and distinct also except when
\[ [d_i, d_2, \ldots, d_{i+1}] = [d_{i+1}, d_i, \ldots, d_i]. \] \hspace{1cm} \text{(B)}

The total number of decompositions obtained from (15), (17) and (18) is readily seen to be
\[ 2 \left( \frac{n}{2} ; \frac{k}{2} \right). \] \hspace{1cm} \text{(19)}

We assert that the number of symmetric $k$-gons obtained from these decompositions is only
\[ \left( \frac{n}{2} ; \frac{k-1}{2} \right). \] \hspace{1cm} \text{(20)}

This will follow if we can show that the decompositions satisfying relations A and B consist of pairs of equivalents.

The following examples cover all the four cases that can arise.

(1) \( n \equiv 2 \pmod{4}, \ k \equiv 0 \pmod{4}; \)
(2) \( n \equiv 0 \pmod{4}, \ k \equiv 0 \pmod{4}; \)
(3) \( n \equiv 0 \pmod{4}, \ k \equiv 2 \pmod{4}; \)
(4) \( n \equiv 2 \pmod{4}, \ k \equiv 2 \pmod{4}. \)

\textit{Case 1} \quad \text{Take} \ n = 18, \ k = 8.

The set A is empty.

The set B consists of the following decompositions:

\[
\begin{array}{cccccccccccc}
1 & 1 & 6 & 1 & 1 & 1 & 6 & 1 & 6 & 1 & 1 & 1 \\
1 & 2 & 4 & 2 & 1 & 2 & 4 & 2 & 4 & 2 & 1 & 2 \\
1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 2 & 3 & 1 & 3 \\
3 & 1 & 4 & 1 & 3 & 1 & 4 & 1 & 4 & 1 & 3 & 1 \\
3 & 2 & 2 & 2 & 3 & 2 & 2 & 2 & 2 & 2 & 3 & 2 \\
5 & 1 & 2 & 1 & 5 & 1 & 2 & 1 & 2 & 1 & 5 & 1 \\
\end{array}
\]

We have written the pairs of equivalents in one line.

\textit{Case 2} \quad \text{Take} \ n = 20, \ k = 8.

Set A consists of the two pairs of equivalents

\[
\begin{array}{cccccccccccc}
1 & 4 & 4 & 1 & 1 & 4 & 4 & 1 & 4 & 1 & 1 & 4 \\
2 & 3 & 2 & 2 & 3 & 2 & 3 & 3 & 2 & 3 & 2 & 3 \\
\end{array}
\]
Set B consists of eight pairs of equivalents:

\[
\begin{array}{cccccccc}
1 & 1 & 7 & 1 & 1 & 1 & 7 & 1 \\
1 & 2 & 5 & 2 & 1 & 2 & 5 & 2 \\
1 & 3 & 3 & 3 & 1 & 3 & 3 & 3 \\
1 & 4 & 1 & 4 & 1 & 4 & 1 & 4 \\
2 & 1 & 6 & 1 & 2 & 1 & 6 & 1 \\
2 & 2 & 4 & 2 & 2 & 2 & 4 & 2 \\
2 & 3 & 2 & 3 & 2 & 3 & 2 & 3 \\
3 & 1 & 5 & 1 & 3 & 1 & 5 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
7 & 1 & 1 & 1 & 7 & 1 & 1 & 1 \\
5 & 2 & 1 & 2 & 5 & 2 & 1 & 2 \\
3 & 3 & 1 & 3 & 3 & 3 & 1 & 3 \\
4 & 1 & 4 & 1 & 4 & 1 & 4 & 1 \\
6 & 1 & 2 & 1 & 6 & 1 & 2 & 1 \\
4 & 2 & 2 & 2 & 4 & 2 & 2 & 2 \\
3 & 2 & 3 & 2 & 3 & 2 & 3 & 2 \\
5 & 1 & 3 & 1 & 5 & 1 & 3 & 1 \\
\end{array}
\]

Case 3 — Take \( n = 20, \ k = 10 \).

Each element of set A pairs with an element of set B:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 1 6 1 1 1 1 1 6 1 1</td>
<td>6 1 1 1 1 6 1 1 1 1</td>
</tr>
<tr>
<td>1 2 4 2 1 1 2 4 2 1</td>
<td>4 2 1 1 2 4 2 1 1 2</td>
</tr>
<tr>
<td>1 3 2 3 1 1 3 2 3 1</td>
<td>2 3 1 1 3 2 3 1 1 3</td>
</tr>
<tr>
<td>2 1 4 1 2 2 1 4 1 2</td>
<td>4 1 2 2 1 4 1 2 2 1</td>
</tr>
<tr>
<td>2 2 2 2 2 2 2 2 2 2</td>
<td>2 2 2 2 2 2 2 2 2 2</td>
</tr>
<tr>
<td>3 1 2 1 3 3 1 2 1 3</td>
<td>2 1 3 3 1 2 1 3 3 1</td>
</tr>
</tbody>
</table>

Case 4 — Take \( n = 18, \ k = 10 \).

