COEFFICIENT ESTIMATES FOR SOME CLASSES OF SPIRAL-LIKE FUNCTIONS

H. S. GOPALAKRISHNA
Department of Mathematics, Karnataka University, Dharwad 580003

AND

PRAKASH G. UMARANI
Nijalingappa College, Rajaji Nagar, Bangalore 560010

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The authors introduce the class $F_\lambda(\alpha, \beta)$ of regular functions $f(z) = z + a_2z^2 + \ldots$, defined in the unit disk $E = \{z : |z| < 1\}$ and satisfying the condition

$$\left| \frac{H(f(z)) - 1}{H(f(z)) + 1} \right| < \lambda$$

where

$$H(f(z)) = \frac{e^{i\alpha} \frac{zf'(z)}{f(z)} - \beta \cos \alpha - i \sin \alpha}{(1 - \beta) \cos \alpha}$$

for all $z$ in $E$ and the class $H_\lambda(\alpha, \beta)$ of regular functions $g(z) = (1/z) + a_1z + \ldots$, defined in the punctured disk $E' = \{z : 0 < |z| < 1\}$ and satisfying the condition

$$\left| \frac{H_1(g(z)) - 1}{H_1(g(z)) + 1} \right| < \lambda$$

where

$$H_1(g(z)) = \frac{-e^{i\alpha} \frac{zg'(z)}{g(z)} - \beta \cos \alpha - i \sin \alpha}{(1 - \beta) \cos \alpha}$$

for all $z$ in $E'$, where $\alpha \in (-\pi/2, \pi/2)$, $\beta \in [0, 1)$ and $\lambda \in (0, 1]$. Using the technique of Clunie (1959) the authors obtain the sharp coefficient estimates for the class $F_\lambda(\alpha, \beta)$ and $H_\lambda(\alpha, \beta)$.

1. INTRODUCTION

Let $A$ denote the class of functions $f(z)$ which are analytic in the unit disk $E = \{z : |z| < 1\}$ and satisfy the conditions $f(0) = 0$ and $f'(0) = 1$. For

$\alpha \in (-\pi/2, \pi/2)$

and $\beta \in [0, 1)$, let $F(\alpha, \beta)$ denote the class of functions $f(z) \in A$ which satisfy the condition

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\[ \text{Re } e^{i\alpha} \frac{zf'(z)}{f(z)} > \beta \cos \alpha \quad \ldots(1.1) \]

for all \( z \in E \). We call functions of \( F(\alpha, \beta) \) as \( \alpha \)-spiral functions of order \( \beta \). Clearly \( F(\alpha, \beta) \subset F(\alpha) \) and \( F(\alpha, 0) \equiv F(\alpha) \), where \( F(\alpha) \) is the well-known class of \( \alpha \)-spiral functions introduced by Špacek (1933). The class \( F(\alpha, \beta) \) was introduced and studied by Libera (1967).

For \( \alpha \in (-\pi/2, \pi/2), \beta \in [0, 1) \) and \( \lambda \in (0, 1] \), let \( F_\lambda(\alpha, \beta) \) denote the class of functions \( f(z) \) belonging to \( A \) which satisfy the condition

\[ \left| \frac{H(f(z)) - 1}{H(f(z)) + 1} \right| < \lambda \quad \ldots(1.2) \]

for \( z \in E \), where

\[ H(f(z)) = \frac{e^{i\alpha} \frac{zf'(z)}{f(z)} - \beta \cos \alpha - i \sin \alpha}{(1 - \beta) \cos \alpha}. \quad \ldots(1.3) \]

It is easy to see that \( F_\lambda(\alpha, \beta) \subset F(\alpha, \beta) \) for every \( \lambda \in (0, 1] \) and \( F_1(\alpha, \beta) \equiv F(\alpha, \beta) \).

In this paper we use a method of Clunie (1959) to obtain sharp bounds for the coefficients of functions of \( F_\lambda(\alpha, \beta) \). In the rest of the paper we always assume that \( \alpha \in (-\pi/2, \pi/2), \beta \in [0, 1) \) and \( \lambda \in (0, 1] \).

2.

The following lemma gives a representation formula for functions of \( F_\lambda(\alpha, \beta) \).

