THE GEOMETRICAL RELATIONSHIP BETWEEN THE QUADRUPLE 
\((P_{n+1}(C), Q_n(C), P_{n+1}(R), V_{n+2,2}(R))\)

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We study the relations between the \((n + 1)\)-complex projective space \(P_{n+1}(C)\), the \((n + 1)\)-real projective space \(P_{n+1}(R)\), the \(n\)-complex quadric \(Q_n(C)\) in \(P_{n+1}(C)\) and the real Stiefel manifold \(V_{n+2,2}(R)\) and prove a few theorems.

1. INTRODUCTION

An \((n + 1)\)-complex projective space \(P_{n+1}(C)\) is defined as the space of complex lines through the origin in \(C^{n+2}\). Let \((z_0, z_1, \ldots, z_{n+1})\) be homogeneous complex coordinates on \(P_{n+1}(C)\) and let \(f\) be a differentiable function on \(P_{n+1}(C)\) defined by

\[f(z_0, z_1, \ldots, z_{n+1}) = \left( \sum_{j=0}^{n+1} z_j \right) \left( \sum_{j=0}^{n+1} \bar{z}_j \right) / \left( \sum_{j=0}^{n+1} | z_j^2 | \right)^2.\]

An \((n + 1)\)-real projective space \(P_{n+1}(R)\) is defined as the space of real lines through the origin in \(R^{n+2}\), the \(n\)-complex quadric \(Q_n(C)\) in \(P_{n+1}(C)\) is given by the equation

\[\sum_{j=0}^{n+1} z_j^2 = 0\]

and the real Stiefel manifold \(V_{n+2,2}\) is defined as the set of all 2-frames in \(R^{n+2}\) (an ordered set of 2-linearly independent vectors of \(R^{n+2}\)) endowed with induced differentiable structure (see Brickell and Clark 1970)

In this paper we study the geometrical relationship between the quadruple \((P_{n+1}(C), Q_n(C), P_{n+1}(R), V_{n+2,2}(R))\). Specifically we prove the following:

Theorem 1 — \(X\) is diffeomorphic to \(V_{n+2,2}(R) \times (0, 1)\), where \(X\) denotes the connected non-compact manifold \(P_{n+1}(C) - Q_n(C) - P_{n+1}(R)\).

On the other hand if \(B = X/\text{SO}(n + 2)\) we can prove the following series of lemmas:

Lemma 1 — \(B\) has a differentiable structure and diffeomorphic to \((0, 1)\).
Lemma 2 — If \( a, b \in B \) provided that \( a \neq b \) then \( f(P) \neq f(q) \) for all \( p \in a \) and \( q \in b \).

Finally we prove the following two theorems:

Theorem 2 — (i) If \( f(p) = 0 \), for all \( p \in P_n+1(C) \), then the level surfaces (see Definition 2.3) of \( f \) are homeomorphic to \( Q_n(C) \).

(ii) If \( f(p) = 1 \), for all \( p \in P_n+1(C) \), then the level surfaces of \( f \) are homeomorphic to \( P_n+1(R) \).

(iii) If \( 0 < f(p) < 1 \), for all \( p \in P_n+1(C) \), then the level surfaces of \( f \) are homeomorphic to \( V_{n+2,2}(R) \).

Theorem 3 — \( P_{n+1}(C) - P_{n+1}(R) \) is diffeomorphic to the normal bundle of \( Q_n(C) \) in \( P_{n+1}(C) \).

2. Preliminaries (Bott 1960)

Let \( M \) be a compact differentiable \( n \)-manifold and

\[ g : M \to R \]

a differentiable real-valued function on \( M \). Since \( \text{dim. } R = 1 \) and \( \text{dim } M \geq 1 \); it follows that the Rank \( r_xg \) of \( g \) at \( P \in M \), i.e., the rank of the linear transformation \( dg : M_p \to R \) induced by the tangent spaces, is either 0 or 1.

Definition 2.1 — If \( r_xg = 0 \), then \( p \) is called a critical point of \( g \).

Definition 2.2 — If \( r_xg = 1 \), then \( p \) is called a regular point of \( g \).

Definition 2.3 — For each \( a \in R \), the set \( g^{-1}(a) \) is called the \( a \)-level surface of \( g \) in \( M \).

