MAPPINGS ON METRIC SPACES

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It is well known that a contraction on a complete metric space has a unique fixed point. Kannan (1968) exhibited another class of maps with this property and investigated about common fixed points of pairs of mappings. Fisher (1976) defined Kannan maps and obtained a relationship between contractions and Kannan maps. Our first result exhibits a wide class of mappings each member of which has a unique fixed point such that contractions and Kannan maps belong to it. We call the members of this class as generalized Kannan maps. Our second result exhibits a relationship between generalized Kannan maps satisfying a condition and contractions.

§1. A contraction mapping is a mapping $T$ on a metric space $(X, d)$ into itself which satisfies $d(Tx, Ty) \leq C d(x, y)$ for all $x, y$ in $X$ where $0 \leq C < 1$. It is well known that a contraction mapping on a complete metric space has a unique fixed point. Kannan (1968) investigated the conditions under which two mappings on a metric space have a common fixed point and proved the following:

Theorem (Kannan 1968) — If $T_1$ and $T_2$ are mappings on a complete metric space $(X, d)$ into itself and if there is a constant $K$ such that $0 \leq K < \frac{1}{2}$ and

$$d(T_1x, T_2y) \leq K(d(x, T_1x) + d(y, T_2y)) \quad \text{...(A)}$$

for all $x, y$ in $X$, then $T_1$ and $T_2$ have a unique common fixed point.

Fisher (1976) defined a mapping $T$ on a metric space $(X, d)$ to be a Kannan mapping if it satisfies (A) with $T_1 = T_2 = T$ and obtained a relation between Kannan mappings and contraction mappings. In fact he proved that if $T$ is a contraction mapping on a metric space into itself then $T^n$ is a Kannan mapping for large $n$ and on the other hand if $T$ is any Kannan mapping on $(X, d)$ satisfying

$$d(x, Tx) + d(y, Ty) \leq h d(x, y)$$

for all $x, y$ in $X$ and a fixed $h > 0$, $T^n$ is a contraction mapping for large $n$.

The purpose of this note is to extend these results to a much wider class of mappings.

§2. Let $(X, d)$ be a metric space.
Definition 1 — A pair of mappings \((T_1, T_2)\) on \(X\) into itself is said to have the Kannan property or simply the property \(K\) if there exist constants \(K_i(1 \leq i \leq 5)\) so that

\[
0 \leq K_i \quad \text{for} \quad 1 \leq i \leq 5 \tag{1.1}
\]

\[
K_1 + K_2 + 2K_4 + K_5 < 1 \text{ for } i = 3 \text{ and } i = 4 \tag{1.2}
\]

and

\[
d(T_1x, T_2y) \leq K_1d(x, T_1x) + K_2d(y, T_2y) + K_4d(x, T_2y) + K_5d(x, y) \tag{1.3}
\]

for all \(x, y\) in \(X\).

Remark : The inequalities (1.2) imply that \(K_2 + K_4 + K_5 < 1\).

Definition 2 — A mapping \(T\) on \(X\) into itself is said to be a generalized Kannan mapping if the pair \((T, T)\) has the property \(K\). In that case we sometimes write that \(T\) has the property \(K\).

Remark : Contraction mappings and Kannan mappings are generalized Kannan mappings.

Theorem 1 — If \(T\) is a generalized Kannan mapping of a complete metric space into itself, then \(T\) has a unique fixed point.

Theorem 1 is an immediate consequence of the following proposition.

Proposition 1 — Let \((X, d)\) be a complete metric space and \(T_1, T_2\) be mappings of \(X\) into itself. If \((T_1, T_2)\) has the property \(K\) then \((T_1, T_2)\) has a unique common fixed point.

Proof : Let \(x \in X\) and write \(x_1 = T_1x, x_2 = T_2x_1, x_3 = T_1x_2, \) and so on. Then

\[
d(x_1, x_2) = d(T_1x, T_2x_1) \leq K_1d(x, T_1x) + K_2d(x_1, T_2x_1) + K_4d(x, T_2x_1) + K_5d(x, x_1) \]

where \(K_i(1 \leq i \leq 5)\) are constants as in Definition 1.

