ON A QUASI-NORMAL $(\psi, g, u, v, e, \lambda)$-STRUCTURE

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In the present paper it is shown that the $(\psi, g, u, v, e, \lambda)$-structure is quasi-normal in the sense of Yano (Yano and Ki 1972) iff the vector fields are harmonic. Also, the induced structure of the invariant submanifold of co-dimension 2 is quasi-normal.

1. GENERAL FORMULA FOR $(\psi, g, u, v, e, \lambda)$-STRUCTURE

Let $V_a$ be a $C^\infty$ differentiable manifold with (1, 1) tensor field $\psi$, Riemannian metric tensor $g$, three 1-forms $u, v, e$ and associated with three vector fields $U, V, E$ and a differentiable function $\lambda$ satisfying (Baik 1972, Yano 1965)

(a) $\psi^j_i \psi^h_i = - \delta^h_j + u^j u^h + v^j v^h + e^j e^h$,

(b) $\psi^j_i g_{is} = g_{ji} - u^j u_i - v^j v_i - e^j e_i$,

(c) $\psi^i_i u_i = \lambda v_i, \psi^i_i v_i = - \lambda u_i, \psi^i_i e_i = 0$,

(d) $\psi^h_i u^i = - \lambda v_i, \psi^h_i v^i = \lambda u_i, \psi^h_i e^i = 0$,

(e) $u_i u^i = 1 - \lambda^2, u_i v^i = 0, u_i e^i = 0$,

(f) $v_i u^i = 0, v_i v^i = 1 - \lambda^2, v_i e^i = 0$,

(g) $e_i u^i = 0, e_i v^i = 0, e_i e^i = 1$. ... (1.1)

Let us put

$\psi_{ji} = \psi^i_j g_{ji}$ (being skew symmetric) ... (1.2)

$h, i, j, ...$ run over $1, 2, ..., n = (2m + 3)$.

Let us define (1.2) type of torsion tensor as follows:

$S^h_{ji} \overset{\text{def}}{=} \psi^i_j \nabla_i \psi^h_j - \psi^i_j \nabla^i \psi^h_j - \psi^h_i (\nabla^i \psi^j_j - \nabla^j \psi^i_i) + u^j u^h + v^j v^h + e^j e^h$ ... (1.3)

where $u_{ji} = \nabla_j u_i - \nabla_i u_j$ is covariant differential w.r.t. Levi Civita connection. If the torsion tensor $S^h_{ji}$ vanishes, the $(\psi, g, u, v, e, \lambda)$-structure is called normal.
Transvecting (1.3) with $u_h$, $v_h$, $e_h$ and also keeping in view (1.1) we obtain

$$S^h_{ji} u_h = u_{hi} + \lambda(u_i \nabla j \lambda - u_j \nabla i \lambda) + \lambda(\psi^i_s v_{is} - \psi^i_s v_{is})$$

$$+ \nabla i \lambda(\psi^i_s v_i - \psi^i_s v_i) - \psi^i_s \psi^i_s u_{is}, \quad \ldots(1.4)$$

$$S^h_{ji} v_h = v_{hi} + \lambda(v_i \nabla j \lambda - v_j \nabla i \lambda) - \lambda(\psi^i_s u_{is} - \psi^i_s u_{is})$$

$$- \nabla i \lambda(\psi^i_s u_i - \psi^i_s u_i) - \psi^i_s \psi^i_s v_{is}, \quad \ldots(1.5)$$

and

$$S^h_{ji} e_h = e_{hi} + \psi^i_s \psi^i_s e_{si}. \quad \ldots(1.6)$$

We have (Yano and Ki 1972)

$$\psi_{ih} = \nabla i \psi_{ih} + \nabla s \psi_{is} + \nabla n \psi_{ih}. \quad \ldots(1.7)$$