**Lemma 1** — If \( f(z) \in A \), then \( f(z) \in F_\lambda(\alpha, \beta) \) if and only if

\[ e^{i\alpha} \frac{zf'(z)}{f(z)} = \cos \alpha \frac{1 + (2\beta - 1)w(z)}{1 + w(z)} + i \sin \alpha \quad \ldots(2.1) \]

for \( z \in E \) for some \( w(z) \) analytic in \( E \) and satisfying \( w(0) = 0 \) and \( |w(z)| < \lambda \) for \( z \in E \).

**Proof:** If \( f(z) \) is given by (2.1), then

\[ H(f(z)) = \frac{1 - w(z)}{1 + w(z)} \quad \text{so that} \quad \frac{H(f(z)) - 1}{H(f(z)) + 1} = -w(z) \]

and so (1.2) holds. Thus \( f(z) \in F_\lambda(\alpha, \beta) \).

If \( f(z) \in F_\lambda(\alpha, \beta) \), then (1.2) holds.

Defining \( w(z) = \frac{1 - H(f(z))}{1 + H(f(z))} \) we obtain (2.1).
Lemma 2 — If $m$ is a natural number $\geq 3$, then

$$
\frac{\cos^2 \alpha}{(m - 1)^2} \left\{ 4\lambda^2(1 - \beta)^2 + \sum_{k=2}^{m-1} \left[ (k + 1 - 2\beta)^2 \lambda^2 + (k - 1)^2 \lambda^2 \tan^2 \alpha - \frac{(k - 1)^2 \sec^2 \alpha}{\prod_{j=0}^{k-2} u_j} \right] \right\} = \prod_{j=0}^{m-2} u_j
$$

...(2.2)

where

$$
u_j = \frac{\lambda^2 | 2(1 - \beta) \cos \alpha e^{-i\alpha} + j |^2}{(j + 1)^2} \text{ for } j = 0, 1, 2, \ldots . \quad \text{...(2.3)}
$$

Proof: We prove the lemma by induction on $m$. For $m = 3$, (2.2) is easily verified directly. Suppose, now, that (2.2) holds for $m = p - 1$ for some $p \geq 4$. Then, for $m = p$, the left member of (2.2) reduces to

$$
\frac{\cos^2 \alpha}{(p - 1)^2} \left\{ 4\lambda^2(1 - \beta)^2 + \sum_{k=2}^{p-2} \left[ (k + 1 - 2\beta)^2 \lambda^2 + (k - 1)^2 \lambda^2 \tan^2 \alpha - \frac{(k - 1)^2 \sec^2 \alpha}{\prod_{j=0}^{k-2} u_j} \right] \right\}

+ (p - 1)^2 \lambda^2 \tan^2 \alpha - (k - 1)^2 \sec^2 \alpha \right\} \prod_{j=0}^{p-3} u_j \right\}

= \frac{1}{(p - 1)^2} \left\{ (p - 2)^2 \prod_{j=0}^{p-3} u_j + \left[ (p - 2\beta)^2 \lambda^2 \cos^2 \alpha + (p - 2)^2 \lambda^2 \sin^2 \alpha \right] \prod_{j=0}^{p-3} u_j \right\} \quad \text{(by the inductive hypothesis)}.

= \frac{\lambda^2}{(p - 1)^2} \left\{ (p - 2\beta)^2 \cos^2 \alpha + (p - 2)^2 \sin^2 \alpha \right\} \prod_{j=0}^{p-3} u_j

= \prod_{j=0}^{p-2} u_j.

Thus (2.2) holds for $m = p$ which proves Lemma 2.
Theorem 1: If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in F_{\lambda}(\alpha, \beta) \), then
\[
| a_n | \leq \prod_{j=0}^{n-2} u_j^{1/2}
\]
...(2.4)
for \( n = 2, 3, \ldots \), where \( u_j \) is defined by (2.3) for \( j = 0, 1, 2, \ldots \). The result is sharp.