Hence \((1 - K_2)\) \(d(x_1, x_2) \leq (K_1 + K_5)\) \(d(x, x_1) + K_5d(x, x_2)\)

\[
\leq (K_1 + K_5)\) d(x, x_1) + K_5d(x, x_1) + d(x_1, x_2)).
\]

\[
\Rightarrow (1 - K_2 - K_5)\) d(x_1, x_2) \leq (K_1 + K_3 + K_5)\) d(x, x_1).
\]

Hence \(d(x_1, x_2) \leq \frac{K_1 + K_3 + K_5}{1 - K_2 - K_5} d(x, x_1)\).
An inductive argument yields the inequality
\[ d(x_n, x_{n+1}) \leq r^n d(x, x_1) \]
where \( r = \frac{K_1 + K_3 + K_5}{1 - K_2 - K_3} \).

Now \( d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + \ldots + d(x_{n+p-1}, x_{n+p}) \)
\[ \leq (r^n + r^{n+1} + \ldots + r^{n+p-1}) d(x, x_1) \]
\[ \leq \frac{r^n}{1 - r} d(x, x_1). \quad \text{...}(B) \]

Since \( 0 \leq r < 1 \), it is now clear that \( \{x_n\} \) is a Cauchy sequence and since \( X \) is complete, there is an \( x_0 \) in \( X \) so that \( d(x_n, x_0) \to 0 \) as \( n \to \infty \).

We show that \( x_0 \) is a common fixed point of \( T_1 \) and \( T_2 \). Let \( n \) be any even integer.
\[ d(x_0, T_1 x_0) \leq d(x_0, x_n) + d(x_n, T_1 x_0) \leq d(x_0, x_n) + K_1 d(x_0, T_1 x_0) \]
\[ + K_2 d(x_{n-1}, T_2 x_{n-1}) + K_3 d(x_0, T_2 x_{n-1}) \]
\[ + K_4 d(T_1 x_0, x_{n-1}) + K_5 d(x_{n-1}, x_0). \]
Hence \( (1 - K_1) d(x_0, T_1 x_0) - K_4 d(T_1 x_0, x_{n-1}) \leq d(x_0, x_n) + K_2 d(x_{n-1}, x_n) \]
\[ + K_3 d(x_0, x_n) + K_5 d(x_{n-1}, x_0). \]

Let \( n = 2p \). Taking the limits as \( p \to \infty \), we get
\[ (1 - K_1) d(x_0, T_1 x_0) - K_4 d(T_1 x_0, x_0) \leq 0. \]

Since \( K_1 + K_4 < 1 \), we now get \( d(x_0, T_1 x_0) = 0 \). Hence \( x_0 = T_1 x_0 \). Now let \( n \) be any odd integer.
\[ d(x_0, T_2 x_0) \leq d(x_0, x_n) + d(x_n, T_2 x_0) = d(x_0, x_n) + d(T_1 x_{n-1}, T_2 x_0) \]
\[ \leq d(x_0, x_n) + K_1 d(x_{n-1}, T_1 x_{n-1}) + K_2 d(x_0, T_2 x_0) \]
\[ + K_3 d(x_{n-1}, T_2 x_0) \]
\[ + K_4 d(x_0, T_1 x_{n-1}) + K_5 d(x_{n-1}, x_0). \]
Hence \( (1 - K_2) d(x_0, T_2 x_0) - K_5 d(x_{n-1}, T_2 x_0) \)
\[ \leq d(x_0, x_n) + K_1 d(x_{n-1}, x_n) + K_4 d(x_0, x_n) + K_5 d(x_{n-1}, x_0). \]
Let \( n = 2p - 1 \). Taking the limits as \( p \to \infty \) we get
\[ (1 - K_2 - K_5) d(x_0, T_2 x_0) \leq 0, \]
and since \( K_2 + K_5 < 1 \) we get \( x_0 = T_2 x_0 \). Thus \( x_0 \) is a common fixed point of \( T_1 \) and \( T_2 \).
We now prove the uniqueness. Suppose \( y_0 \) is also a common fixed point of \( T_1 \) and \( T_2 \). Then \( d(x_0, y_0) = d(T_1x_0, T_2y_0) \). Hence
\[
d(x_0, y_0) \leq K_1d(x_0, T_1x_0) + K_2d(y_0, T_2y_0) + K_3d(x_0, T_2y_0) + K_4d(y_0, T_1x_0) + K_5d(x_0, y_0).
\]
Hence \( d(x_0, y_0) \leq (K_3 + K_4 + K_5) d(x_0, y_0) \). Since \( K_3 + K_4 + K_5 < 1 \), it now follows that \( d(x_0, y_0) = 0 \); hence \( x_0 = y_0 \).