Using (1.3) and (1.7) the covariant components $S_{iinh}$ can be written as

$$S_{iinh} = (\psi^i_s \psi_{iinh} - \psi^i_s \psi_{iinh}) = \psi^i_s \nabla h \psi_{iinh} - \psi^i_s \nabla h \psi_{iinh} + u_i \nabla i u_h$$

$$+ v_i \nabla i v_h + e_i \nabla i e_h - u_i \nabla v_h - v_i \nabla v_i - e_i \nabla e_i. \quad \ldots(1.8)$$

Transvecting (1.8), with $u^t$, $v^t$, $e^t$, we get respectively

$$\{S_{iinh} - (\psi^i_s \psi_{iinh} - \psi^i_s \psi_{iinh})\} u^t = -\lambda^2 u_{ih} + \lambda \psi^i_s \lambda^2 u_{ih} + \lambda \psi^i_s \lambda^2 u_{ih}$$

$$- e_i u^t e_{ih} + \lambda \psi^i_s \lambda^2 u_{ih} - v_{ih}(v^i u^t + \lambda \psi^i_s \lambda^2 u_{ih}) \quad \ldots(1.9)$$

$$\{S_{iinh} - (\psi^i_s \psi_{iinh} - \psi^i_s \psi_{iinh})\} v^t = -\lambda^2 v_{ih} + \lambda \psi^i_s \lambda^2 v_{ih} + \lambda \psi^i_s \lambda^2 v_{ih}$$

$$- e_i v^t e_{ih} - \lambda \psi^i_s \lambda^2 v_{ih} + (\lambda \psi^i_s - u_i v^t) u_{ih} \quad \ldots(1.10)$$

and

$$\{S_{iinh} - (\psi^i_s \psi_{iinh} - \psi^i_s \psi_{iinh})\} e^t = \lambda \psi^i_s \lambda^2 v_{ih} - e_i e^t \lambda \psi^i_s \lambda^2 v_{ih}$$

$$- v_i e^t v_{ih} - u_i e^t u_{ih}. \quad \ldots(1.11)$$

where $l$, $l$, $l$ denote the Lie derivative w.r.t. the vector field $u$, $v$, $e$ respectively.

2. **Quasi-normal ($\psi$, $g$, $u$, $v$, $e$, $\lambda$)-structure**

($\psi$, $g$, $u$, $v$, $e$, $\lambda$)-structure is called quasi-normal if (Yano and Ki 1972)

$$S_{iinh} - (\psi^i_s \psi_{iinh} - \psi^i_s \psi_{iinh}) = 0. \quad \ldots(2.1)$$
Using (2.1), eqns. (1.9) – (1.11) reduce to
\[
\begin{align*}
l g_{ih} - u_i u^i l g_{ih} + \lambda \psi^j_i \lambda^j_i l g_{ih} - e_i u^i e_{ih} \\
= \lambda^2 u_{ih} + v_{ih} (v_i u^i + \lambda \psi^i_i) \quad \ldots(2.2)
\end{align*}
\]
\[
\begin{align*}
l g_{ih} - v_i u^i l g_{ih} - \lambda \psi^i_i \lambda^i_i l g_{ih} - e_i v^i e_{ih} \\
= \lambda^2 v_{ih} - u_{ih} (\lambda \psi^i_i - u_i v^i) \quad \ldots(2.3)
\end{align*}
\]

and
\[
\begin{align*}
l g_{ih} - e_i e^i l g_{ih} - v_i e^i v_{ih} - u_i e^i u_{ih} = 0. \quad \ldots(2.4)
\end{align*}
\]
Substituting (2.2) and (2.3) in (2.4) and on taking account of (1.1), we obtain
\[
\begin{align*}
\lambda^2 l g_{ih} - \lambda^2 e_i e^i l g_{ih} + v_i v^i e_{sh} - v_i e^i l g_{ih} - u_i e^i l g_{ih} + u_i u^s e_{ih} = 0.
\end{align*}
\]
\ldots(2.5)