Again each element of A pairs with an element of B:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 1 5 1 1 1 1 1 5 1 1</td>
<td>5 1 1 1 1 5 1 1 1 1</td>
</tr>
<tr>
<td>1 2 3 2 1 1 2 3 2 1</td>
<td>3 2 1 1 2 3 2 1 1 2</td>
</tr>
<tr>
<td>1 3 1 3 1 1 3 1 3 1</td>
<td>1 3 1 1 3 1 3 1 3 1</td>
</tr>
<tr>
<td>2 1 3 1 2 2 1 3 1 2</td>
<td>3 1 2 2 1 3 1 2 2 1</td>
</tr>
<tr>
<td>2 2 1 2 2 2 2 1 2 2</td>
<td>1 2 2 2 2 1 2 2 2 2</td>
</tr>
<tr>
<td>3 1 1 1 3 3 1 1 1 3</td>
<td>1 1 3 3 1 1 3 3 1</td>
</tr>
</tbody>
</table>

The reader will find that the general case needs no new technique.

(b) When \( n \) is odd.

The Diophantine equation we have now to consider is

\[ d_1 + 2d_2 + \ldots + 2d_i + d_{i+1} = n. \]

Since \( n \) is odd, we can avoid duplication by assuming \( d_1 \) to be odd and \( d_{i+1} \) to be even.

The number of symmetric \( k \)-gons is readily found to be

\[
\binom{n - \frac{1}{2}}{\frac{k}{2}}.
\]

...(21)
Putting together (20) and (21), we can state that

For $k$ even, the number of symmetric $k$-gons is given by

$$\left(\left\lfloor \frac{n}{2} \right\rfloor ; \left\lfloor \frac{k}{2} \right\rfloor \right).$$

3. The Problem of Reis

3.1. By far the best way of stating the problem of Reis will be to ask:

If a circle is drawn with an arbitrarily fixed radius and its circumference is divided into $n$ equal parts, find $R(n, k)$—the number of mutually incongruent convex $k$-gons that can be obtained by joining $k$ of the $n$ points of division.

Alternatively, one can ask

What is the number $R(n, k)$ of equivalence classes into which the $(n - 1; k - 1)$ decompositions of $n$ into $k$ parts can be decomposed.

We can easily prove two interesting theorems concerning $R(n, k)$.

**Theorem 1** — For $n > k$,

$$R(n, k) \geq (n - 1; k - 1)/(2k).$$

**Proof:** Since no equivalence class into which the decompositions of $n$ into $k$ parts can be decomposed can have more than $2k$ elements, the theorem follows immediately.

**Theorem 2** — For each $k < n$,

$$R(n, k) = R(n, n - k).$$

**Proof:** Every time we select $k$ of the $n$ points of division on our basic circle, we are left with $(n - k)$ points which when joined to form a convex $(n - k)$-gon produce a unique figure corresponding to the given $k$-gon. Moreover, if the $k$-gons are incongruent, so also are the corresponding $(n - k)$-gons. Hence the theorem follows.

Evidently

$$R(n, n) = 1 \text{ and } R(n, n - 1) = 1.$$

We can, therefore, take

$$R(n, 0) = 1 \text{ and } R(n, 1) = 1.$$
them into equivalence classes. But the labour this will involve will be prohibitive even for small values of \( n \) and \( k \). A short-cut will be to consider only those decompositions which start with a suitably chosen element and break these up into equivalence classes. The number of classes so obtained will be \( R(n, k) \). We have already stated how such an element can best be chosen.

The following example will show how one proceeds along these lines.

Take \( n = 11, k = 5 \).

The partitions of 11 into 5 parts are:

1. 11117  
2. 11126  
3. 11135  
4. 11144  
5. 11225  
6. 11234  
7. 11333  
8. 12224  
9. 12233  
10. 22223

First note that the contribution which the decompositions arising from any of these partitions, make to \( R(n, k) \) depends only on the type of the partition and not on the size of the parts. We will, therefore, do well to put together those partitions which are of one type.

<table>
<thead>
<tr>
<th>Type</th>
<th>Partitions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1 4)</td>
<td>11117; 22223</td>
</tr>
<tr>
<td>(1 1 3)</td>
<td>11126; 11135; 12224</td>
</tr>
<tr>
<td>(2 3)</td>
<td>11144; 11333</td>
</tr>
<tr>
<td>(1 2 2)</td>
<td>11225; 12233</td>
</tr>
<tr>
<td>(1 1 1 2)</td>
<td>11234</td>
</tr>
</tbody>
</table>

We need consider only one member of each type, say the first from the left. Moreover, we take only those decompositions which start with one of the most suitable elements.

<table>
<thead>
<tr>
<th>Partition</th>
<th>Decompositions</th>
<th>Classes</th>
<th>Number of classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>11117</td>
<td>71111</td>
<td>71111</td>
<td>1</td>
</tr>
<tr>
<td>11126</td>
<td>61112</td>
<td>61112 ( { )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>61121</td>
<td>62111 ( } )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>61211</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>62111</td>
<td>61211 ( { )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>61211 ( } )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11144</td>
<td>41114</td>
<td>41114 ( { )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>41141</td>
<td>44111 ( } )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>41411</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>44111</td>
<td>41411 ( { )</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>41411 ( } )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Enumeration of Incongruent Cyclic $k$-gons

