FIXED POINT THEOREMS IN TYCHONOFF SPACES

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Extensions of Banach's contraction principle are considered in the non-metric setting of a topology generated by a family of pseudometrics. A class of mappings on such spaces, called contingent contractions, is defined and used to prove a fixed point theorem from which several recent results follow as corollaries.

For selfmappings \( T \) of a complete metric space \((X, d)\) that satisfy the condition 
\[ d(Tx, Ty) < qd(x, y) \]
for all \( x, y \in X, x \neq y \), \( q \) being a constant \( 0 \leq q < 1 \), a well-known theorem of Banach (1922) states that there exists a unique point \( \xi \in X \) such that \( T\xi = \xi \). The usefulness of this "principle of contraction mappings" in analysis has been well illustrated by Kolmogorov and Fomin (1957, pp. 43-51).

In this paper we consider a double generalization of this theorem. Firstly the contractive nature of the mapping is generalized after the style of Ćirić (1972), and secondly the underlying space is freed to a non-metrizable situation. The topological space \((X, \mathcal{T})\) is called a gauge space if the topology \( \mathcal{T} \) is generated by a family \( \{d_\alpha : \alpha \in \Gamma\} \) of pseudometrics on \( X \), in the sense that the family of balls
\[ \{B(x, d_\alpha, \varepsilon) : x \in X, \alpha \in \Gamma, \varepsilon > 0\} \]
where \( B(x, d_\alpha, \varepsilon) = \{y \in X : d_\alpha(x, y) < \varepsilon\} \) is a subbase for \( \mathcal{T} \). It is well known that \((X, \mathcal{T})\) is a gauge space if and only if it is completely regular. Furthermore, \((X, \mathcal{T})\) is Hausdorff (and hence Tychonoff) if and only if for \( x, y \in X, x \neq y \) there is an \( \alpha \in \Gamma \) such that \( d_\alpha(x, y) > 0 \) [see Dugundji (1966, pp. 198-200)]. Tan (1972) has considered fixed point theorems in this non-metric setting, so that this paper may be regarded as a continuation of his work. Throughout this paper, unless otherwise stated, \( X \) is a gauge space generated by the family \( \{d_\alpha : \alpha \in \Gamma\} \) of pseudometrics.

If \( \{x_n : n = 0, 1, 2, \ldots\} \) is a sequence in \( X \) and \( x \in X \), then \( \{x_n\} \) converges to \( x \), written as \( x_n \rightarrow x \) as \( n \rightarrow \infty \) or \( \lim_{n \to \infty} x_n = x \), if and only if for each \( \alpha \in \Gamma \),
\[ d_\alpha(x_n, x) \rightarrow 0 \text{ as } n \rightarrow \infty. \]
The sequence \( \{x_n : n = 0, 1, 2, \ldots\} \) in \( X \) is said to be Cauchy if and only if for each \( \alpha \in \Gamma \), \( d_\alpha(x_n, x_m) \rightarrow 0 \) as \( n, m \rightarrow \infty \).

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Definition 1 — A selfmapping $T$ of a gauge space $(X, \mathcal{G})$ is a contingent contraction (henceforth abbreviated as c. contraction) if for each $\alpha \in \Gamma$ there exist non-negative bounded real valued functions $q_\alpha, r_\alpha, s_\alpha, t_\alpha$ and $u_\alpha$ defined on $X \times X$ such that for all $x, y \in X$ we have

$$d_\alpha(Tx, Ty) \leq q_\alpha(x, y) d_\alpha(x, y) + r_\alpha(x, y) d_\alpha(x, Tx) + s_\alpha(x, y) d_\alpha(y, Ty) + t_\alpha(x, y) d_\alpha(x, Ty) + u_\alpha(x, y) d_\alpha(y, Tx)$$

and either

(i) \[ \limsup_{n \to \infty} \sup_{x \in X} \left\{ \left[ \frac{q_\alpha(T^n x, T^{n+1} x) + r_\alpha(T^n x, T^{n+1} x) + t_\alpha(T^n x, T^{n+1} x)}{1 - s_\alpha(T^n x, T^{n+1} x) - t_\alpha(T^n x, T^{n+1} x)} \right] : x \in X \right\} = \lambda_\alpha < 1 \]

and

\[ \limsup_{n \to \infty} \sup_{x, y \in X} \{q_\alpha(T^n x, y) + t_\alpha(T^n x, y) : x, y \in X \} < 1 \]

or

(i') \[ \limsup_{n \to \infty} \sup_{x \in X} \left\{ \left[ \frac{q_\alpha(T^{n+1} x, T^n x) + s_\alpha(T^{n+1} x, T^n x) + u_\alpha(T^{n+1} x, T^n x)}{1 - r_\alpha(T^{n+1} x, T^n x) - u_\alpha(T^{n+1} x, T^n x)} \right] : x \in X \right\} = \lambda'_\alpha < 1 \]

and

\[ \limsup_{n \to \infty} \sup_{x, y \in X} \{r_\alpha(y, T^n x) + u_\alpha(y, T^n x) : x, y \in X \} < 1. \]