Definition 2.4 — The set \([p : p \in M ; g(p) \leq a, \text{ for each } a \in R]\) will be denoted by \( g^aM \), or just by \( M^a \) if the function \( g \) is understood, and is called a half-space for \( g \) on \( M \).

Let \( f \) and \( X \) be as stated above. Suppose that \( a, b \in f(P_{n+1}(C)) \) as \( f^{-1}(a), f^{-1}(b) \subseteq X \). Then it is well known that both \( f^{-1}(a) \) or \( f^{-1}(b) \) are submanifolds of \( P_{n+1}(C) \) with codimension one. Let \( p \) be either a point of \( f^{-1}(a) \) or \( f^{-1}(b) \). Thus \( p \) is a regular point of \( f \) (see Definition 2.2). Since \( X \) is an open submanifold of \( P_{n+1}(C) \), we can choose a neighbourhood \( N \) (say) of \( p \) such that \( NX \). Given a chart \( \phi : N \to C^{n+2} \) on \( N \) and suppose, without any loss of generality, that \( \phi(p) = o \) (\( o \) denotes the origin in \( C^{n+2} \)). Write \( F = f\phi \) which is so clear that \( F \) has rank one everywhere. Hence \( \partial F(o)/\partial x_i \neq 0 \) and \( \partial F(o)/\partial y_j \neq 0 \), where \( (x_1, y_1, ..., x_{n+1}, y_{n+1}) \) are the underlying real coordinate systems in \( C^{n+2} \), for some \( j = 1, 2, ..., n + 1. \)
By ordering the coordinates so that \( j = n + 1 \), then \( \partial F(q)/\partial x_{n+1} \neq 0 \) and \( \partial F(q)/\partial y_{n+1} \neq 0 \) for all \( q \in C^{n+2} \). Suppose further, for \( a < b \), that

\[
P_{n+1}(C)^b_a = \overline{P_{n+1}(C)^b - P_{n+1}(C)^a} \subset X.
\]

Then by the compactness of \( f^{-1}(a) \), together with the fact that \( X \) is open in \( P_{n+1}(C) \), there is an \( 0 < \epsilon < a \) as \( P_{n+1}(C)^b_{a-\epsilon} \subset CX \).

But it is well known that every \( \lambda \)-level surface of \( f \); for \( a - \epsilon \leq \lambda \leq b \), is a compact differentiable \( 2n + 1 \)-dimensional submanifold of \( P_{n+1}(C) \). But since \( P_{n+1}(C) \) carries a Riemannian structure (see Brickell and Clark 1970). Thus at each point \( p \in P_{n+1}(C)^b_{a-\epsilon} \) there is exactly one such curve passing through each point and cuts \( \lambda \)-level surfaces transversally. These curves are given in each coordinate neighbourhood by a system of differential equations:

\[
\begin{align*}
\frac{dx_i(t)/dt}{dt} &= \rho_1 g^{ik}(x(t), y(t)) \partial F'(x(t), y(t))/\partial x_k, \\
\frac{dy_j(t)/dt}{dt} &= \rho_2 g^{ik}(x(t), y(t)) \partial F'(x(t), y(t))/\partial y_k,
\end{align*}
\]

where \( F' \) is defined to be \( F - m \) \((m = (0, 1)\) which is either \( a \) or \( b \)\) which has the same properties as \( F \) and \( g_{ik} \) is the reciprocal of the positive definite Riemannian metric \( g_{1b} \) on \( P_{n+1}(C) \). Here we choose

\[
\begin{align*}
\rho_1 g^{ik} \partial F'(x(t), y(t))/\partial x_i \cdot \partial F'(x(t), y(t))/\partial x_k &= -1, \\
\rho_2 g^{ik} \partial F'(x(t), y(t))/\partial y_i \cdot \partial F'(x(t), y(t))/\partial y_k &= -1
\end{align*}
\]

to get parametrizations for which

\[
\begin{align*}
\frac{dF'(x(t), y(t))/dt}{dt} &= \partial F'(x(t), y(t))/\partial x_k \cdot dx_k(t)/dt = -1, \\
\frac{dF'(x(t), y(t))/dt}{dt} &= \partial F'(x(t), y(t))/\partial y_k \cdot dy_k(t)/dt = -1.
\end{align*}
\]

Definition 2.5 — Any orthogonal trajectory parametrized in this way is called an ortho-\( f \)-arc.

