SUBCLASSES OF CONVEXOID OPERATORS II

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Using the notation of operator radii $w_p(\cdot)$, we define new classes $S_p$ for each $p \geq 1$. An operator $T \in S_p$ if $w_p((T - zI)^{-1}) = 1/d(z, \sigma(T))$, for all $z \notin \tilde{W}(T)$. It is shown that, class $S_p$ is a subset of class of convexoid operators and there exists a non-normal operator on a finite dimensional Hilbert space $H$, in this class.

INTRODUCTION

Let $B(H)$ denote the set of all bounded linear transformations on a complex Hilbert space $H$. Let $\sigma(T)$, con $\sigma(T)$ and $\tilde{W}(T)$ respectively denote the spectrum, convex-hull of $\sigma(T)$ and the closure of numerical range of $T$. According to Fujii (1973) hen-spectrum of $T$ (denoted by $\tilde{\sigma}(T)$) is the complement, of the unbounded component, of the complement of $\sigma(T)$, i.e. $\tilde{\sigma}(T) = [(\sigma(T))^c]_\infty$,

$$\sigma(T) \subseteq \tilde{\sigma}(T) \subseteq \text{con } \sigma(T) \subseteq \tilde{W}(T) \quad \text{and} \quad \partial \tilde{\sigma}(T) \subseteq \sigma(T),$$

where $\partial M$ denotes the boundary of set $M$. Clearly for $z \notin \tilde{\sigma}(T)$,

$$d(z, \sigma(T)) = d(z, \tilde{\sigma}(T)).$$

Let $r(T)$, $w(T)$ and $\|T\|$ denote the spectral radius, the numerical radius and the norm of $T$. An operator $T$ is normaloid if $w(T) = \|T\|$, spectraloid if $r(T) = w(T)$ and convexoid if $\text{con } \sigma(T) = \tilde{W}(T)$.

$$T \in G_1 \text{ if } \| (T - zI)^{-1} \| = 1/d(z, \sigma(T)), \text{ for all } z \notin \sigma(T)$$

$$T \in H_1 \text{ if } \| (T - zI)^{-1} \| = 1/d(z, \tilde{\sigma}(T)), \text{ for all } z \notin \tilde{\sigma}(T)$$

$$T \in S \text{ if } \| (T - zI)^{-1} \| = 1/d(z, \sigma(T)), \text{ for all } z \notin \tilde{W}(T).$$

Let $C_p(\rho > 0)$ be the class of all operators with unitary $\rho$-dilations (Sz-Nagy and Foias 1970). According to Holbrook (1968), an operator radius $w_p(T)$ is defined as

$$w_p(T) = \inf \{u : u > 0 \text{ and } u^{-1}T \in C_p\},$$

in particular $w_1(T) = \|T\|$ and $w_1(T) = w(T)$. Further $w_p(\cdot)$ is homogeneous, i.e. $w_p(zT) = |z| w_p(T)$ for all complex numbers $z$. 
$T \in B(H)$ is an operator of class $M_\rho(\rho \geq 1)$ if, for all $z \in \sigma(T)$,

$$w_\rho((T - zI)^{-1}) = 1/d(z, \sigma(T)) \ (\text{Patel 1976}).$$

$T$ is an operator of class $H_\rho(\rho \geq 1)$, if $w_\rho((T - zI)^{-1}) = 1/d(z, \sigma(T))$, for all $z \in \sigma(T)$ (Acharya 1980). It is further known that $M_1 = G_1$ (Patel 1976) and for $\rho \geq 1$, $M_\rho \subseteq H_\rho$; $H_\rho$ is a proper subset of convexoid operators (Acharya 1980).

We define, classes $S_\rho$ for each $\rho \geq 1$, as follows:

**Definition**: Let $\rho \geq 1$. $T \in S_\rho$ if $w_\rho((T - zI)^{-1}) = 1/d(z, \sigma(T))$ for all $z \in \sigma \bar{W}(T)$.

It is clear that,

(i) For $\rho = 1$, $w_1(T) = \|T\|$. Therefore class $S_\rho$ coincides with class $S$, defined by Furuta (1977). In this case $(T - zI)^{-1}$ is normaloid for each $z \in \sigma \bar{W}(T)$.

(ii) For $\rho = 2$, $w_\rho((T - zI)^{-1}) = w_1((T - zI)^{-1}) = r((T - zI)^{-1})$ for all $z \in \sigma \bar{W}(T)$. Hence, $(T - zI)^{-1}$ is spectraloid for $z \in \sigma \bar{W}(T)$.