<table>
<thead>
<tr>
<th>Partition</th>
<th>Decompositions</th>
<th>Classes</th>
<th>Number of classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>11225</td>
<td>51122</td>
<td>51122</td>
<td></td>
</tr>
<tr>
<td></td>
<td>51212</td>
<td>52211</td>
<td></td>
</tr>
<tr>
<td></td>
<td>51221</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>52112</td>
<td>51212</td>
<td></td>
</tr>
<tr>
<td></td>
<td>52121</td>
<td>52121</td>
<td></td>
</tr>
<tr>
<td></td>
<td>52211</td>
<td>51221</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>52112</td>
<td></td>
</tr>
</tbody>
</table>

| 11234     | 41123         | 41123   |
|           | 41132         | 43211   |
|           | 41213         |         |
|           | 41231         | 41132   |
|           | 41312         | 42311   |
|           | 41321         |         |
|           | 42113         | 41213   |
|           | 42131         | 43121   |
|           | 42311         |         |
|           | 43112         | 41231   |
|           | 43121         | 41321   |
|           | 43211         |         |
|           |               | 41312   |
|           |               | 42131   |
|           |               | 42113   |
|           |               | 43112   |

Hence

\[ R(11, 5) = 2.1 + 3.2 + 2.2 + 2.4 + 1.6 = 26. \]

From the above, it will be clear that to find $R(n, k)$, we have to determine two things:

**One :** What contribution does a given type of partition make to $R(n, k)$?

**Two :** How many partitions of $n$ into $k$ parts belong to that type?

For one, we need not consider the given $n$ at all. It will be enough to consider the least $n$ for which a partition of that type exists. The importance of knowing such an $n$ will be realized a little later. We will denote such an $n$ by $n_o$.

**Example** — For the type $(1 2 2 3)$, the $n_o$ is the least number which can be written in the form

\[ 3u_1 + 2u_2 + 2u_3 + u_4 \]

with $u$'s all distinct positive integers. Evidently, for $n_o$ we must take $u_1 = 1$, $u_2 = 2$, $u_3 = 3$, $u_4 = 4$.

This gives $n_o = 17$. 
3.3. It will not be out of place here to give a few easy-to-prove rules, for determining $C(T)$ — the contribution which a partition of type $T$ will make to $R$ for any $n$.

(i) When $T$ has at least three odd frequencies, no symmetric polygons can arise. Each class will, therefore, contain the same number of decompositions, if the g.c.d. of the frequencies is 1. To find this number, it will be enough to consider only one of the classes.

*Example* — Take $T = (3 3 3 2)$.

Consider the partition 1111222233344.

The number of decompositions starting with 4 is given by

$$12!/(5! 3! 3! 1!).$$

The class to which the decomposition 44333222211111 belongs has the four members

44333222211111, 3333222111114, 41111122223334, 44111112222333.

Hence $C(5 3 3 2) = 12!/(5! 3! 3! 1! 4)$.

(ii) When the partition has just one unreported summand and the frequencies of all other summands are even, we consider the decompositions which start with the unreported element. Then each ordinary class has two decompositions belonging to it, while each decomposition representable by a symmetric $k$-gon forms a class by itself.

Let $T = (1 2a_1 2a_2 \ldots 2a_i)$; where each $a > 0$.

Then, we readily have

$$2C(T) = (2a_1 + 2a_2 + \ldots + 2a_i)!/(2a_1)! \cdot (2a_2)! \cdots (2a_i)!$$

$$+ (a_1 + a_2 + \ldots + a_i)!/a_1! \cdot a_2! \ldots a_i!.$$

*Example* — Take $T = (1 2 2 4)$, then

$$2C(1 2 2 4) = 8!/2! \cdot 2! \cdot 4! + 4!/1! \cdot 1! \cdot 2!.$$

(iii) When $T = (1 2a_1 - 1 2a_2 2a_3 \ldots 2a_i)$; with each $a > 0$;

we have

$$2C(T) = (2a_1 - 1 + 2a_2 + 2a_3 + \ldots + 2a_i)!/(2a_1 - 1)! \cdot (2a_2)! \cdots (2a_i)!$$

$$+ (a_1 - 1 + a_2 + \ldots + a_i)!/(a_1 - 1)! \cdot a_2! \ldots a_i!.$$

Recall that $0!$ is taken as 1.

This rule covers the case when the partition has two unreported summands.

*Example* — $2C(1 3 2 4) = 9!/3! \cdot 2! \cdot 4! + 4!/1! \cdot 1! \cdot 2!$.

(iv) When there are three or more unreported summands in the partition, $C(T)$ is half the number of decompositions which start with one of the unreported summands, this one remaining fixed.
Example — $T = (1\ 1\ 2\ 3)$, \( C(T) = \frac{1}{3} (7!/1!1!2!3)! \).

3.4. We give below, for reference, \( C(T) \) for each \( T \), when \( k = 5 \) or \( 6 \).

\[
\begin{array}{cccc}
T & C(T) & T & C(T) \\
(5) & 1 & (6) & 1 \\
(1\ 4) & 1 & (1\ 5) & 1 \\
(2\ 3) & 2 & (2\ 4) & 3 \\
(1\ 1\ 3) & 2 & (3\ 3) & 3 \\
(1\ 2\ 2) & 4 & (1\ 1\ 4) & 3 \\
(1\ 1\ 1\ 2) & 6 & (1\ 2\ 3) & 6 \\
(1\ 1\ 1\ 1\ 1) & 12 & (2\ 2\ 2) & 11 \\
 & & (1\ 1\ 1\ 3) & 10 \\
 & & (1\ 1\ 2\ 2) & 16 \\
 & & (1\ 1\ 1\ 1\ 2) & 30 \\
 & & (1\ 1\ 1\ 1\ 1) & 60 \\
\end{array}
\]

Note that the total number of \( T \)'s for any \( k \) is \( p(k) \).

