AN ANISOTROPIC MAGNETOHYDRODYNAMIC UNIVERSE IN GENERAL
RELATIVITY

T. SINGH AND R. B. S. YADAV

Applied Mathematics Section, Institute of Technology, Banaras Hindu University,
Varanasi 221005

(Received 23 January 1978; after revision 15 June 1979)

Considering the cylindrically symmetric metric of Marder, we have constructed
a non-static cylindrically symmetric cosmological model which is spatially
homogeneous nondegenerate Petrov type I. The energy momentum tensor
has been assumed to be that of a perfect fluid with an electromagnetic field
and the 4-current is either zero or space-like. The model represents an
expanding and shearing but non-rotating fluid flow which is also geodetic.
The requirement of positive conductivity for a physically realistic model
imposes an additional restriction on the metric potentials. Various physical
and geometrical properties of the model have been discussed.

1. INTRODUCTION

In recent years there has been a lot of interest in magnetohydrodynamic cosmo-
logies in general relativity. Cosmological models in the presence of a magnetic
field have been studied by Zeldovich (1965) and Thorne (1967). Galaxies and inter-
stellar spaces exhibit the presence of strong magnetic fields (Zeldovich and Novikov
1971). Monoghan (1966) and Seymour (1966) have discussed the magnetic field in
stellar bodies and Ginzburg (1965) has studied the gravitational collapse of the
magnetic star.

Jacobs (1967) has studied the behaviour of the general Bianchi type I cosmo-
logical model in the presence of the spatially homogeneous magnetic field. This
problem has been studied again by De (1975) with a different approach. This work
has been further extended by Tupper (1977a) to include Einstein-Maxwell fields in
which the electric field is non-zero. He has also interpreted certain type VI cosmo-
logies with electromagnetic field (Tupper 1977b).

Recently Roy and Prakash (1978) taking the cylindrically symmetric metric of
Marder (1958) have constructed a spatially homogeneous cosmological model in the
presence of an incident magnetic field which is also anisotropic and nondegenerate
Petrov type I. In this paper the energy momentum tensor has been assumed to be
that of a perfect fluid with an electromagnetic field and a spatially homogeneous
cosmological model has been obtained. It is found that the model represents an
expanding and shearing but non-rotating fluid flow which is also geodetic. We have
also shown that the model has a 4-current which is either zero or space-like. The latter corresponds to the case of magnetohydrodynamics (MHD). The requirement that the conductivity be positive imposes an additional restriction on the metric potentials. It is found that the electromagnetic field gives positive contributions to the expansion, shear and free gravitational field which die out for large values of time at a slower rate than the corresponding quantities in the absence of the electromagnetic field. When the cosmological constant \( \Lambda = 0 \), it is found that in the absence of electromagnetic field pressure and density become equal and conversely if pressure and density are equal (stiff matter) there is no electromagnetic field.

2. Solution of the Field Equations

We consider here the cylindrically symmetric metric in the form given by Marder (1958)

\[
ds^2 = A^2(dt^2 - dx^2) - B^2dy^2 - C^2dz^2 \tag{2.1}
\]

where \( A, B, C \) are functions of \( t \) only. The distribution consists of a perfect fluid and an electromagnetic field. Thus

\[
G_{it} + \Lambda g_{it} = -K \left[(\rho + p) \lambda_i \lambda_t - pg_{it} + E_{it}\right] \tag{2.2}
\]

\[
g_{it} \lambda_i \lambda_t = 1 \tag{2.3}
\]

\[
E_{it} = g^{\alpha \beta} F_{i \alpha} F_{i \beta} - \frac{1}{4} g_{it} F_{\alpha \beta} F^{\alpha \beta} \tag{2.4}
\]

\[
F_{[it, k]} = 0 \tag{2.5}
\]

\[
F^i_{;i} = J^i \tag{2.6}
\]

where \( E_{it} \) is the electromagnetic energy-momentum tensor, \( F_{it} \) the electromagnetic field tensor, \( \Lambda \) cosmological constant, \( J^i \) the current 4-vector and \( \rho \) and \( p \) are the density and pressure of the distribution. The coordinates are chosen to be comoving so that

\[
\lambda^1 = \lambda^2 = \lambda^3 = 0, \lambda^4 = \frac{1}{A}. \tag{2.7}
\]

We label the coordinates \((x, y, z, t) = (x^1, x^2, x^3, x^4)\).