(iii) For $\rho \geq 1$, $w_\rho((T - zI)^{-1}) = r((T - zI)^{-1})$, for all $z \in \sigma \bar{W}(T)$ implies that $(T - zI)^{-1}$ is $\rho$-oid for all $z \in \sigma \bar{W}(T)$.

(iv) Since $w_\rho(.)$ is non-increasing as $\rho$ increases, we have $S_\rho \subseteq S_\rho'$ whenever $\rho \leq \rho'$.

**Theorem 1** — (i) Class $S_\rho$ is arcwise connected.

(ii) Class $S_\rho$ is translation invariant.

**Proof**: We have $\sigma(\alpha T + \beta) = \alpha \sigma(T) + \beta$ and $W(\alpha T + \beta) = \alpha W(T) + \beta$ for complex numbers $\alpha$ and $\beta$. Further it is shown in (Holbrook 1968) that

$$w_\rho(\alpha T) = |\alpha| w_\rho(T).$$

Now it is easy to see that $T \in S_\rho$ implies that $\alpha T \in S_\rho$ for every complex number $\alpha$. Hence the ray in $B(H)$ through $T$ is contained in $S_\rho$. Therefore $S_\rho$ is arcwise connected.

Since, $\sigma(T)$ and $W(T)$ are translation invariants, it is easy to see that class $S_\rho$ is also translation invariant.

It is known that $H_1$ is a subset of $S$ (Furuta 1977) i.e. $H_\rho \subseteq S_\rho$ is true for $\rho = 1$. We generalize the same for each $\rho \geq 1$.

**Theorem 2** — For each $\rho \geq 1$, class $H_\rho$ is contained in class $S_\rho$. 

Proof: If $T \in H_{\rho}$, then $w_{\rho}((T - zI)^{-1}) = 1/d(z, \sigma(T))$ for all $z \notin \sigma(T)$. Now for $z \in \sigma(T)$, $d(z, \sigma(T)) = d(z, \tilde{\sigma}(T)) \cdot \sigma(T) \subseteq \tilde{W}(T)$. Hence for $z \notin \tilde{W}(T)$,
$$w_{\rho}((T - zI)^{-1}) = 1/d(z, \sigma(T)) \quad \text{or} \quad T \in S_{\rho}. $$

Therefore, we have $H_{\rho} \subseteq S_{\rho}$. 

In our next result, we show that above inclusion is proper for $1 \leq \rho \leq 2$. We require Theorem 3.2 (Furuta 1977).

Theorem A (Furuta 1977) — If $A$ is an operator and $B \in S$ such that
$$d(z, \sigma(B)) \leq d(z, W(A))$$
for all $z \notin \tilde{W}(B)$ then $T = A \oplus B \in S$.

Example 1 — Let $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ and $B$ be a normal operator with
$$\sigma(B) = \{z : |z - 3| = 4, \text{Re } z \leq 3\} \cup \{z : |z - 3| = 2, \text{Re } z \leq 3\}.$$ 

Consider $T = A \oplus B$. Since $d(z, \sigma(B)) \leq d(z, W(A))$ for all $z \notin \tilde{W}(B)$ and $B$ is normal (hence belonging to $S$), by Theorem A, $T \in S = S_{1} \subseteq S_{\rho}$. Now
$$w((T - zI)^{-1}) = \max \{w((A - zI)^{-1}), w((B - zI)^{-1})\},$$
$$w((A - zI)^{-1}) = (|z| + 1)/|z|^2 \quad \text{and} \quad w((B - zI)^{-1}) = 1/d(z, \sigma(B)).$$

For $z = 1/10 \notin \tilde{\sigma}(T) = \{0\} \cup \sigma(B)$, $w((T - zI)^{-1}) = 110$ and $1/d(z, \sigma(T)) = 10$. Hence $T \notin H_{\rho}$, i.e., $T \notin H_{\rho}$ for $1 \leq \rho \leq 2$.

To show that the operators in class $S_{\rho}$ are convexoid operators, we require the following generalization of Theorem 3 (Patel 1976).

Theorem B (Furuta 1978) — $T$ is convexoid if and only if
$$w_{\rho}((T - zI)^{-1}) \leq 1/d(z, \text{con } \sigma(T))$$
for all $z$ whose absolute values are sufficiently large.

Theorem 3 — Class $S_{\rho}$ is a subset of class of convexoid operators.

Proof: Let $T \in S_{\rho}$. Hence $w_{\rho}((T - zI)^{-1}) = 1/d(z, \sigma(T))$ for all $z \notin \tilde{W}(T)$.