This notion is a generalization of the concept of $\lambda$-generalized contraction for metric spaces introduced by Ćirić (1972, Definition 2.1). If $X$ is a gauge space, a selfmapping $T$ on $X$ is called a $\lambda$-generalized contraction if for each pair of points $x, y \in X$ and for each $\alpha \in \Gamma$ there are non-negative numbers $q_\alpha(x, y), r_\alpha(x, y), s_\alpha(x, y), t_\alpha(x, y)$ such that

$$\sup \{q_\alpha(x, y) + r_\alpha(x, y) + s_\alpha(x, y) + 2t_\alpha(x, y) : x, y \in X, \alpha \in \Gamma \} = \lambda < 1$$

and

$$d_\alpha(Tx, Ty) \leq q_\alpha(x, y) d_\alpha(x, y) + r_\alpha(x, y) d_\alpha(x, Tx) + s_\alpha(x, y) d_\alpha(y, Ty) + t_\alpha(x, y) (d_\alpha(x, Ty) + d_\alpha(y, Tx)).$$

The following example shows that for metric spaces there are c. contractions which are not $\lambda$-generalized contractions.

Example — Let $X = [0, 10] \cup \{11.5, 12\}$ regarded as a subspace of the real line with the usual metric. Define the mapping $T$ by $Tx = \frac{3}{4} x$ for all $x \in X \setminus \{12\}$, and $T(12) = 11.5$. Then $T$ is a c. contraction. For if $x, y \neq 12$ take

$$q(x, y) = \frac{3}{4}, \quad r(x, y) = s(x, y) = t(x, y) = u(x, y) = \frac{1}{20},$$

$$q(x, 12) = \frac{3}{4}, \quad r(x, 12) = s(x, 12) = t(x, 12) = \frac{1}{20}, \quad u(x, 12) = \frac{2}{3},$$
and \[ q(12, x) = \frac{3}{4}, \quad r(12, x) = s(12, x) = u(12, x) = \frac{1}{20}, \quad t(12, x) = \frac{2}{3}. \]

However, \( T \) is not a \( \lambda \)-generalized contraction, for consider the points \( x = 10, \ y = 12. \)

**Definition 2** — Let \( T \) be a selfmapping of a gauge space \( (X, \mathcal{G}) \). Then \( X \) is said to be \( T \)-orbitally complete if every sequence of the form \( \{T_{\mathcal{G}}^{n}(x) : i = 0, 1, 2, \ldots\} \) where \( x \in X \), which is a Cauchy sequence has a limit point in \( X \).

A gauge space \( X \) is sequentially complete [see Tan (1972, Definition 1.6)] if every Cauchy sequence in \( X \) converges to some point of \( X \). If \( X \) is a gauge space, then \( X \) is sequentially compact implies that \( X \) is countably compact which implies that \( X \) is sequentially complete which implies that \( X \) is \( T \)-orbitally complete for any self-mapping \( T \) of \( X \). It is possible for a non-complete gauge space to be orbitally complete with respect to one selfmapping but not another. [See Cirić (1971, Example 3) for an example in the metric case].

**Theorem 1** — If \( T \) is a c. contraction of a \( T \)-orbitally complete Hausdorff gauge space \( X \), then \( T \) has a fixed point in \( X \). Moreover, this fixed point is unique if \[ \sup \{q_{a}(x, y) + t_{a}(x, y) + u_{a}(x, y) : x, y \in X, a \in \Gamma \} < 1. \]

**Proof:** Let \( x_{0} \) be any point of \( X \) and define the sequence \[ x_{1} = Tx_{0}, \quad x_{2} = Tx_{1} = T^{2}x_{0}, \quad \ldots \quad x_{n} = Tx_{n-1} = T^{n}x_{0}, \quad \ldots. \]