The off-diagonal components of (2.2) are

\[
\begin{align*}
(a) & \quad F_{12} F_{24} B^{-2} + F_{13} F_{34} C^{-2} = 0 \\
(b) & \quad F_{12} F_{14} A^{-2} - F_{23} F_{34} C^{-2} = 0 \\
(c) & \quad F_{13} F_{14} A^{-2} + F_{23} F_{24} B^{-2} = 0 \\
(d) & \quad F_{14} F_{24} A^{-2} - F_{13} F_{34} C^{-2} = 0 \\
(e) & \quad F_{14} F_{34} A^{-2} + F_{12} F_{23} B^{-2} = 0 \\
(f) & \quad F_{24} F_{34} - F_{13} F_{13} = 0
\end{align*} \tag{2.8}
\]
which lead to three possible cases:

(i) \( F_{24} = F_{34} = F_{12} = F_{13} = 0 \) at least one of \( F_{14}, F_{23} \) non-zero i.e. when the field \( F_{i4} \) is in \( x \)-direction only.

(ii) \( F_{14} = F_{34} = F_{12} = F_{23} = 0 \) at least one of \( F_{24}, F_{13} \) non-zero i.e. when the field is in \( y \)-direction only.

(iii) \( F_{14} = F_{24} = F_{13} = F_{23} = 0 \) at least one of \( F_{34}, F_{12} \) non-zero i.e. when the field is in \( z \)-direction only.

Hence the electromagnetic field is non-null and consists of an electric and/or magnetic field both of which are in the direction of same space axis. Without loss of generality we may consider only case (i) in which the fields are in the \( x \)-direction. We write

\[
F_{14}^2 A^{-4} + F_{23}^2 B^{-2} C^{-2} = L^2. \tag{2.9}
\]

The diagonal components of the eqn. (2.2) may be written as

\[
\frac{2}{A^2} \left[ \frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{C_{44}}{C} - \frac{A_{4}C_{4}}{AC} - \frac{A_{4}B_{4}}{AB} - \frac{A_{4}^2}{A^2} \right] - 2\Delta = - K \left[ -L^2 + (\rho - 3\rho) \right] \tag{2.10}
\]

\[
- \frac{2}{A^2} \left[ \frac{A_{44}}{A} + \frac{A_{4}B_{4}}{AB} + \frac{A_{4}C_{4}}{AC} - \frac{A_{4}^2}{A^2} \right] + 2\Delta = - K \left[ L^2 + (\rho - p) \right] \tag{2.11}
\]

\[
- \frac{2}{A^2} \left[ \frac{B_{44}}{B} + \frac{B_{4}C_{4}}{BC} \right] + 2\Delta = - K \left[ -L^2 + (\rho - p) \right] \tag{2.12}
\]

\[
- \frac{2}{A^2} \left[ \frac{C_{44}}{C} + \frac{B_{4}C_{4}}{BC} \right] + 2\Delta = - K \left[ L^2 + (\rho - p) \right] \tag{2.13}
\]

where the suffix 4 after the symbols \( A, B, C \) stands for ordinary differentiation with respect to time. It is evident from these equations that \( L^2, \rho \) and \( p \) are each functions of time alone. From eqns. (2.5) and (2.9) it follows that \( F_{23} \) is a constant and \( F_{14} \) is a function of \( t \) only i.e.

\[
F_{23} = k, \quad F_{14} = \pm A^2 \left( L^2 - k^2 B^{-2} C^{-2} \right)^{1/2} \tag{2.14}
\]

where \( k \) is a constant.