Now $\sigma(T) \subseteq \text{con } \sigma(T)$ and $\tilde{W}(T)$ is compact, implies that
$$w_{\rho}((T - zI)^{-1}) \leq 1/d(z, \text{con } \sigma(T)),$$
for all $z$ whose absolute values are sufficiently large. Hence $T$ is convexoid by Theorem B.
We give an example to show that the above inclusion is proper for $1 \leq \rho \leq 2$.

**Example 2** — Let $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ and $B$ be a diagonal operator with

$$ \text{dig } B = \{ \sqrt{2}, -\sqrt{2}, \sqrt{2}i, -\sqrt{2}i \}. $$

Consider $T = A \oplus B$. Since $\overline{W}(A) \subseteq \overline{W}(B)$ and $B$ is convexoid, by Fujii (1971), $T$ is convexoid. Now we show that $T \not\in S_2$. We have

$$ w((T - zI)^{-1}) = \max \{ w((A - zI)^{-1}), w((B - zI)^{-1}) \}, $$

$$ w((A - zI)^{-1}) = (1 + |z|^2)/(1 + |z|^2), \quad w((B - zI)^{-1}) = 1/d(z, \sigma(B)). $$

For $z = 1 + i \not\in \overline{W}(T)$, $w((T - zI)^{-1}) = (1 + \sqrt{2})/2$

and

$$ 1/d(z, \sigma(T)) = (2 + \sqrt{2})/4. $$

Hence $T \not\in S_2$. i.e. $T \not\in S_\rho$ for $1 \leq \rho \leq 2$.

It is known that, if dim $H < \infty$ or $\sigma(T)$ is finite, then $T \in H_\rho$ implies that $T$ is normal (Acharya 1980). It is interesting to know that this property does not hold for class $S_\rho$.

**Theorem 4** — There exists a non-normal operator (on finite dimensional Hilbert space $H$) belonging to class $S_\rho$, even if dim $H < \infty$.

**Proof** : Let $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ and $B$ be a diagonal operator with

$$ \text{dig } B = \{ z_i : z_i = \frac{1}{2} e^{\pi i/4}, i = 1, 2, ..., 21. \} $$

Consider $T = A \oplus B$. Clearly $T$ is an operator on a finite dimensional Hilbert space $H$. Since $A$ is non-normal, $T$ is also non-normal. Here $d(z, \sigma(B)) \leq d(z, \overline{W}(A))$, for all $z \not\in \overline{W}(B)$, hence by Theorem A, $T \in S = S_1 \subset S_\rho$.

It is shown (Patel and Gupta 1975) that if $T \in M_\rho$ and $\sigma(T) \subset C$, where $C$ is the unit circle in the complex plane, then $T$ is unitary. It is shown (Acharya 1980) that if $\sigma(T)$ is a proper subset of $C$ and $T \in H_\rho$ when $T$ is unitary. Here we show that:

**Theorem 5** — Let $T \in S_\rho$. If $\sigma(T) \subset \{ z : z = e^{i\theta}, \alpha < \theta < \alpha + \pi \}$, then $T$ is unitary.

**Proof** : Since $T \in S_\rho$, and $\sigma(T)$ is as defined above, we have

$$ w_\rho((T - zI)^{-1}) = 1/d(z, \sigma(T)) \leq 1/(1 - |z| - 1), $$

for all $z \not\in \overline{W}(T)$. In particular we have $w_\rho(T^{-1}) \leq 1$

and

$$ w_\rho((T - zI)^{-1}) \leq 1/(1 - |z| - 1) $$
for $|z| > 1$. Therefore the result follows from Theorem 2 and Corollary 1 of Patel and Gupta (1975).

Lastly we show that the class $S_p$ is not closed under multiplication and inversion.

**Theorem 6** — There exists a non-singular operator $T$ contained in class $S_p$ such that
(i) $T^q \not\in S_p$ for any $p \geq 1$,
(ii) $T^{-1} \not\in S_p$ for any $p \geq 1$.

**Proof:** Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B$ be a normal operator with $\sigma(B) = W(A)$. Consider $T = A \oplus B$. Here $T \in G_1$ by (Luecke 1972). Now

$$G_1 = M_1 \subseteq M_p \subseteq H_p \subseteq S_p$$

implies that $T \in S_p$. Further it is shown Luecke (1972) that $T^2$ is not convexoid. It is further proved by Patel and Gupta (1975) that $T^{-1}$ is not convexoid. Since $S_p$ is a subclass of convexoid operators, we get the required result.

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**References**

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