Let \( a \in \Gamma \), and suppose that condition (i) of Definition 1 is satisfied. Then we have

\[
\begin{align*}
    d_{a}(x_{n}, x_{n+1}) &= d_{a}(T_{\mathcal{G}}^{n}(x_{0}), T_{\mathcal{G}}^{n}(x_{0})) \\
    &= q_{a}(x_{n-1}, x_{n}) d_{a}(x_{n-1}, x_{n}) \\
    &+ r_{a}(x_{n-1}, x_{n}) d_{a}(x_{n-1}, T_{\mathcal{G}}^{n}(x_{0})) + s_{a}(x_{n-1}, x_{n}) d_{a}(x_{n}, T_{\mathcal{G}}^{n}(x_{0})) \\
    &+ t_{a}(x_{n-1}, x_{n}) d_{a}(x_{n-1}, T_{\mathcal{G}}^{n}(x_{0})) + u_{a}(x_{n-1}, x_{n}) d_{a}(x_{n}, T_{\mathcal{G}}^{n-1}(x_{0})) \\
    &= q_{a}(x_{n-1}, x_{n}) d_{a}(x_{n-1}, x_{n}) + r_{a}(x_{n-1}, x_{n}) d_{a}(x_{n-1}, x_{n}) \\
    &+ s_{a}(x_{n-1}, x_{n}) d_{a}(x_{n-1}, x_{n+1}) + t_{a}(x_{n-1}, x_{n}) d_{a}(x_{n-1}, x_{n+1}).
\end{align*}
\]

Using the triangle inequality we have

\[
\begin{align*}
    d_{a}(x_{n}, x_{n+1}) &\leq \{q_{a}(x_{n-1}, x_{n}) + r_{a}(x_{n-1}, x_{n})\} d_{a}(x_{n-1}, x_{n}) \\
    &+ s_{a}(x_{n-1}, x_{n}) d_{a}(x_{n}, x_{n+1}) + t_{a}(x_{n-1}, x_{n}) \{d_{a}(x_{n-1}, x_{n}) \\
    &+ d_{a}(x_{n}, x_{n+1})\}.
\end{align*}
\]

Therefore

\[
\begin{align*}
    d_{a}(x_{n}, x_{n+1}) &\leq \frac{q_{a}(x_{n-1}, x_{n}) + r_{a}(x_{n-1}, x_{n}) + t_{a}(x_{n-1}, x_{n})}{1 - s_{a}(x_{n-1}, x_{n}) - t_{a}(x_{n-1}, x_{n})} d_{a}(x_{n-1}, x_{n}).
\end{align*}
\]
Thus, there is a positive integer $N$ such that
\[
d_a(x_{N+n}, x_{N+n+1}) \leq \lambda_n d_a(x_{N+n-1}, x_{N+n}) \leq \lambda^n d_a(x_N, x_{N+1}).
\]
Hence $d_a(x_n, x_m) \to 0$ as $n, m \to \infty$, since $\lambda < 1$.

If, on the other hand, condition (i)' of Definition 1 is satisfied, then by considering $d_a(x_{n+1}, x_n)$ we have that $d_a(x_n, x_m) \to 0$ as $n, m \to \infty$. Thus in both cases $\{x_n\}$ is a Cauchy sequence, and since $X$ is $T$-orbitally complete, there is a point $\xi$ in $X$ such that $\xi = \lim_{n \to \infty} T^n x_0 = \lim_{n \to \infty} x_n$.

We claim that $\xi$ is a fixed point of $T$ in $X$. In case (i) we have
\[
d_a(Tx_n, T\xi) \leq q_a(x_n, \xi) d_a(x_n, \xi) + r_a(x_n, \xi) d_a(x_n, x_{n+1})
+ s_a(x_n, \xi) \{d_a(\xi, x_{n+1}) + d_a(x_{n+1}, T\xi)\}
+ t_a(x_n, \xi) \{d_a(x_n, x_{n+1}) + d_a(x_{n+1}, T\xi)\}
+ u_a(x_n, \xi) d_a(\xi, x_{n+1}).
\]
Hence
\[
\{1 - s_a(x_n, \xi) - t_a(x_n, \xi)\} d_a(Tx_n, T\xi) \leq q_a(x_n, \xi) d_a(x_n, \xi)
+ \{r_a(x_n, \xi) + t_a(x_n, \xi)\} d_a(x_n, x_{n+1}) + \{s_a(x_n, \xi)
+ u_a(x_n, \xi)\} d_a(\xi, x_{n+1}).
\]
Now there is a positive integer $N'$ such that
\[
\sup \{s_a(x_n, y) + t_a(x_n, y) : x, y \in X\} < 1 \text{ for } n > N'.
\]
Moreover, the functions $q_a, r_a, s_a, t_a$ and $u_a$ are bounded, so that $d_a(Tx_n, T\xi) \to 0$ as $n \to \infty$.