Proof: We have by Lemma 1,
\[
e^{i\alpha} \frac{zf''(z)}{f(z)} = \cos \alpha \frac{1 + (2\beta - 1) \omega(z)}{1 + \omega(z)} + i \sin \alpha
\]
for \( z \in E \) where \( \omega(z) \) is analytic in \( E \), \( \omega(0) = 0 \) and \( | \omega(z) | < \lambda \) for \( z \in E \). This yields
\[
\{ e^{i\alpha} \sec \alpha \cdot zf'(z) + (1 - 2\beta - i \tan \alpha) f(z) \} \omega(z)
= (1 + i \tan \alpha) f(z) - e^{i\alpha} \sec \alpha \cdot zf'(z).
\]
That is,
\[
\left\{ \sum_{k=1}^{\infty} [ke^{i\alpha} \sec \alpha + (1 - 2\beta - i \tan \alpha)] a_k z^k \right\} \omega(z)
= -(1 + i \tan \alpha) \sum_{k=2}^{\infty} (k - 1) a_k z^k \text{, where } a_1 = 1.
\]
Since \( \omega(z) \) is of the form \( \omega(z) = \sum_{k=1}^{\infty} b_k z^k \) we obtain for \( n \geq 2 \),
\[
\left\{ \sum_{k=1}^{n-1} [ke^{i\alpha} \sec \alpha + (1 - 2\beta - i \tan \alpha)] a_k z^k \right\} \omega(z)
= -(1 + i \tan \alpha) \sum_{k=2}^{n} (k - 1) a_k z^k + \sum_{k=n+1}^{\infty} d_k z^k \text{ } \ldots \text{(2.5)}
\]
where \( \sum_{k=n+1}^{\infty} d_k z^k \) converges in \( E \).

Denoting the right member of (2.5) by \( G(z) \) and the factor multiplying \( \omega(z) \) in the left member of (2.5) by \( F(z) \), (2.5) assumes the form
\[
G(z) = F(z) \omega(z) \text{ for } z \in E.
\]
Since \( | \omega(z) | < \lambda \) for \( z \in E \) this yields for \( 0 < r < 1 \),
\[
\frac{1}{2\pi} \int_{0}^{2\pi} |G(re^{i\theta})|^2 \, d\theta \leq \lambda^2 \frac{1}{2\pi} \int_{0}^{2\pi} |F(re^{i\theta})|^2 \, d\theta,
\]
whence, using the definitions of $G(z)$ and $F(z)$

$$\sec^2 \alpha \sum_{k=2}^{n} (k - 1)^2 \ | a_k |^2 \ r^{2k} + \sum_{k=n+1}^{\infty} \ | d_k |^2 \ r^{2k}$$

$$\leqslant \lambda^2 \left\{ \sum_{k=1}^{n-1} \ | k e^{i\alpha} \sec \alpha + (1 - 2\beta - i \tan \alpha) |^2 \ | a_k |^2 \ r^{2k} \right\}.$$ 

Letting $r \to 1$, we obtain for $n \geq 2$

$$\sec^2 \alpha \sum_{k=2}^{n} (k - 1)^2 \ | a_k |^2$$

$$\leqslant \lambda^2 \left\{ \sum_{k=1}^{n-1} \ [(k + 1 - 2\beta)^2 + (k - 1)^2 \tan^2 \alpha] \ | a_k |^2 \right\}.$$ 

which may be written as

$$\sec^2 \alpha (n - 1)^2 \ | a_n |^2 \leqslant \sum_{k=1}^{n-1} \ [(k + 1 - 2\beta)^2 \lambda^2 + (k - 1)^2 \lambda^2 \tan^2 \alpha$$

$$- (k - 1)^2 \sec^2 \alpha] \ | a_k |^2$$

$$| a_n |^2 \leqslant \frac{\cos^2 \alpha}{(n - 1)^2} \sum_{k=1}^{n-1} \ [(k + 1 - 2\beta)^2 \lambda^2 + (k - 1)^2 \lambda^2 \tan^2 \alpha$$

$$- (k - 1)^2 \sec^2 \alpha] \ | a_k |^2$$

For $n = 2$, (2.6) yields $| a_2 |^2 \leqslant 4(1 - \beta)^2 \lambda^2 \cos^2 \alpha = u_0$ which proves (2.4) for $n = 2$.