Transvecting (2.5) with \( u^i \) and using (1.1), we get
\[
\begin{align*}
u^s e_{sh} = e^i l g_{ih} - \frac{\lambda^2}{1 - \lambda^2} u^i l g_{ih}. \quad \ldots(2.6)
\end{align*}
\]
Substituting the value of \( u^s e_{sh} \) in eqn. (2.5) and then transvecting the resulting equation with \( e^i \) and \( v^i \) respectively, we find
\[
\begin{align*}
l g_{ih} = 0 \quad \ldots(2.7a)
\end{align*}
\]
and
\[
\begin{align*}
v^s e_{sh} = e^i l g_{ih}. \quad \ldots(2.7b)
\end{align*}
\]
From (1.1), (2.2) and (2.3), we have
\[
\begin{align*}
\lambda^2 (1 - \lambda^2) v_{ih} + v_{sh} (u_i u^s - \lambda^2 v_i v^s - \lambda^2 e_i e^s) + \{\lambda (1 - \lambda^2) \psi^j_i - \lambda^2 v_i u^i - u_i v^i \} l g_{ih} + \lambda^2 e_i e^i l g_{ih} - \lambda^2 e_i v^i e_{ih} = 0 \quad \ldots(2.8)
\end{align*}
\]
which on transvecting with \( u^i \) and in view of (1.1) becomes
\[
\begin{align*}
u^s v_{sh} = v^s l g_{sh} \quad (\lambda \neq \pm 1, \pm i) \quad \ldots(2.9)
\end{align*}
\]
From (2.9), it can be seen that
\[
\begin{align*}
l g_{sh} \quad u^i v^i = 0. \quad \ldots(2.10)
\end{align*}
\]
Transvecting (2.2) and (2.3) with \( e^t \), we get
\[
e^t l_x g_{th} - u^t e_{th} = \lambda^2 u_{th} e^t
\]...(2.11a)
and
\[
e^t l_y g_{th} - v^t e_{th} = \lambda^2 v^t v_{th}.
\]...(2.11b)

Transvecting (2.2) with \( v^t \) and also using (1.1) and (2.9), we find
\[
u^t l_x g_{th} = v^t u_{th}.
\]...(2.12)

Making use of (2.6) and (2.7a, b) in (2.11a, b) we obtain
\[
\begin{align*}
(a) & \quad u_{th} = 0, & (b) & \quad v_{th} = 0, & (c) & \quad e_{th} = 0.
\end{align*}
\]...(2.13)

Similarly from (2.9), (2.12) and (2.13), we have
\[
\begin{align*}
(a) & \quad l_x g_{th} = 0, & (b) & \quad l_y g_{th} = 0.
\end{align*}
\]...(2.14)

Since
\[
l_x g_{th} = \nabla_x u_t + \nabla_y u_t = 0.
\]

which can be written as
\[
\nabla_x u^t = 0.
\]

Thus in a quasi-normal \((\psi, g, u, v, e, \lambda)\)-structure, we get \( u_{th} = 0, v_{th} = 0, e_{th} = 0 \) and \( \nabla_x u^t = 0, \nabla_y v^t = 0 \) and \( \nabla_x e^t = 0 \).

Now if we substitute the last set of equations in (1.9) – (1.11), we find
\[
S_{th} = (\psi^t \psi_{th} - \psi^t \psi_{th}) = 0
\]
which is the condition for a structure to be quasi-normal. Hence

\textit{Theorem 2.1} — If the \((\psi, g, u, v, e, \lambda)\)-structure is quasi-normal, it is necessary and sufficient that \( U, V, E \) are harmonic vectors \((\lambda \neq \pm 1, \pm i)\).