We leave it to the reader to compute \( C(T) \)'s for \( k = 7 \).

3.5. The Number of Partitions of \( n \) of a given Type \( T \)

Suppose, we wish to find the number of those partitions of \( n \) which are of the type \((2\ 3)\).

Let the summand which is repeated twice in the partition be denoted by \( u \) and that which is repeated thrice by \( v \). Then the required number of partitions is the same as the number of solutions of the Diophantine equation

\[
2u + 3v = n \quad \ldots (22)
\]

where \( u \) and \( v \) are distinct positive integers.

From the theory of partitions, it is well known that the number of solutions of (22) including those with \( u = v \), is the same as the coefficient of

\[ x^n \] in the ascending power expansion of \( x^3/(1 - x^2) \ (1 - x^3) \).

Since the number of solutions of (22) with \( u = v \), is in the same manner the same as the coefficient of

\[ x^n \] in the ascending power expansion of \( x^3/(1 - x^3) \),
it follows that the required number is the coefficient of

\[ x^n \] in \( x^3 \{(1 - x^2)^{-1} \ (1 - x^3)^{-1} - (1 - x^6)^{-1}\} \).

Writing \( p(n, (2\ 3)) \) for the number of partitions of \( n \) which are of the type \((2\ 3)\), we thus have
\[ \sum_{n \geq 5} p(n, (2 \ 3)) x^n = x^5 \frac{(1 - x^5) - (1 - x^2)(1 - x^3)}{(1 - x^2)(1 - x^3)(1 - x^5)} . \] ...\(23\)

The expression on the right of (23) is called the generating function of \( p(n, (2 \ 3)) \).

For any given \( k \), the best denominator to use for all types of partitions is
\[ D_k = (1 - x) (1 - x^2) \ldots (1 - x^k) \] ...\(24\)
and this we shall use henceforth. We shall thus be left to record only the numerators of generating functions. For this we shall adopt the notation:
\[ (b_0 + b_1 + b_2 + \ldots + b_i)_x = b_0 + b_1 x + b_2 x^2 + \ldots + b_i x^i \] ...\(25\)
where the \( b \)'s are integers not necessarily positive.

Thus, (23) will take the form
\[ \sum_{n \geq 7} p(n, (2 \ 3)) x^n = x^7(1 + 0 - 1 - 2 + 1 + 0 + 1 + 2 - 2)_x / D_5. \] ...\(26\)
This implies that (22) has no solution for \( n < 7 \). (So the least \( n \) has asserted itself).

We shall denote the numerator in the generator of \( p(n, T) \) for any type \( T \) of partitions by \( P(T) \).

If \( T = (a_1 \ a_2 \ a_3 \ldots \ a_i) \)
where \( a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_i \); and \( a_1 + a_2 + a_3 + \ldots + a_i = k \);
then \( P(T) \) is of the form
\[ x^{n_0}(1 + b_1 + b_2 + \ldots + b_r)_x \]
where
\[ n_0 = a_i + 2a_{i-1} + 3a_{i-2} + \ldots + ja_1 \] ...\(27\)
and
\[ r = (k + 1; 2) - n_0. \] ...\(28\)
We might also state that the total number of solutions of the Diophantine equation
\[ a_1 u_1 + a_2 u_2 + a_3 u_3 + \ldots + a_i u_i = n \]
in positive integers \( u \) is the coefficient of \( x^n \) in the expansion (in ascending powers of \( x \)) of
\[ x^k/(1 - x^{a_1}) \ (1 - x^{a_2}) \ldots (1 - x^{a_i}). \] ...\(29\)
Write (29) with \( D_k \) as the denominator and let \( Q(T) \) denote the numerator. Then \( Q(T) \) will be of the form
\[ x^k(1 + c_1 + c_2 + \ldots + c_s), \quad s = (k; 2) \quad \ldots (30) \]

where the \( c \)'s are integers but not necessarily positive.

To obtain \( P(T) \) from \( Q(T) \), we have to remove somehow from \( Q(T) \) all terms of degree less than \( n_0 \) in \( x \). We have also to bear in mind that the coefficient of \( x^{n_0} \) in the final answer has to be 1. How this is done, we illustrate by an example in the next sub-section.

3.6. The Generating Functions for \( R(n, 5) \) and \( R(n, 6) \)

Write \( Q(T) = x^kN(T), \quad P(T) = x^{n_0}M(T) \).

(i) Case \( k = 5 \).

We take each type in turn.

Type (5), we have

\[ n_0(5) = 5. \]

\[ N(5) = (1 - 1 - 1 + 0 + 0 + 2 + 0 + 0 - 1 - 1 + 1)_x = M(5). \]

Type (1 4), we have

\[ n_0(1 4) = 6. \]

\[ N(1 4) = (1 + 0 - 1 - 1 + 0 + 0 + 0 + 1 + 1 + 0 - 1)_x. \]

To get rid of the 5th degree term from \( Q(1 4) \), we subtract \( M(5) \) from \( N(1 4) \). This we do most conveniently as follows:

\[
\begin{array}{c}
1 + 0 - 1 - 1 + 0 + 0 + 0 + 1 + 1 + 0 - 1 \\
- 1 + 1 + 1 + 0 + 0 - 2 - 0 - 0 + 1 + 1 - 1 \\
\hline
1 + 0 - 1 - 1 + 0 + 0 + 1 + 2 + 1 - 2
\end{array}
\]