The case when there is no electric field i.e. when \( F_{14} = 0 \), we have \( J^t = 0 \). It is the case considered by Roy and Prakash (1978). Here we assume that \( F_{14} \neq 0 \) and find the only non-zero component of \( J^t \) to be

\[
J^1 = \pm \frac{1}{A^2 BC} \frac{\partial}{\partial t} \left[ BC \left( L^2 - k^2 B^{-2} C^{-2} \right)^{1/2} \right]. \tag{2.15}
\]
Equation (2.15) shows that \( J^i \) is space-like, unless \( L^2 = fB^{-2}C^{-2} \) where \( f \) is a constant in which case \( J^i = 0 \). The 4-current \( J^i \) is in general the sum of the convection current and the conduction current (Lichnerowicz 1967 and Greenberg 1971):

\[
J^i = \epsilon_0 \lambda^i + \zeta \lambda^i F^{ij} \quad \ldots(2.16)
\]

where \( \epsilon_0 \) is the rest charge density and \( \zeta \) is the conductivity. In the case considered here we have \( \epsilon_0 = 0 \) i.e. magnetohydrodynamics. Thus

\[
\zeta = -\frac{1}{A} I_4 I^{-1}, \quad \ldots(2.17)
\]

where

\[
I = BC(L^2 - k^2 B^{-2}C^{-2})^{1/2}.
\]

The requirement of positive conductivity in (2.17) puts further restrictions on \( A, B, C \). Hence in magnetohydrodynamic case metric potentials are restricted not only by the field equations and energy conditions (Hawking and Penrose 1973) they are also restricted by the requirement that the conductivity be positive for a realistic model.

Equations (2.10) – (2.13) are four equations in six unknowns \( A, B, C, \rho, p \) and \( L \). For complete determinacy of this system of equations, we make two assumptions viz.,

(i) \( F_{14} \) is such that

\[
L^2 = l^2 B^{-4}C^{-4} \quad \ldots(2.18)
\]

where \( l \) is a constant.

(ii) The space time is Petrov type I degenerate (the degeneracy being in \( y \) and \( z \) directions) which requires that

\[
C_{12} = C_{15} \quad \text{with} \quad B \neq C. \quad \ldots(2.19)
\]

Thus from (2.19) we have

\[
\frac{B_{44}}{B} - \frac{C_{44}}{C} + \frac{2A_4}{A} \left( \frac{C_4}{C} - \frac{B_4}{B} \right) = 0. \quad \ldots(2.20)
\]

Equations (2.12) and (2.13) yield

\[
\frac{B_{44}}{B} = \frac{C_{44}}{C} \quad \ldots(2.21)
\]

which on integration gives

\[
B_4 C - BC_4 = k_1 \quad \ldots(2.22)
\]

\( k_1 \) being an arbitrary constant.
Further form (2.20) and (2.21) we get
\[
\frac{A}{A} \left( \frac{C}{C} - \frac{B}{B} \right) = 0. \tag{2.23}
\]

Since \( B \neq C \), eqn. (2.23) gives
\[
A = N \text{ (a constant)}. \tag{2.24}
\]

From eqns. (2.11), (2.12) and (2.24) we have
\[
\frac{B_4}{B} + \frac{B_4 C_4}{BC} = -KN^2L^2. \tag{2.25}
\]

Putting \( B/C = \alpha \) and \( BC = \beta \), eqn. (2.22) reduces to
\[
\left( \frac{\alpha_4}{\alpha} \right) \beta = k_1 \tag{2.26}
\]
and eqn. (2.25) turns into
\[
\frac{1}{\beta} \left[ \left( \frac{\alpha_4}{\alpha} + \frac{\beta_4}{\beta} \right) \beta \right]_4 = -2KN^2L^2. \tag{2.27}
\]

From eqns. (2.26) and (2.27) we have
\[
\frac{\beta_4}{\beta} = -2KN^2L^2 \tag{2.28}
\]

which after the use of eqn. (2.18) goes to the from
\[
\beta_4 = -\frac{2KN^2L^2}{\beta^2}. \tag{2.29}
\]

Equation (2.29) on integration, gives
\[
[\beta_4]^2 = \frac{2KN^2L^2}{\beta^2} + k_3^2 \tag{2.30}
\]

where \( k_3^2 \) is an arbitrary constant.