\section{Invariant Submanifold of Co-dimension 2 of a Manifold with \((\psi, g, u, v, e, \lambda)\)-Structure}

We consider a submanifold \(V_m\) of \(V_n\) represented by
\[
x^h = x^h(y^a)
\]...(3.1)
and put
\[
B^h_a \partial_h x^h, \partial_a = \partial/\partial y^a.
\]...(3.2)

\(a, b, c, \ldots\) run over 1, 2, \ldots, \(n\) and \(h, i, j\) run over 1 to \(m\).
The induced Riemannian metric is given by

$$g_{eb} = g_{hi} B_{e}^{f} B_{h}^{i}.$$  \hspace{1cm} ...(3.3)

We denote by \(C_{a}^{h}, 2m - n\) mutually orthogonal unit normals to \(N\). The equations of Gauss and those of Weingarten are given by (Yoshiko 1972)

$$\nabla_{e} B_{b}^{h} = \sum_{x} h_{ebx} C_{a}^{x} \hspace{1cm} ...(3.4)$$

$$\nabla_{e} C_{a}^{h} = - h_{ca}^{a} B_{a}^{h} + \sum_{y} l_{cxy} C_{a}^{h} \hspace{1cm} ...(3.5)$$

where

$$\nabla_{e} B_{b}^{h} = \partial_{e} B_{b}^{h} + \left\{ \begin{array}{c} h_{j}^{i} \end{array} \right\} B_{e}^{j} B_{b}^{i} - \left\{ \begin{array}{c} a \end{array} \right\} B_{a}^{h} \hspace{1cm} ...(3.6)$$

is the Vander Warden-Borotolotti covariant differential of \(B_{b}^{h}\), \(\left\{ \begin{array}{c} a \end{array} \right\} \) being Christoffel symbols formed with \(g_{eb} \).

$$\nabla_{e} C_{a}^{h} = \partial_{e} C_{a}^{h} + \left\{ \begin{array}{c} h_{j}^{i} \end{array} \right\} B_{e}^{j} C_{a}^{i} \hspace{1cm} ...(3.7)$$

\(h_{ebx}\) are the components of the second fundamental tensors w.r.t. the normals \(C_{a}^{h}\).

$$h_{ca}^{a} = h_{ebx} g_{ba} \hspace{1cm} ...(3.8)$$

\(g_{ba}\) being contravariant components of the induced metric and \(l_{cxy}\) components of the third fundamental tensor w.r.t. the normal \(C_{a}^{h}\).

We assume that the submanifold \(V_{m}\) of \(V_{a}\) is \(\psi\)-invariant, hence we have (Yoshiko 1972, Yano and Okumura 1970)

$$\psi_{b}^{b} B_{b}^{i} = \psi_{a}^{a} B_{a}^{h} \hspace{1cm} ...(3.9)$$

\(\psi_{a}^{a}\) is a (1,1) tensor field of \(V_{m}\). This shows that

$$\psi_{eb} B_{b}^{i} C_{a}^{h} = 0.$$

\(\psi_{i}^{h} C_{a}^{i}\) is normal to the submanifold \(V_{m}\). Thus

$$\psi_{i}^{h} C_{a}^{i} = \sum_{y} \gamma_{xy} C_{y}^{h}.$$

Since

$$\psi_{eb} C_{a}^{i} C_{b}^{h} = - \gamma_{xy}$$
we see that
\[ \gamma_{x
u} = -\gamma_{\nu x}. \] ...(3.10)

Let
\[ u^h = \mathcal{B}_a^h u^a + \sum_x \alpha_x C_x^h \] ...(3.11)
\[ v^h = \mathcal{B}_a^h v^a + \sum_x \beta_x C_x^h \] ...(3.12)
and
\[ e^h = \mathcal{B}_a^h e^a + \sum_x \epsilon_x C_x^h \] ...(3.13)
u^a, v^a, e^a being vector fields of \( V_m \) and \( \alpha_x, \beta_x, \epsilon_x \) are functions in \( V_m \).