In case (i)', by considering $d_a(T\xi, Tx_n)$ we show that $d_a(T\xi, Tx) \to 0$ as $n \to \infty$. Thus for all $\alpha \in \Gamma$, $d_a(T\xi, Tx_n) \to 0$ as $n \to \infty$, so that $T\xi = \lim_{n \to \infty} x_n$. But $X$ is Hausdorff and $\xi = \lim_{n \to \infty} x_n$, so that $T\xi = \xi$ as desired.

If $\eta \in X$ and $\eta = T\eta$ then we have for all $\alpha \in \Gamma$ that
\[
d_a(\xi, \eta) = d_a(T\xi, T\eta) \leq q_a(\xi, \eta) d_a(\xi, \eta) + t_a(\xi, \eta) d_a(\xi, \eta)
+ u_a(\xi, \eta) d_a(\eta, \xi),
\]
that is
\[
d_a(\xi, \eta) \leq \{q_a(\xi, \eta) + t_a(\xi, \eta) + u_a(\xi, \eta)\} d_a(\xi, \eta).
\]
If $\xi \neq \eta$ there is a $\beta \in \Gamma$ such that $d_\beta(\xi, \eta) > 0$.

By hypothesis $q_\beta(\xi, \eta) + t_\beta(\xi, \eta) + u_\beta(\xi, \eta) < 1$, so that we have the contradiction
\[
d_\beta(\xi, \eta) < d_\beta(\xi, \eta).
\]
We now state the preceding result for a metric space. This seems to be a new result.

**Corollary** — If $T$ is a c. contraction of a $T$-orbitally complete metric space $X$, then $T$ has a fixed point in $X$ which will be unique if

$$\sup \{q(x, y) + t(x, y) + u(x, y) : x, y \in X\} < 1.$$  

The metric version of the next theorem has been proven by Cirić (1972, Theorem 2.6). It follows immediately from Theorem 1 by observing that a $\lambda$-generalized contraction is a c. contraction such that

$$\sup \{q_\alpha(x, y) + t_\alpha(x, y) + u_\alpha(x, y) : x, y \in X, \alpha \in \Gamma\} < 1$$

since $t_\alpha(x, y) = u_\alpha(x, y)$ for all $x, y \in X$ and $\alpha \in \Gamma$.

**Theorem 2** — If $T$ is a $\lambda$-generalized contraction of a $T$-orbitally complete Hausdorff gauge space, then $T$ has a unique fixed point in $X$.

The following result is an extension to gauge spaces of a theorem proved by Zamfirescu (1972, Theorem 1) for metric spaces.

**Theorem 3** — Let $X$ be a sequentially complete Hausdorff gauge space such that for each $\alpha \in \Gamma$ there are non-negative real numbers $a_\alpha, b_\alpha, c_\alpha$ which satisfy

$$\sup \{a_\alpha : \alpha \in \Gamma\} < 1, \sup \{b_\alpha : \alpha \in \Gamma\} < \frac{1}{2} \quad \text{and} \quad \sup \{c_\alpha : \alpha \in \Gamma\} < \frac{1}{2}.$$  

Let $T$ be a selfmapping of $X$ such that for each pair of points $x, y \in X$ and each $\alpha \in \Gamma$ at least one of the following conditions is satisfied:

1. $d_\alpha(Tx, Ty) \leq a_\alpha d_\alpha(x, y),$
2. $d_\alpha(Tx, Ty) \leq b_\alpha [d_\alpha(x, Tx) + d_\alpha(y, Ty)],$
3. $d_\alpha(Tx, Ty) \leq c_\alpha [d_\alpha(x, Ty) + d_\alpha(y, Tx)].$

Then $T$ has a unique fixed point in $X$.

The proof follows by noting that such a mapping $T$ is a c. contraction of $X$ and appealing to Theorem 1.

The next three results follow immediately from Theorem 3. Theorem 4 is Banach's classical contraction principle for gauge spaces [see Tan (1972, Theorem 2.3)]. Theorems 5 and 6 have been proven in the metric case by Kannan (1968) and Zamfirescu (1972, Corollary 3) respectively.