From eqns. (2.26) and (2.30) we get
\[
\frac{dx}{\alpha} = \frac{k_1}{k_2} \frac{d\beta}{(\beta^2 + k_3^2)^{1/2}} \tag{2.31}
\]

where
\[
k_3^2 = \frac{2KN^2L^2}{k_3^2}. \tag{2.32}
\]

Integration of eqn. (2.31) gives
\[
\alpha = k_4 \left[ \beta + (\beta^2 + k_3^2)^{1/2} \right]^{1/2} \tag{2.33}
\]
$k_4$ being a constant of integration. Therefore

$$B^2 = k_4 \beta \left[ \beta + (\beta^2 + k_3^2)^{1/2} \right]^{k_1/k_2}$$  \hspace{1cm} (2.34)

and

$$C^2 = \frac{\beta}{k_4} \left[ \beta + (\beta^2 + k_3^2)^{1/2} \right]^{-k_1/k_2}.$$  \hspace{1cm} (2.35)

Hence the metric (2.1) can be written as

$$ds^2 = A^2 \left[ \frac{d\beta^2}{(d\beta/dt)^2} - dx^2 \right] - B^2 dy^2 - C^2 dz^2$$  \hspace{1cm} (2.36)

which by use of eqns. (2.24), (2.30), (2.34) and (2.35) takes the form

$$ds^2 = N^2 \left[ - dx^2 + \frac{d\beta^2}{(k_3^2/\beta^2) (\beta^2 + k_3^2)} \right]$$

$$- k_4 \beta \left[ \beta + (\beta^2 + k_3^2)^{1/2} \right]^{k_1/k_2} dy^2$$

$$- \frac{\beta}{k_4} \left[ \beta + (\beta^2 + k_3^2)^{1/2} \right]^{-k_1/k_2} dz^2.$$  \hspace{1cm} (2.37)

The transformation

$$N \alpha \rightarrow X, \ k_4 \gamma \rightarrow Y, \ k_4^{-1} z \rightarrow Z, \ \beta \rightarrow \sqrt{(T^2 - k_3^2)}$$  \hspace{1cm} (2.38)

reduces (2.37) to the form

$$ds^2 = \frac{N^2 dT^2}{k_3^2} - dX^2 - \left( T^2 - k_3^2 \right)^{1/2} \left[ T + (T^2 - k_3^2)^{1/2} \right]^{k_1/k_2} dy^2$$

$$- \left( T^2 - k_3^2 \right)^{1/2} \left[ T + (T^2 - k_3^2)^{1/2} \right]^{-k_1/k_2} dz^2$$  \hspace{1cm} (2.39)

which can be further transformed to the metric

$$ds^2 = dT^2 - dX^2 - \left( T^2 - P^2 \right)^{1/2} \left[ T + (T^2 - P^2)^{1/2} \right] dy^2$$

$$- \left( T^2 - P^2 \right)^{1/2} \left[ T + (T^2 - P^2)^{1/2} \right]^{-1} dz^2.$$  \hspace{1cm} (2.40)

This metric has no singularity and will be real only when $T^2 > P^2$.

3. SOME PHYSICAL FEATURES

(a) The Distribution in the Model

For the model (2.40) pressure $p$ and density $\rho$ are given by

$$K \rho = \frac{T^2}{4} \left( T^2 - P^2 \right)^{-2} - \frac{q^2}{4} \left( T^2 - P^2 \right)^{-1} + \frac{3Kl^2}{2} \left( T^2 - P^2 \right)^{-3} + \Delta$$  \hspace{1cm} (3.1)
\[ K_\rho = \frac{T^2}{4} (T^2 - P^2)^{-2} - \frac{q^2}{4} (T^2 - P^2)^{-1} + \frac{Kl^2}{2} (T^2 - P^2)^{-2} - \Delta. \]  

...(3.2)

The model has to satisfy the reality conditions (Ellis 1971)

(i) \[ \rho + p > 0 \]

(ii) \[ \rho + 3p > 0 \]

which requires that

\[ P^2 < T^2 < \frac{P^2 q^2 + 4Kl^2}{(1 - q^2)} \]  

...(3.3)

and

\[ \Delta > \frac{-1}{2(T^2 - P^2)^2} [(1 - q^2) T^2 + P^2 q^2 + 5Kl^2]. \]  

...(3.4)

The condition (3.3) holds only when \( q^2 < 1 \).