This means that

\[ M(1 4) = (1 + 0 - 1 + 0 - 2 + 0 + 1 + 2 + 1 - 2)_x \]

Type (2 3) : \( n_0(2 3) = 7 \);

\[ N(2 3) = (1 - 1 + 0 + 1 + 0 + 1 + 0 + 0 + 1 - 1)_x. \]

Processing:

\[
\begin{array}{c}
1 - 1 + 0 + 0 - 1 + 0 + 1 + 0 + 0 + 1 - 1 \\
- 1 + 1 + 1 - 0 - 0 + 0 + 1 + 1 - 1 \\
\hline
1 + 0 - 1 - 2 + 1 + 0 + 1 + 2 - 2
\end{array}
\]

Hence

\[ M(2 3) = (1 + 0 - 1 - 2 + 1 + 0 + 1 + 2 - 2)_x. \]
Type (1 1 3): \( n_0(1 1 3) = 8; \)

\[
N(1 1 3) = (1 + 1 + 0 + 0 - 1 - 2 - 1 + 0 + 0 + 1 + 1)_z.
\]

Processing:

\[
\begin{align*}
1 + 1 + 0 + 0 - 1 - 2 - 1 &+ 0 + 0 + 1 + 1 & N(1 1 3) \\
- 1 + 1 + 1 - 0 - 2 - 0 &- 0 + 1 + 1 - 1 & - M(5) \\
- 2 - 0 + 2 + 0 + 4 + 0 - 2 - 4 - 2 + 4 & & - 2M(1 4) \\
- 1 + 0 + 1 + 2 - 1 + 0 &- 1 - 2 + 2 & - M(2 3) \\
) 2 + 0 + 2 - 2 - 2 - 4 - 2 + 6 & & \text{Divide by 2} \\
1 + 0 + 1 - 1 - 1 - 2 - 1 + 3 & &
\end{align*}
\]

Thus

\[
M(1 1 3) = (1 + 0 + 1 - 1 - 1 - 2 - 1 + 3)_z.
\]

Type (1 2 2): \( n_0(1 2 2) = 9. \)

\[
N(1 2 2) = (1 + 0 + 1 - 1 + 0 - 2 + 0 - 1 + 1 + 0 + 1)_z.
\]

Processing:

\[
\begin{align*}
1 + 0 + 1 &- 1 + 0 - 2 + 0 - 1 + 1 + 0 + 1 & N(1 2 2) \\
- 1 + 1 + 1 + 0 + 0 - 2 + 0 + 0 + 1 + 1 - 1 & & - M(5) \\
- 1 + 0 + 1 + 0 + 2 + 0 - 1 - 2 - 1 + 2 & & - M(1 4) \\
- 2 + 0 + 2 + 4 - 2 + 0 - 2 - 4 + 4 & & - 2M(2 3) \\
) 2 + 2 - 2 - 2 - 2 + 4 + 6 & & \text{Divide by 2} \\
1 + 1 - 1 - 1 - 2 + 3 & &
\end{align*}
\]

i.e. \( M(1 2 2) = (1 + 1 - 1 - 1 - 2 + 3)_z. \)

Type (1 1 1 2): \( n_0(1 1 1 2) = 11. \)

\[
N(1 1 1 2) = (1 + 2 + 3 + 3 + 2 + 0 - 2 - 3 - 3 - 2 - 1)_z.
\]

Processing:

\[
\begin{align*}
1 + 2 + 3 + 3 + 2 + 0 &- 2 - 3 - 3 - 2 - 1 & N(1 1 1 2) \\
- 1 + 1 + 1 + 0 + 0 - 2 + 0 + 0 + 1 + 1 - 1 & & - M(5) \\
- 3 + 0 + 3 + 0 + 6 + 0 - 3 &- 6 - 3 + 6 & - 3M(1 4) \\
- 4 + 0 + 4 + 8 - 4 + 0 - 4 - 8 + 8 & & - 4M(2 3) \\
- 6 + 0 - 6 + 6 + 6 + 12 + 6 - 18 & & - 6M(1 1 3) \\
- 6 - 6 + 6 + 6 + 6 + 12 - 18 & & - 6M(1 2 2) \\
) 6 + 6 + 6 + 6 - 24 & & \text{Divide by 6} \\
1 + 1 + 1 + 1 + 1 &- 4 &
\end{align*}
\]

so that \( M(1 1 1 2) = (1 + 1 + 1 + 1 - 4)_z. \)
Type (1 1 1 1 1) : \( n_6(1 1 1 1 1) = 15 \).

\[ N(1 1 1 1 1) = (1 + 4 + 9 + 15 + 20 + 22 + 20 + 15 + 9 + 4 + 1) \times \]

Processing:

\[
\begin{align*}
1 + 4 + 9 + 15 + 20 + 22 + 20 + 15 + 9 + 4 + 1 & \quad N(1 1 1 1 1) \\
-1 + 1 + 1 + 0 + 0 - 2 + 0 + 0 + 1 + 1 - 1 & \quad - M(5) \\
-5 + 0 + 5 + 0 + 10 + 0 - 5 - 10 - 5 + 10 & \quad - 5M(1 4) \\
-10 + 0 + 10 + 20 - 10 + 0 - 10 - 20 + 20 & \quad - 10M(2 3) \\
-20 + 0 - 20 + 20 + 20 + 40 + 20 - 60 & \quad - 20M(1 1 3) \\
-30 - 30 + 30 + 30 + 30 + 60 - 90 & \quad - 30M(1 2 2) \\
-60 - 60 - 60 - 60 + 240 & \quad - 60M(1 1 1 2) \\
\end{align*}
\]

\[
\frac{) \quad 120 \quad \text{Divide by 120}}{1}
\]

Thus \( M(1 1 1 1 1) = (1)_0 \).