Now from (1.1), (3.9) and (3.11) to (3.12), we obtain the following relations
\[ \psi_a^h \psi_b^c = -\delta_b^a + u_b u^a + v_b v^a + e_b e^a \] ...(3.14)
\[ u_b \alpha_x + v_b \beta_x + e_b \epsilon_x = 0 \] ...(3.15)
\[ \psi_a^h \psi_b^d g_{ad} = g_{eb} - u_b u_e - v_b v_e - e_b e_e \] ...(3.16)
\[ \alpha_x u^a + \beta_x v^a + \epsilon_x e^a = 0 \] ...(3.17)
\[ \sum_y \gamma_{x \nu} \gamma_{y \nu} = -\delta_{x \nu} + \alpha_x \alpha_x + \beta_x \beta_x + \epsilon_x \epsilon_x \] ...(3.18)
\[ \psi_a^h u^a = -\lambda \nu^h, \psi_a^h v^a = \lambda u^h, \psi_a^h e^a = 0 \] ...(3.19)
\[ \sum_x \alpha_x \gamma_{x \nu} = -\lambda \beta_x, \sum_x \epsilon_x \gamma_{x \nu} = 0 \] \[ \sum_x \beta_x \gamma_{x \nu} = \lambda \alpha_x \] \[ \sum_x \epsilon_x \gamma_{x \nu} = 0 \] \[ \sum_x \lambda \gamma_{x \nu} = 0 \] \[ \sum_x \lambda \gamma_{x \nu} = 0 \] ...(3.20)
\[ u_a u^a = 1 - \lambda^2 - \sum_x \alpha_x^2 \] ...(3.21)
\[ v_a v^a = 1 - \lambda^2 - \sum_x \beta_x^2 \] ...(3.21)
\[ e_a e^a = 1 - \sum_x \epsilon_x^2 \] ...(3.21)
\[ u_a v^a = -\sum_x \beta_x \alpha_x \] ...(3.22)
\[ u_a e^a = -\sum_x \alpha_x \epsilon_x \] ...(3.22)
it can be easily seen that \( \psi_{cb} \) is skew-symmetric, since
\[ \psi_{hc} B_c^h B_{bc} = \psi_{cb}. \] ...(3.23)
Equations (3.14) – (3.22) show that a necessary and sufficient condition for $\psi^a_b$, $g_{ob}$, $u_b$, $v_b$, $e_b$ and $\lambda$ to admit a $(\psi, g, u, v, e, \lambda)$-structure is that $\Sigma_{x} a^2_x = 0$, $\Sigma_{x} \beta^2_x = 0$ and $\Sigma_{x} \epsilon^2_x = 0$. $\alpha_x$, $\beta_x$ and $\epsilon_x$ are zero or we say that $u^h$, $v^h$ and $e^h$ are always tangent to the submanifold. Since

$$\left(\nabla u_i - \nabla_s u_i\right) B^t_\alpha B^i_\beta = \nabla_s u_b - \nabla_b u_c$$ ...(3.24a)

and

$$\left(\nabla u_i + \nabla_s u_t\right) B^t_\alpha B^i_\beta = \nabla_s u_b + \nabla_b u_c$$ ...(3.24b)

hence

$$S^h_{ji} B^t_\alpha B^i_\beta = (N^a_{ob} + u_{ob} u^a + v_{ob} v^a + e_{ob} e^a) B^h_\alpha$$

$$+ \{\Sigma_{x} (\alpha_x u_{ob} + v_{ob} \beta_x + e_{ob} \epsilon_x)\} C^h_x.$$ ...(3.25)

where $N^a_{ob}$ is the Nijenhuis tensor $\psi^a_b$.

This leads to the fact that if manifold with $(\psi, g, u, v, e, \lambda)$-structure is normal then the induced structure $(\psi, g, u, v, e, \lambda)$ on the submanifold $V_m$ is also normal.

In view of (3.24a, b) we can also say that $u^a$, $v^a$, $e^a$ will be harmonic vector fields on induced submanifold, if $u^h$, $v^h$ and $e^h$ are the harmonic vectors on the manifold $V_n$ with $(\psi, g, u, v, e, \lambda)$-structure. Therefore

**Theorem 3.1** — The induced $(\psi, g, u, v, e, \lambda)$-structure of the invariant submanifold $V$ of co-dimension 2 with quasi-normal $(\psi, g, u, v, e, \lambda)$-structure is also quasi-normal.

**References**