We now prove (2.4) for all $n \geq 2$ by induction on $n$. Suppose (2.4) holds for $n \leq p - 1$ for some $p \geq 3$. Then for $n = p$, (2.6) yields,

$$| a_p |^2 \leqslant \frac{\cos^2 \alpha}{(p - 1)^2} \sum_{k=1}^{p-1} \ [(k + 1 - 2\beta)^2 \lambda^2 + (k - 1)^2 \lambda^2 \tan^2 \alpha$$

$$- (k - 1)^2 \sec^2 \alpha] \ | a_k |^2$$

$$\leqslant \frac{\cos^2 \alpha}{(p - 1)^2} \left\{ 4(1 - \beta)^2 \lambda^2 + \sum_{k=2}^{p-1} \ [(k + 1 - 2\beta)^2 \lambda^2$$

$$+ (k - 1)^2 \lambda^2 \tan^2 \alpha - (k - 1)^2 \sec^2 \alpha \right\} \prod_{j=2}^{k-2} u_j \right\}.$$
\[ p-2 = \prod_{j=0}^{n} u_j, \text{ by Lemma 2.} \]

So (2.4) holds for all \( n = p \). Thus (2.4) holds for all \( n \geq 2 \).

Equality holds in (2.4) for each \( n \geq 2 \) for the function \( f(z) \) in \( A \) defined by (2.1) with \( \omega(z) = \lambda z \).

This completes the proof of Theorem 1.

For \( \lambda = 1 \), Theorem 1 reduces to a result of Libera (1967). For \( \lambda = 1 \) and \( \alpha = 0 \) in Theorem 1, we obtain a result of Schild (1965). For \( \lambda = 1 \) and \( \beta = 0 \), Theorem 1 yields a result of Zamorski (1962).

3. Meromorphic Spiral-like Functions

Let \( E' \) denote the punctured disk \( \{ z : 0 < |z| < 1 \} \). For \( \alpha \in (-\pi/2, \pi/2) \) and \( \beta \in [0, 1) \), let \( H(\alpha, \beta) \) denote the family of functions \( g(z) \) analytic in \( E' \) and of the form

\[ g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_nz^n \quad \text{...(3.1)} \]

for \( z \in E' \) which satisfy the condition

\[ \text{Re} \left\{ -e^{i\alpha}z \frac{g'(z)}{g(z)} \right\} > \beta \cos \alpha. \quad \text{...(3.2)} \]

Functions of the class \( H(\alpha, \beta) \) are called meromorphic \( \alpha \)-spiral-like of order \( \beta \). Kaczmarz (1969) obtained sharp coefficient estimates for such functions.

Now, for \( \alpha \in (-\pi/2, \pi/2), \beta \in [0, 1) \) and \( \lambda \in (0, 1] \), let \( H_\lambda(\alpha, \beta) \) denote the class of all functions \( g(z) \) analytic in \( E' \) and of the form (3.1) which satisfy the condition

\[ \left| \frac{H_1(g(z)) - 1}{H_1(g(z)) + 1} \right| < \lambda \quad \text{...(3.3)} \]

for \( z \in E' \), where

\[ H_1(g(z)) = \frac{-e^{i\alpha}z \frac{g'(z)}{g(z)} - \beta \cos \alpha - i \sin \alpha}{(1 - \beta) \cos \alpha}. \]

It is clear that \( H_\lambda(\alpha, \beta) \subseteq H(\alpha, \beta) \) for all \( \lambda \in (0, 1] \) and \( H_1(\alpha, \beta) \equiv H(\alpha, \beta) \).

It can be shown that \( g(z) \in H_\lambda(\alpha, \beta) \) if and only if

\[ -e^{i\alpha}z \frac{g'(z)}{g(z)} = \cos \alpha \frac{1 + (2\beta - 1) \omega(z)}{1 + \omega(z)} + i \sin \alpha \]
for $z \in E'$, for some function $\omega(z)$ analytic in $E$ and satisfying $\omega(0) = 0$ and $|\omega(z)| < \lambda$ for $z \in E$.

Proceeding as in the proof of Theorem 1, we can prove the following

**Theorem 2** — If

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \in H_{\lambda}(\alpha, \beta),$$

then

$$|b_n| \leq \frac{2\lambda(1 - \beta) \cos \alpha}{n + 1} \quad \text{for} \quad n = 1, 2, 3, \ldots \quad \ldots(3.4)$$

The result is sharp.

In fact equality holds in (3.4) for fixed $n$ for the function

$$g(z) = \frac{1}{z} \left(1 - \lambda z^{n+1}\right)^2(1 - \beta \cos \omega e^{-i\alpha} / (n + 1))$$

$$= \frac{1}{z} - \frac{2\lambda(1 - \beta) \cos \omega e^{-i\alpha}}{n + 1} z^n + \ldots$$

For $\lambda = 1$, Theorem 2 yields a result of Kaczmarski (1969).

**References**