Using the table of \( C(T) \)'s in section 3.4, we have

\[
\begin{align*}
1 \ M(5) & \quad = 1 - 1 - 1 + 0 + 0 + 2 + 0 + 0 - 1 - 1 - 1 \\
1 \ M(1 4) & \quad = 1 + 0 - 1 + 0 - 2 + 0 + 1 + 2 + 1 - 2 \\
2 \ M(2 3) & \quad = 2 + 0 - 2 - 4 + 2 + 0 + 2 + 4 - 4 \\
2 \ M(1 1 3) & \quad = 2 + 0 + 2 - 2 - 2 - 4 - 2 + 6 \\
4 \ M(1 2 2) & \quad = 4 + 4 - 4 - 4 - 4 - 4 - 4 - 4 + 12 \\
6 \ M(1 1 1 2) & \quad = 6 + 6 + 6 + 6 - 24 \\
12 \ M(1 1 1 1 1) & \quad = 1 + 0 + 1 + 1 + 2 + 2 + 2 + 1 + 1 + 0 + 1 \\
\end{align*}
\]

This shows that \( R(n, 5) \) is the coefficient of \( x^n \) in

\[ x^5(1 + 0 + 1 + 1 + 2 + 2 + 2 + 1 + 1 + 0 + 1)_0/D_5. \]

Letting

\[ (1 + 0 + 1 + 1 + 2 + 2 + 2 + 1 + 1 + 0 + 1)_0 = D_5 \sum_{j > 5} R(j, 5) x^{j-5}, \]

... (31)

and comparing the coefficients of like powers of \( x \) on the two sides, the values of \( R(n, 5) \) can be computed in succession. In fact for values of \( n > 15 \), one gets a recurrence relation from which the values of \( R(n, 5) \) can be readily computed. This is about half as laborious as the method of Reis.

Proceeding on the same lines, one can show that
\((1 + 0 + 2 + 2 + 5 + 4 + 9 + 6 + 9 + 6 + 7 + 3 + 4 + 1 + 1) \equiv D_{k} \sum_{j \geq 6} R(j, 6) x^{j-6}. \) ...(32)

3.7. Closed Formulae for \(R(n, k)\)

The break-through came unexpectedly, when I tried to express the generating functions for \(R(n, 5)\) and \(R(n, 6)\) in terms of \(N\)'s in place of \(M\)'s.

From the processing in 3.6, it will be seen that

\[
\begin{align*}
N(5) &= M(5) \\
N(1\ 4) &= M(5) + M(1\ 4) \\
N(2\ 3) &= M(5) + M(2\ 3) \\
N(1\ 1\ 3) &= M(5) + 2M(1\ 4) + M(2\ 3) + 2M(1\ 1\ 3) \\
N(1\ 2\ 2) &= M(5) + M(1\ 4) + 2M(2\ 3) + 2M(1\ 2\ 2) \\
N(1\ 1\ 1\ 2) &= M(5) + 3M(1\ 4) + 4M(2\ 3) + 6M(1\ 1\ 3) + 6M(1\ 2\ 2) + 6M(1\ 1\ 1\ 2)
\end{align*}
\]

and finally

\[
N(1\ 1\ 1\ 1\ 1) = M(5) + 5M(1\ 4) + 10M(2\ 3) + 20M(1\ 1\ 3) + 30M(1\ 2\ 2) + 60M(1\ 1\ 1\ 2) + 120M(1\ 1\ 1\ 1\ 1).
\]

In the form of a matrix equation, these relations can be written as

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 2 & 0 & 0 & 0 \\
1 & 1 & 2 & 0 & 2 & 0 & 0 \\
1 & 3 & 4 & 6 & 6 & 6 & 0 \\
1 & 5 & 10 & 20 & 30 & 60 & 120
\end{bmatrix}
\begin{bmatrix}
M(5) \\
M(1\ 4) \\
M(2\ 3) \\
M(1\ 1\ 3) \\
M(1\ 2\ 2) \\
M(1\ 1\ 1\ 2) \\
M(1\ 1\ 1\ 1\ 1)
\end{bmatrix}
\]

Hence, we get
10 \[ \begin{array}{cccccccc} M(5) \\ M(1\ 4) \\ 2M(2\ 3) \\ 2M(1\ 1\ 3) \\ 4M(1\ 2\ 2) \\ 6M(1\ 1\ 1\ 2) \\ 12M(1\ 1\ 1\ 1\ 1) \end{array} \]

\[
\begin{pmatrix}
10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-10 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\
-20 & 0 & 20 & 0 & 0 & 0 & 0 & 0 \\
20 & -20 & -10 & 10 & 0 & 0 & 0 & 0 \\
40 & -20 & -40 & 0 & 20 & 0 & 0 & 0 \\
-60 & 60 & 50 & -30 & -30 & 10 & 0 & 0 \\
24 & -30 & -20 & 20 & 15 & -10 & 1 & 0 \\
\end{pmatrix}
\]