In the case of disordered radiation \( (\rho = 3p) \) we have

\[ \Delta = \frac{-1}{8(T^2 - P^2)^2} [(1 - q^2) T^2 + P^2 q^2 + 8Kl^2] \]  

...(3.5)

and

\[ K_\rho = 3K_\rho = \frac{1}{8(T^2 - P^2)^2} [5(1 - q^2) T^2 + 5P^2 q^2 + 4Kl^2] \]  

...(3.6)

and in the case of stiff matter \( (\rho = p) \)

\[ \Delta = \frac{-Kl^2}{2(T^2 - P^2)^2} \]  

...(3.7)

and

\[ K_\rho = K_\rho = \left[ \frac{T^2}{4} (T^2 - P^2)^{-2} - \frac{q^2}{4} (T^2 - P^2)^{-1} \right] + \frac{Kl^2}{2} (T^2 - P^2)^{-2}. \]  

...(3.8)

The flow vector \( \lambda^i \) is given by

\[ \lambda^1 = \lambda^2 = \lambda^3 = 0, \lambda^4 = 1. \]  

...(3.9)

The flow vector \( \lambda^i \) satisfies \( \lambda^i \lambda^i = 0 \). Thus the lines of flow are geodesics. Tensor of rotation \( W_{ij} \) defined by

\[ W_{ij} = \lambda^i \lambda^j - \lambda^j \lambda^i \]  

...(3.10)

is identically zero. Thus the fluid filling the universe is non-rotational.
The scalar of expansion \( \Theta = \lambda^i_{;i} \) is given by

\[
\Theta = \frac{T}{(T^2 - P^2)^{3/2}} \tag{3.11}
\]

which tends to zero when \( T \to \infty \).

The components of the shear tensor defined by

\[
\sigma_{ij} = \frac{1}{2} (\lambda_{i;j} + \lambda_{j;i}) - \frac{1}{3} \Theta (g_{ij} - \lambda_i \lambda_j) \tag{3.12}
\]

are

\[
\sigma_{11} = \frac{1}{3T} (T^2 - P^2)^{-3/2},
\]

\[
\sigma_{22} = [T + (T^2 - P^2)^{1/2}] q \left\{ - \frac{1}{2} \left[ T(T^2 - P^2)^{-1/2} + q \right] + \frac{1}{2} T(T^2 - P^2)^{-1} \right\},
\]

\[
\sigma_{33} = [T + (T^2 - P^2)^{1/2}]^{-q} \left\{ - \frac{1}{2} \left[ T(T^2 - P^2)^{-1/2} - q \right] + \frac{1}{2} T(T^2 - P^2)^{-1} \right\},
\]

\[
\sigma_{44} = 0; \tag{3.13}
\]

the other components of \( \sigma_{ij} \) being zero.

The non-vanishing components of the conformal curvature tensor \( C_{\mu
\nu\kappa\lambda} \) are

\[
C_{123}^{12} = C_{132}^{13} = - \frac{1}{3} C_{232}^{2} = - \frac{1}{3} \left[ \frac{1}{2} (T^2 - P^2)^{-1/2} - 3T^2(T^2 - P^2)^{-3/2} + \frac{1}{2} q^2(T^2 - P^2)^{-1} - \frac{1}{2} T^2(T^2 - P^2)^{-1} \right]. \tag{3.14}
\]

The non-vanishing component of the charge current 4-vector is given by

\[
J^1 = l^2 T(T^2 - P^2)^{-2} [l^2 - k^2(T^2 - P^2)]^{-1/2}. \tag{3.15}
\]

