HYPERSURFACES OF ALMOST HYPERBOLIC HERMITE MANIFOLDS

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In this paper it is shown that a hypersurface of hyperbolic Hermite manifold admits an induced hyperbolic contact structure. A condition for hypersurface of an almost hyperbolic Hermite manifold with vanishing curvature tensor to be conformally flat is also obtained.

1. INTRODUCTION

Let $V_m$ be an $m$-dimensional differentiable manifold in which there exist a vector-valued linear function $F$ and a Riemannian metric $G$, satisfying

$$F^2 \lambda = \lambda$$  \hspace{2cm} (1.1)

$$G(F\lambda, F\mu) = - G(\lambda, \mu)$$  \hspace{2cm} (1.2)

for arbitrary vector fields $\lambda, \mu$ in $V_m$. Then $V_m$ is called an almost hyperbolic Hermite manifold (Prvanović 1971, Dube 1973).

An almost hyperbolic Hermite manifold $V_m$ for which

$$(E_\lambda F) \mu = 0$$  \hspace{2cm} (1.3)

where $E$ is the Riemannian connexion, is satisfied, is called hyperbolic Kählerian manifold (Prvanović 1971).

Let $V_n$ be a differentiable manifold of dimension $n$. Let there exist a tensor field $f$ of the type $(1, 1)$, a 1-form $A$, a vector field $T$ and the Riemannian metric tensor $g$ satisfying

$$f^2X = X + A(X) T$$  \hspace{2cm} (1.4)

$$fT = 0$$  \hspace{2cm} (1.5)

$$A(fX) = 0$$  \hspace{2cm} (1.6)

$$A(T) = -1$$  \hspace{2cm} (1.7)

$$g(X, T) = A(X)$$  \hspace{2cm} (1.8)

$$g(fX, fY) = -g(X, Y) - A(X) A(Y).$$  \hspace{2cm} (1.9)

Then $V_n$ is called hyperbolic contact metric manifold (Upadhyay and Dube 1973).
The Nijenhuis tensor $N(X, Y)$ of the hyperbolic contact metric manifold is given by the relation

$$N(X, Y) = (D_{Xf}) Y - (D_{Yf}) X + f(D_Y f) X - f(D_X f) Y. \quad \ldots(1.10)$$

$D$ is the Riemannian connexion of $V_n$.

**Definition 1.1** — When the tensor field

$$P(X, Y) = N(X, Y) + (dA)(X, Y) T \quad \ldots(1.11)$$

vanishes, where $N$ is the Nijenhuis tensor formed with $f$ then hyperbolic contact metric manifold $V_n$ is said to be normal.

2. **HYPERSURFACES**

Let us consider a hypersurface $V_n$, $(n = m - 1)$ of an almost hyperbolic Hermite manifold with the immersion map $b : V_n \rightarrow V_m$ such that a point

$$p \in V_n \Rightarrow bp \in V_m.$$ 

Let $B$ be the corresponding Jacobian map, such that a vector field $X$ in $V_n$ at $p \Rightarrow BX$ in $V_m$ at $bp$. Let $g$ be the induced Riemannian metric of $V_n$. Let $N$ be a unit normal vector to $V_n$. Then we have

\[
\begin{align*}
(a) \quad & G(BX, BY) \circ b = g(X, Y) \\
(b) \quad & G(BX, N) \circ b = 0 \\
(c) \quad & G(N, N) = 1.
\end{align*}
\]

\ldots(2.1)

Let us express the transformation of $BX$ and $N$ by $F$ as the sum of tangential and normal parts in the form

\[
\begin{align*}
(a) \quad & FBX = BfX + A(X) N \\
(b) \quad & FN = -BT.
\end{align*}
\]

\ldots(2.2)

**Theorem 2.1** — A hypersurface $V_n$ of an almost hyperbolic Hermite manifold $V_m$ is a hyperbolic contact metric manifold.

**Proof:** Premultiplying (2.2a) and (2.2b) by $F$ and using (1.1) and (2.2) and collecting tangential and normal parts, we have

\[
\begin{align*}
(a) \quad & f^2 X = X + A(X) T, \quad (b) \quad A(fX) = 0. \quad \ldots(2.3) \\
(a) \quad & fT = 0, \quad (b) \quad A(T) = -1. \quad \ldots(2.4)
\end{align*}
\]

Also from (1.2), we have

$$G(FBX, FBY) = -G(BX, BY). \quad \ldots(2.5)$$
Making use of (2.1) and (2.2) in (2.5), we have

\[ g(fX, fY) = -g(X, Y) - A(X) A(Y). \] ...(2.6)

Equations (2.3), (2.4) and (2.6) prove the statement.