Whence, adding the entries in each column of the square matrix, it is readily seen that 10 \( R(n, 5) \) is the coefficient of \( x^n \) in

\[
\frac{4N(5) + 5N(1\ 2\ 2) + N(1\ 1\ 1\ 1\ 1)}{D^5}
\]
i.e. in

\[
4(1 - x^3)^{-1} + 5(1 - x)^{-1} (1 - x^2)^{-2} + (1 - x)^{-5}.
\]

...(33)

It simplifies matters, if we write (33) in the form

\[
5(1 + x) (1 - x^2)^{-3} + (1 - x)^{-5} + 4(1 - x^5)^{-1}.
\]

...(34)

Hence

\[
10 \ R(n, 5) = 5(\{(n - 1)/2\}; 2) + (n - 1; 4) + 4 \text{ if } (n, 5) = 5;
\]

\[
= 5(\{(n - 1)/2\}; 2) + (n - 1; 4) \text{ otherwise.}
\]

For \( k = 6 \), we get the matrix equation:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 3 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\
1 & 3 & 3 & 2 & 6 & 3 & 0 & 6 & 0 & 0 \\
1 & 2 & 3 & 4 & 2 & 4 & 6 & 0 & 4 & 0 \\
1 & 4 & 7 & 8 & 12 & 16 & 18 & 24 & 24 & 0 \\
1 & 6 & 15 & 20 & 30 & 60 & 90 & 120 & 180 & 360 & 720
\end{pmatrix}
\]

\( M^* = N^* \)
where \[ M^* = \begin{bmatrix} M(6) \\ M(15) \\ M(24) \\ M(33) \\ M(114) \\ M(123) \\ M(222) \\ M(1113) \\ M(1122) \\ M(11112) \\ M(111111) \end{bmatrix} \quad \text{and} \quad N^* = \begin{bmatrix} N(6) \\ N(15) \\ N(24) \\ N(33) \\ N(114) \\ N(123) \\ N(222) \\ N(1113) \\ N(1122) \\ N(11112) \\ N(111111) \end{bmatrix} \]

Proceeding as in the case of \( k = 5 \), we finally find that \( 12 \, R(n, 6) \) is the coefficient of \( x^{n-6} \) in

\[
2(1 - x^6)^{-1} \div 2(1 - x^3)^{-2} + 4(1 - x^2)^{-3} + 3(1 - x)^{-2} (1 - x^2)^{-2} + (1 - x)^{-6}
\]

i.e. in

\[
6(1 + x) (1 - x^2)^{-4} + (1 - x)^{-6} + (1 - x^2)^{-3} + 2(1 - x^2)^{-2} + 2(1 - x^6)^{-1}
\]

...(35)

Results (34) and (35) are very suggestive and in view of the prediction made by Reis, led me to the conjecture:

"\( 2k \, R(n, k) \) is the coefficient of \( x^{n-k} \) in

\[
k(1 + x) (1 - x^2)^{-[k+2]/2} + \sum_{d \mid g} \phi(d) (1 - x^d)^{-n/d}
\]

...(36)

where \( g = (n, k) \)."

To prove the conjecture, all that is necessary is to show that the conjecture is not at variance with the fundamental relation

\[
R(n, k) = R(n, n - k).
\]

...(37)

Let \( n - k = h \), then we have to show that for each divisor \( d \) of \( g \), relation (37) holds good.

Now, we have

\[
2kh R(n, k) = hk \left( \left[ \frac{k + h - t}{2} \right]; \left[ \frac{k}{2} \right] \right) + h \sum_{d \mid g} \phi(d) \left( \frac{n}{d} - 1; \frac{k}{d} - 1 \right)
\]
where \( t = 0 \) or \( 1 \) according as \( k \) is even or odd; and

\[
2hk R(n, h) = kh \left( \left[ \frac{k + h - s}{2} \right] \left[ \frac{h}{2} \right] \right) + k \sum_{d \mid g} \phi(d) \left( \frac{n}{d} - 1; \frac{h}{d} - 1 \right)
\]

where \( s = 0 \) or \( 1 \) according as \( h \) is even or odd.

It is easy to see that

\[
\left( \left[ \frac{k + h - t}{2} \right] \left[ \frac{k}{2} \right] \right) = \left( \left[ \frac{k + h - s}{2} \right] \left[ \frac{h}{2} \right] \right) \text{ for all } h \text{ and } k.
\]

Also

\[
h \left( \frac{n}{d} - 1; \frac{k}{d} - 1 \right) = h \left( \frac{k + h}{d} - 1; \frac{k}{d} - 1 \right)
\]

\[
= h \left( \frac{k + h}{d} - 1; \frac{h}{d} \right)
\]

\[
= h \left( \frac{k + h}{d} - 1 \right)! \left( \frac{k}{d} - 1 \right)! \left( \frac{h}{d} \right)!
\]

\[
= d \left( \frac{k + h}{d} - 1 \right)! \left( \frac{k}{d} - 1 \right)! \left( \frac{h}{d} - 1 \right)!
\]

\[
= k \left( \frac{k + h}{d} - 1 \right)! \left( \frac{h}{d} - 1 \right)! \left( \frac{k}{d} \right)!
\]

and we are through.

Of course, induction takes care of the rest.

4. **The Function \( R'(n, k) \)**

4.1. Our account will not be complete, if we do not consider, at least very briefly, the function \( R'(n, k) \) which is closely related to \( R(n, k) \) which has been dealt with at length in the foregoing pages.

Two decompositions of \( n \) into \( k \) parts are defined to be weakly equivalent if the \( k \)-gons representing them are directly congruent.