**Theorem 2** — Suppose \( T \) is a mapping of a metric space \((X, d)\) into itself having the property \( K \). If there is a constant \( h > 0 \) such that
\[
d(x, Tx) + d(y, Ty) \leq hd(x, y)
\]
for all \( x, y \) in \( X \), then there exists a positive integer \( n \) such that \( T^n \) is a contraction mapping.

**Proof:** Since \( T \) has the property \( K \), there exist constants \( K_i (1 \leq i \leq 5) \) that satisfy (1.1) through (1.3). Now assume first that \( X \) is complete. Putting \( T_1 = T_2 = T \) in the inequality (B) of Proposition 1 we get
\[
d(x_n, x_{n+p}) \leq \frac{r^n}{1 - r} d(x, x_1) \text{ where } x_k = T^kx \text{ and } r = \frac{K_1 + K_3 + K_5}{1 - K_2 - K_4}.
\]
Letting \( p \) tend to infinity in the above inequality, we get
\[
d(x_n, x_0) = d(T^n x, x_0) \leq \frac{r^n}{1 - r} d(x, Tx) \text{ where } x_0 = \lim_n x_n.
\]
Similarly \( d(T^n y, x_0) \leq \frac{r^n}{1 - r} d(y, Ty) \) and so
\[
d(T^n x, T^n y) \leq \frac{r^n}{1 - r} [d(x, Tx) + d(y, Ty)]
\]
\[
\leq \frac{r^n}{1 - r} hd(x, y).
\]
But \( r^n \to 0 \) as \( n \to \infty \) and hence if \( 0 < K < 1 \), then \( \frac{r^n}{1 - r} h < K \) for large \( n \) and if we fix any such \( n \) we get \( d(T^n x, T^n y) \leq Kd(x, y) \) for all \( x, y \), yielding that \( T^n \) is a contradiction.

Suppose now \( X \) is not complete.
\[
d(Tx, Ty) \leq K_1d(x, Tx) + K_2d(y, Ty) + K_3d(x, Ty) + K_4d(y, Tx) + K_5d(x, y)
\]
\[ \leq d(x, Tx) + d(y, Ty) + K_2 d(x, Tx) + K_3 d(Tx, Ty) \]
\[ + K_4 d(y, Ty) + K_4 d(Tx, Ty) + K_5 d(x, y) \]

Hence
\[ (1 - K_3 - K_4) d(Tx, Ty) \leq 2d(x, Tx) + 2d(y, Ty) + K_5 d(x, y). \]
\[ \leq (2h + K_5) d(x, y). \]

Therefore
\[ d(Tx, Ty) \leq \frac{2h + K_5}{1 - K_3 - K_4} d(x, y). \]

Thus \( T \) is uniformly continuous. Let \( \bar{X} \) be the completion of \( X \) and let \( \bar{T} \) be the completion of \( T \). It is clear that \( \bar{T} \) has the property \( K \). By what we have just proved, \( \bar{T}^n \) is a contraction on \( \bar{X} \) for large \( n \). It follows that \( T^n \) is a contraction for large \( n \) on \( X \) completing the proof of the theorem.

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