The conductivity is given by

\[
\zeta = l^2 T(T^2 - P^2)^{-1} [l^2 - k^2(T^2 - P^2)]^{-1}. \tag{3.16}
\]

For a physically realistic MHD model \( \zeta \) has to be positive which requires that

\[
0 < T < (k^2 P^2 + l^2)^{1/2}/k.
\]

(b) The Doppler Effect in the Model

The track of a light pulse in the model (2.40) is obtained by setting

\[
\frac{ds^2}{d\tau} = 0 \quad \text{i.e.}
\]

\[
\left( \frac{dX}{dT} \right)^2 + (T^2 - P^2)^{1/2} [T + (T^2 - P^2)^{1/2}] \left( \frac{dY}{dT} \right)^2 + (T^2 - P^2)^{1/2} [T + (T^2 - P^2)^{1/2}]^{-q} \left( \frac{dZ}{dT} \right)^2 = 1. \tag{3.17}
\]
For the case when velocity is along z-axis, eqn. (3.17) gives
\[
\frac{dZ}{dT} = \pm \left( T^2 - P^2 \right)^{-1/4} \left[ T + \left( T^2 - P^2 \right)^{1/2} \right]^{q/2}
\]
\[
= \pm \phi(T).
\]
...(3.18)

Hence the light pulse leaving a particle at \((0, 0, z)\) at time \(T_1\) would arrive at a later time \(T_2\) given by
\[
\int_{T_1}^{T_2} \phi(T) \, dT = \int_0^Z dZ.
\]
...(3.19)

Hence
\[
\phi_2(T) \delta T_2 = \phi_1(T) \delta T_1 + \frac{dZ}{dT} \delta T_1
\]
\[
= \phi_1(T) \delta T_1 + UZ \delta T_1
\]
...(3.20)

where \(\frac{dZ}{dT} = U_Z\) is the z-component of the velocity of the particle at the time of emission and \(\phi_1(T)\) and \(\phi_2(T)\) are the values of \(\phi(T)\) for \(T = T_1\) and \(T = T_2\) respectively. From the above equation we get
\[
\delta T_2 = \left\{ \frac{\phi_1(T) + UZ}{\phi_2(T)} \right\} \delta T_1.
\]
...(3.21)

The proper time interval \(\delta T_1^0\) between successive wave crests as measured by the local observer moving with the source is given by
\[
\delta T_1^0 = \left\{ 1 - \left( \frac{dX}{dT} \right)^2 - \left( T^2 - P^2 \right)^{1/2} \left[ T + \left( T^2 - P^2 \right)^{1/2} \right]^q \left( \frac{dY}{dT} \right)^2 \right\}^{1/2} \delta T_1
\]
...(3.22)

This can be written as
\[
\delta T_1^0 = \left\{ 1 - U^2 \right\}^{1/2} \delta T_1
\]
...(3.23)

where \(U\) is the velocity of the source at the time of emission. Similarly we may write
\[
\delta T_2^0 = \delta T_2
\]
...(3.24)

as the proper time interval between the reception of two successive wave crests by an observer at rest at the origin. Hence following Tolman (1962), the red shift in this case is given by
\[
\frac{\lambda + \delta \lambda}{\lambda} = \frac{\delta T_2^0}{\delta T_1^0}
\]
\[
= \frac{\left( T_2^1 - P^2 \right)^{-1/4} \left[ T_1 + \left( T_2^1 - P^2 \right)^{1/2} \right]^{q/2} + UZ}{\left( T_2^0 - P^2 \right)^{-1/4} \left[ T_0 + \left( T_2^0 - P^2 \right)^{1/2} \right]^{q/2} \left\{ 1 - U^2 \right\}^{1/2}}.
\]
...(3.25)
(c) *Newtonian Analogue of Force in the Model*