\( R'(n, k) \) denotes the number of equivalence classes into which the decompositions of \( n \) into \( k \) parts, can now be divided.

Besides replacing the table in section 3.4 by the following, no new technique is required.
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</table>

We give only the final matrix in the case of $k = 6$:

\[
\begin{bmatrix}
6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-18 & 0 & 18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-12 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\
30 & -30 & -15 & 0 & 15 & 0 & 0 & 0 & 0 & 0 \\
120 & -60 & -60 & -60 & 0 & 60 & 0 & 0 & 0 & 0 \\
32 & 0 & -48 & 0 & 0 & 0 & 16 & 0 & 0 & 0 \\
-120 & 120 & 60 & 40 & -60 & -60 & 0 & 20 & 0 & 0 \\
-270 & 180 & 225 & 90 & -45 & -180 & -45 & 0 & 45 & 0 \\
360 & -360 & -270 & -120 & 180 & 300 & 45 & -60 & -90 & 15 \\
-120 & 144 & 90 & 40 & -90 & -120 & -15 & 40 & 45 & -15 \\
\end{bmatrix}
\]

The column sums are

\[2 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1\]

From this we conclude that $6R'(n, 6)$ is the coefficient of $x^{n-6}$ in

\[
2(1 - x^6)^{-1} + 2(1 - x^3)^{-2} + (1 - x^2)^{-3} + (1 - x)^{-6}.
\]

As in the case of $R(n, k)$, we finally get

\[
k \ R'(n, k) = \sum_{d \mid (n, k)} \phi(d) \left( \frac{n}{d} - 1; \frac{k}{d} - 1 \right).
\]... (38)
This is in conformity with the fundamental relation
\[ R'(n, k) = R'(n, n - k). \]
We further have
\[ 2 R(n, k) = S(n, k) + R'(n, k) \]... (39)
where \( S(n, k) \) denotes the number of symmetric \( k \)-gons for the given \( n \). A combinatorial proof of (39) is easy to give.

We might here note that the fundamental relation can be used to find the expression for \( R'(n, k) \) for a given \( k \) after the values for \( R'(n, m) \) have been obtained for each \( m < k \), as the following example will illustrate.

Take \( k = 10 \).

Let
\[ 10 R'(n, 10) = \sum_{d \mid 10} a_d \left( \frac{n}{d} - 1; \frac{k}{d} - 1 \right). \]
To determine the \( a \)'s, take \( n = 10, 11, 12, 15 \) in turn.

We thus get the relations:
\[ a_1 + a_2 + a_5 + a_{10} = 10 R'(10, 0) = 10; \]
\[ 10 a_1 = 10 R'(11, 1) = 10; \]
\[ 55 a_1 + 5 a_2 = 10 R'(12, 2) = 60; \]
\[ 2002 a_1 + 2a_5 = 10 R'(15, 5) = 2010. \]
These give
\[ a_1 = 1 = 1; a_2 = 1 = \phi(2); a_5 = 4 = \phi(5); \]
and \( a_{10} = 4 = \phi(10) \).

4.2. There is an interesting formula for
\[ B'(n) = \sum_{k=0}^{n} R'(n, k). \]
This we proceed to find.

We have
\[ k R'(n, k) = \sum_{d \mid (n, k)} \phi(d) \left( \frac{n}{d} - 1; \frac{k}{d} - 1 \right). \]... (40)
Therefore

\[(n - k) R'(n, n - k) = \sum_{d \mid (n, n-k)} \phi(d) \left( \frac{n}{d} - 1; \frac{n - k}{d} - 1 \right)\]

or what is the same thing

\[(n - k) R'(n, k) = \sum_{d \mid (n, k)} \phi(d) \left( \frac{n}{d} - 1; \frac{k}{d} \right). \quad \ldots(41)\]

Adding (40) and (41), we get

\[n R'(n, k) = \sum_{d \mid (n, k)} \phi(d) \left( \frac{n}{d}; \frac{k}{d} \right). \quad \ldots(42)\]

Whence

\[n \sum_{k=0}^{n} R'(n, k) = \sum_{d \mid n} \phi(d) \{ (n/d; 0) \div (n/d; 1) \div \ldots \div (n/d; n/d) \} \]

\[n B'(n) = \sum_{d \mid n} \phi(d) 2^{n/d}. \quad \ldots(43)\]

We leave it to the reader to find a similar formula for

\[B(n) = \sum_{k=0}^{n} R(n, k).\]

It will thus be seen that the problem of Reis is directly related to the bead-stringing problem* when the beads are available in two different colours.

5. TABLES

The tables that follow give for \(n \leq 100, \ 3 \leq k \leq 12\), the values of \(S(n, k)\)—the number of symmetric \(k\)-gons for any given \(n\) and \(k\) in the range and also the values of \(R'(n, k)\) in the said range.

Note that

\[R'(n, 0) = 1 = R'(n, 1)\]

and

\[R'(n, 2) = \lfloor n/2 \rfloor.\]

In preparing these tables, the Royal Society "Tables of Binomial Coefficients" [University Press, Cambridge (1954)] have been freely used.

---

TABLE I

Table for $S(n, k)$

(This table will enable the reader to find $S(n, k)$—the number of symmetric $k$-gons for $n \leq 100$, $k \leq 12$. The table actually gives the values of $(m; r)$ for $m \leq 50$, $r \leq 6$.)

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