**Theorem 4** — If $T$ is a selfmapping of a sequentially complete Hausdorff gauge space $X$ such that for each $\alpha \in \Gamma$ there is a real number $a_\alpha$ with $0 \leq a_\alpha < 1$ and $d_\alpha(Tx, Ty) \leq a_\alpha d_\alpha(x, y)$ for all $x, y \in X$, then $T$ has a unique fixed point in $X$. 
Theorem 5 — If $T$ is a selfmapping of a sequentially complete Hausdorff gauge space $X$ such that for each $\alpha \in \Gamma$ there is a real number $b_\alpha$ with $0 \leq b_\alpha < \frac{1}{2}$ and $d_\alpha(Tx, Ty) \leq b_\alpha\{d_\alpha(x, Tx) + d_\alpha(y, Ty)\}$ for all $x, y \in X$ then $T$ has a unique fixed point in $X$.

Theorem 6 — If $T$ is a selfmapping of a sequentially complete Hausdorff gauge space $X$ such that for each $\alpha \in \Gamma$ there is a real number $c_\alpha$ with $0 \leq c_\alpha < \frac{1}{2}$ and

$$d_\alpha(Tx, Ty) \leq c_\alpha\{d_\alpha(x, Ty) + d_\alpha(y, Tx)\}$$

then $T$ has a unique fixed point in $X$.

Theorems 4, 5 and 6 also follow immediately from the next result which generalizes a theorem of Hardy and Rogers (1973) for metric spaces.

Theorem 7 — Let $X$ be a sequentially complete Hausdorff gauge space and $T$ a selfmapping of $X$ such that for each $\alpha \in \Gamma$ there are non-negative real numbers $a_\alpha, b_\alpha, c_\alpha, e_\alpha$ and $f_\alpha$ such that

$$\sup\{a_\alpha + b_\alpha + c_\alpha + e_\alpha + f_\alpha : \alpha \in \Gamma\} = \lambda < 1$$

and

$$d_\alpha(Tx, Ty) \leq a_\alpha d_\alpha(x, y) + b_\alpha d_\alpha(x, Tx) + c_\alpha d_\alpha(y, Ty) + e_\alpha d_\alpha(x, Ty) + f_\alpha d_\alpha(y, Tx),$$

for all $x, y \in X$. Then $T$ has a unique fixed point in $X$.

Such a mapping $T$ is a c. contraction, since we may, without loss of generality, take $e_\alpha = f_\alpha$ for each $\alpha \in \Gamma$. Then

$$a_\alpha + b_\alpha + c_\alpha + e_\alpha + \lambda e_\alpha < \lambda \text{ yields } \frac{a_\alpha + b_\alpha + e_\alpha}{1 - c_\alpha - e_\alpha} < \lambda,$$

so that $T$ satisfies condition (i) of Definition 1.

By taking $e_\alpha = f_\alpha = 0$ for each $\alpha \in \Gamma$, we obtain a gauge space version of the theorem of Reich (1971a).

Another similar application of Theorem 1 gives the following result which extends to completely regular spaces the series of results of Rakotch (1962), Reich (1971b, Theorem 3) and Hardy and Rogers (1973, Theorem 2).

Theorem 8 — Let $X$ be a sequentially complete Hausdorff gauge space, and for each $\alpha \in \Gamma$ let $a_\alpha, b_\alpha, c_\alpha, e_\alpha$ and $f_\alpha$ be monotonically decreasing functions from $[0, \infty)$ to $[0, 1)$ such that $a_\alpha(t) + b_\alpha(t) + c_\alpha(t) + e_\alpha(t) + f_\alpha(t) < 1$. Suppose $T$ is a self-mapping of $X$ such that
\[ d_\alpha(Tx, Ty) \leq a_\alpha(d_\alpha(x, y)) d_\alpha(x, y) + b_\alpha(d_\alpha(x, y)) d_\alpha(x, Tx) \\
+ c_\alpha(d_\alpha(x, y)) d_\alpha(y, Ty) + e_\alpha(d_\alpha(x, y)) d_\alpha(x, Ty) \\
+ f_\alpha(d_\alpha(x, y)) d_\alpha(y, Tx) \]

for all \( x, y \in X \) and \( \alpha \in \Gamma \). Then \( T \) has a unique fixed point in \( X \).

REFERENCES