Here we study the effect of electromagnetic field in the force terms $R_i$ and $S_i$ (Narlikar and Singh 1951). The vector $R_i$ and $S_i$ are defined as follows (Narlikar and Singh 1951):

\[ R_i = \Delta^i_{\ell} = H_{i\ell}/H \]  \hspace{1cm} ...(3.26)

\[ S_i = \Delta^i_{\ell k} g^{12} g_{i\ell} \]
\[ = g^{12} g_{i\ell,\kappa} - H_{i\ell}/H \]  \hspace{1cm} ...(3.27)

where

\[ H = \sqrt{g/\gamma}. \]

For the line element (2.40) we have

\[
\begin{align*}
g_{11} &= -1 \\
g_{22} &= -(T^2 - P^2)^{1/2} [T + (T^2 - P^2)^{1/2}]^q \\
g_{33} &= -(T^2 - P^2)^{1/2} [T + (T^2 - P^2)^{1/2}]^{-q} \\
g_{44} &= 1
\end{align*}
\]  \hspace{1cm} ...(3.28)

and

\[
\begin{align*}
g^{11} &= -1 \\
g^{22} &= - (T^2 - P^2)^{-1/2} [T + (T^2 - P^2)^{1/2}]^{-q} \\
g^{33} &= - (T^2 - P^2)^{-1/2} [T + (T^2 - P^2)^{1/2}]^{q} \\
g^{44} &= 1 \\
g &= -(T^2 - P^2)
\end{align*}
\]  \hspace{1cm} ...(3.29)

The corresponding flat metric $\gamma_{\mu\nu}$ is taken to be that of special relativity

\[ ds^2 = dT^2 - dX^2 - dY^2 - dZ^2. \]  \hspace{1cm} ...(3.31)

Thus

\[ \gamma_{\mu} = [-1, -1, -1, +1] \]  \hspace{1cm} ...(3.32)

and

\[ \gamma = -1. \]  \hspace{1cm} ...(3.33)

From (3.30) and (3.33)

\[ H = \sqrt{g/\gamma} = (T^2 - P^2)^{1/2}. \]  \hspace{1cm} ...(3.34)
From (3.26) and (3.27) we get
\[ R_i = [0, 0, 0, T(T^2 - P^2)^{-1}] \] \hspace{1cm} \text{...(3.35)}
and
\[ S_i = [0, 0, 0, - T(T^2 - P^2)^{-1}] . \] \hspace{1cm} \text{...(3.36)}

Thus we find that Newtonian analogue of \( R_i \) and \( S_i \) both are null force vectors. \( R_4 \) and \( S_4 \) have no Newtonian analogues.

In the absence of electromagnetic field the model is given by the metric
\[ ds^2 = dT^2 - dX^2 - \frac{1}{2}(2T)^{q+1} dY^2 - \frac{1}{2} (2T)^{-q-1} dZ^2 \] \hspace{1cm} \text{...(3.37)}
for which the pressure \( p_0 \) and density \( \delta_0 \) are given by
\[ Kp_0 = \frac{1 - q^2}{4T^2} + \Lambda \] \hspace{1cm} \text{...(3.38)}
\[ K\delta_0 = \frac{1 - q^2}{4T^2} - \Lambda. \] \hspace{1cm} \text{...(3.39)}

The reality conditions (Ellis 1971) require that
\[ q^2 < 1 \text{ and } \Lambda > \frac{q^2 - 1}{2T^2} . \] \hspace{1cm} \text{...(3.40)}

Therefore vectors \( R_i \) and \( S_i \) reduce to
\[ R_i = \left[ 0, 0, 0, \frac{1}{T} \right] \] \hspace{1cm} \text{...(3.41)}
and
\[ S_i = \left[ 0, 0, 0, - \frac{1}{T} \right] . \] \hspace{1cm} \text{...(3.42)}

The flow vector \( \lambda^i \) satisfies \( \lambda^i_j \lambda^j = 0 \). Thus the lines of flow are geodesics. The tensor of rotation is identically zero. The scalar of expansion is given by
\[ \Theta = 1/T^2 . \] \hspace{1cm} \text{...(3.43)}