Equations of Gauss and Weingarten are given by

\[
\begin{align*}
(a) & \quad E_{B}X_B Y = BD_X Y + 'H(X, Y) N \\
(b) & \quad E_{B}X_N = -BH_X
\end{align*}
\] ...(2.7)

where \( E \) and \( D \) are Riemannian connexions in \( V_m \) and \( V_n \) respectively. \( 'H(X, Y) \) is a symmetric second fundamental tensor with respect to normal \( N \) and \( H(X) \) is a tensor field of the type \((1, 1)\) defined by

\[ 'H(X, Y) \overset{\text{def}}{=} g(HX, Y). \] ...(2.8)

Let us suppose that \( V_m \) be a hyperbolic Kählerian manifold, then (1.3) implies.

\[ (E_{B}X_B) BY = 0 \Rightarrow E_{B}X_FB Y = FE_{B}X_B Y. \] ...(2.9)

Substituting in (2.9) from (2.2) and (2.7) and collecting tangential and normal parts, we get

\[
\begin{align*}
(a) & \quad (D_X f) Y = -'H(X, Y) T + A(Y) HX \\
(b) & \quad (D_X A) (Y) = -'H(X, fY).
\end{align*}
\] ...(2.10)

**Theorem 2.2** — If \( V_n \) is a hypersurface of a hyperbolic Kählerian manifold \( V_m \) and \( H \) commutes with \( f \), then hyperbolic contact metric manifold \( V_n \) is normal.

**Proof:** Rewrite (1.11)

\[ P(X, Y) = N(X, Y) + (dA) (X, Y) T. \]

In consequence of (1.10), we have

\[ P(X, Y) = (D_X f) Y - (D_Y f) X + f(D_X f) X - f(D_X f) Y \\
+ \{(D_X A) (Y) - (D_Y A) (X)\} T. \] ...(2.11)

In consequence of (2.10), we have

\[ P(X, Y) = A(Y) \{H fX - fHX\} - A(X) \{H fY - fHY\} . \] ...(2.12)

If \( H \) commutes with \( f \), then we have \( P(X, Y) = 0 \).

Hence, we have proved the theorem.

Let \( R \) be the curvature tensor of the Riemannian connexion \( E \) and \( K \) be the curvature tensor of the induced Riemannian connexion \( D \), then we have
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\[ 'R(BX, BY, BZ, BU) = 'K(X, Y, Z, U) + 'H(X, Z) 'H(Y, U) \]
\[ - 'H(Y, Z) 'H(X, U) \]  \hspace{1cm} (2.13a)

\[ 'R(BX, BY, BZ, N) = (Dx'H) (Y, Z) - (Dy'H) (X, Z) \]  \hspace{1cm} (2.13b)

where

\[ 'R(BX, BY, BZ, BU) \overset{\text{def}}{=} G(R(BX, BY, BZ), BU) \]  \hspace{1cm} (2.14a)

\[ 'K(X, Y, Z, U) \overset{\text{def}}{=} g(K(X, Y, Z), U). \]  \hspace{1cm} (2.14b)

**Theorem 2.3** — A hypersurface \( V_n \) of an almost hyperbolic Hermite manifold \( V_m \) with vanishing curvature tensor be conformally flat if \( 'H(X, Y) = -g(fX, fY) \).

**Proof** : Since the curvature tensor of an almost hyperbolic Hermite manifold \( V_m \) vanishes, then from (2.13a), we have

\[ 'K(X, Y, Z, U) = 'H(X, U) 'H(Y, Z) - 'H(X, Z) 'H(Y, U) \]  \hspace{1cm} (2.15)

from the supposition and using (1.9), we have

\[ 'K(X, Y, Z, U) = g(Y, Z) g(X, U) - g(X, Z) g(Y, U) \]
\[ + g(Y, Z) A(X) A(U) - g(X, Z) A(Y) A(U) \]
\[ + g(X, U) A(Y) A(Z) - g(Y, U) A(X) A(Z) \]  \hspace{1cm} (2.16a)

or

\[ K(X, Y, Z) = g(Y, Z) X - g(X, Z) Y + g(Y, Z) A(X) T \]
\[ - g(X, Z) A(Y) T + A(Y) A(Z) X - A(X) A(Z) Y. \]  \hspace{1cm} (2.16b)

Contracting the above equation with respect to \( X \), we set

\[ \text{Ric} \ (Y, Z) = (n - 2) \ (g(Y, Z) + A(Y) A(Z)) \]  \hspace{1cm} (2.17a)

or

\[ R(Y) = (n - 2) \ (Y + A(Y) T). \]  \hspace{1cm} (2.17b)

Contracting the above equation with respect to \( Y \), we get

\[ R = (n - 2) \ (n - 1) \]  \hspace{1cm} (2.18)

where \( R \) is a scalar curvature tensor of \( V_n \).

On making use of (2.17) and (2.18) in (2.16), we get

\[ K(X, Y, Z) - \frac{1}{(n - 2)} \ \{ \text{Ric} \ (Y, Z) X - \text{Ric} \ (X, Z) Y + g(Y, Z) R(X) - \]

(equation continued on p. 632)
\[- g(X, Z) R(Y) - \frac{R}{(n - 1) (n - 2)} \{ g(Y, Z) X - g(X, Z) Y \} = 0 \]
\[ \Rightarrow V(X, Y, Z) = 0 \]

where \( V(X, Y, Z) \) is a conformal curvature tensor. Hence, we have the statement.

**Theorem 2.4** — In a hyperbolic contact metric hypersurface \( V_n \) of a hyperbolic Kählerian manifold \( V_m \), the induced structure tensor \( f \) is covariant constant if \( HX \) vanishes.

**Proof**: Rewrite (2.10a)

\[ (D_x f) \ Y = - H(X, Y) T + A(Y) HX. \]

If we suppose \( HX \) vanishes, then we have

\[ (D_x f) \ Y = 0 \]

Hence, we have the statement.

**References**