The non-zero components of the shear tensor are
\[ \sigma_{11} = \frac{1}{3T^4} \]
\[ \sigma_{22} = \frac{1}{2} (2T)^{-q-1} \left[ -3T(1 + q) + 2 \right] \]
\[ \sigma_{33} = \frac{1}{2} (2T)^{-q-1} \left[ -3T(1 - q) + 2 \right] . \] \hspace{1cm} \text{...(3.44)}
The red shift in the model is given by

$$\frac{\lambda + \delta \lambda}{\lambda} = \frac{[(2)^{q/2} T_1^{(q-1)/2} + U_Z]}{[(2)^{q/2} T_2^{(q-1)/2}] (1 - U^2)^{1/2}}$$

...(3.45)

where

$$U_Z = \frac{dZ}{dT} = (2)^{q/2} (T)^{(q-1)/2}$$

is the velocity at the time of emission.

The non-vanishing components of the conformal curvature tensor are

$$C_{\perp_{\perp}} = C_{\perp_{3}} = -\frac{1}{2} C_{\perp_{3} \parallel} = -\frac{1}{24} \left[ \frac{10}{T} + \frac{(2q^2 - 1)}{T^2} \right].$$

...(3.46)

As $T \to \infty$, shear, expansion and free gravitational fields vanish.

Thus the electromagnetic field gives positive contributions to expansion, shear and free gravitational field which die out for large values of $T$ at a slower rate than the corresponding quantities in the absence of the electromagnetic field.

ACKNOWLEDGEMENT

The authors are thankful to the State Council of Science and Technology, U.P., for financial support. Thanks are also due to the referee for pointing out an error in the calculation.

REFERENCES


AN ANISOTROPIC MAGNETOHYDRODYNAMIC UNIVERSE IN GENERAL RELATIVITY


APPENDIX

The non-vanishing components of the mixed Ricci tensor $R^l_j$ for the metric (2.1) are given by

\[(A.1) \qquad R^1_1 = \frac{1}{A^2} \left[ \frac{A_{44}}{A} + \frac{A_4 B_4}{AB} + \frac{A_4 C_4}{AC} - \frac{A_4^2}{A^2} \right] \]

\[(A.2) \qquad R^2_2 = \frac{1}{A^2} \left[ \frac{B_{44}}{B} + \frac{B_4 C_4}{BC} \right] \]

\[(A.3) \qquad R^3_3 = \frac{1}{A^2} \left[ \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} \right] \]

\[(A.4) \qquad R^4_4 = \frac{1}{A^2} \left[ \frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{C_{44}}{C} - \frac{A_4 C_4}{AC} - \frac{A_4 B_4}{AB} - \frac{A_4^2}{A^2} \right] \]

and

\[(A.5) \qquad R = \frac{2}{A^2} \left[ \frac{A_{44}}{A} - \frac{A_4^2}{A^2} + \frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} \right] \]

The non-vanishing components of the Weyl conformal curvature tensor $C^{4l}_{ik}$ or the metric (2.1) are given by

\[(A.6) \qquad C^{44}_{44} = C^{33}_{33} = \frac{1}{6A^2} \left[ \frac{B_{44}}{B} + \frac{C_{44}}{C} - \frac{2A_{44}}{A} + \frac{2A_4}{A^2} - \frac{2B_4 C_4}{BC} \right] \]
\[ C_{12}^{12} = C_{23}^{23} = \frac{1}{6A^2} \left[ \frac{A_{44}}{A} + \frac{B_{44}}{B} - \frac{2C_{44}}{C} + \frac{3A_4C_4}{AC} - \frac{A_4^2}{A^2} \right] - \frac{3A_4B_4}{AB} + \frac{B_4C_4}{BC} \]

\[ C_{13}^{13} = C_{24}^{24} = \frac{1}{6A^2} \left[ \frac{A_{44}}{A} + \frac{C_{44}}{C} - \frac{2B_{44}}{B} + \frac{3A_4B_4}{AB} - \frac{A_4^2}{A^2} \right] - \frac{3A_4C_4}{AC} + \frac{B_4C_4}{BC} \]