3. THE PROOFS OF THE THEOREMS AND THE LEMMAS

We quote in this section the following two lemmas [see Morsy (1966) for their proofs] which are needed in our proofs of the above theorems and lemmas.

**Lemma A** — \( Q_n(C) \) and \( P_{n+1}(R) \) are non-degenerate manifolds of \( f \) and they contain all the critical points of \( f \).

**Lemma B** — Let \( M \) be the orbit of \( p \in P_{n+1}(C) \) under \( SO(n + 2) \)

(i) If \( p \in Q_n(C) \), then \( SO(2) \times SO(n) \) is the stabilizer [see Morsy (1966) for the definition of the stabilizer of a point] of \( p \) and so \( M \) is homeomorphic to \( Q_n(C) \).
(ii) If \( p \in P_{n+1}(R) \), then \( O(1) \times O(n + 1) \) is the stabilizer of \( p \) and so \( M \) is homeomorphic to \( P_{n+1}(R) \).

(iii) If \( p \in X \), then \( SO(n) \) is the stabilizer of \( p \) and so \( M \) is homeomorphic to \( V_{n+2,2}(R) \).

Proof of Theorem 1 — We will show that the triple \( (X, V_{n+2,2}, (0, 1)) \) is in fact a normal vector bundle of \( V_{n+2,2} \) in \( P_{n+1}(C) \). Let \( N_\varepsilon \subset X \) be an \( \varepsilon \)-tube of the compact oriented manifold \( V_{n+2,2}(R) \), so small that the mapping \( P : N_\varepsilon \rightarrow V_{n+2,2} \) defined by the orthogonal projection of points of \( N_\varepsilon \) onto \( V_{n+2,2} \) is well defined, since \( X \) has a Riemannian structure induced from \( P_{n+1}(C) \), and constitutes fibre decomposition of \( N_\varepsilon \) with \( V_{n+2,2} \) as base space and the orthogonal 1-dimensional cells \( F_b(= (0, 1)) \) for all \( b \in V_{n+2,2}(R) \) as fibres. Thus the quadruple \( (N_\varepsilon, V_{n+2,2}(R), p, (0, 1)) \) is a normal vector bundle of \( V_{n+2,2}(R) \) in \( P_{n+1}(C) \). By lemma (B), it follows that \( X = \bigcup_{\varepsilon \in (0, 1)} N_\varepsilon \) and by the compactness of \( N_\varepsilon \) together with the regularity of \( f \) on each fibre (see lemma (A)) we get that the quadruple \( (X, V_{n+2,2}, p, (0, 1)) \) is a normal vector bundle of \( V_{n+2,2} \) in \( P_{n+1}(C) \).

Proof of Lemma 1 — The proof of this lemma goes in two steps as follows:

(i) \( B \) has a differentiable structure: Choose \( \varepsilon \) small enough and let \( N_\varepsilon \) be an \( \varepsilon \)-tubular neighbourhood of \( V_{n+2,2} \), then by Theorem 1, it follows that \( X \) is diffeomorphic to \( V_{n+2,2} \times (0, 1) \). Since \( f \) is regular on \( X \), let \( \gamma, \gamma' \) be two ortho-f-arcs passing through two distinct points \( p \) and \( q \) of \( V_{n+2,2}(R) \). Such \( \gamma \) and \( \gamma' \) cuts \( N_\varepsilon \) in a section which is still a product. Hence \( \gamma \) and \( \gamma' \) are diffeomorphic. But this in turn defines a differentiable structure on a neighbourhood of every point of \( B \). Such neighbourhoods constitute bases for \( B \). Therefore \( B \) carries a differentiable structure.

(ii) \( B \) is diffeomorphic to \( (0, 1) \): Since \( B = X/\text{SO}(n + 2) \), i.e., in the sense of Lemma B, it follows that \( B \) is a 1-dimensional noncompact connected differentiable manifold. Hence the result. This completes the proof of the lemma.

Proof of Lemma 2 — By Theorem 1 and Lemma 1 we can think of \( f \) as the composite mapping: \( X \twoheadrightarrow B \twoheadrightarrow (0, 1) \), where \( \pi \) is a natural map that sends \( b \in X \) into \( V_{n+2,2} \) (orbit contains it) i.e., if \( U \) is open in \( X \), then \( \pi^{-1}(\text{SO}(n + 2) U) = \text{SO}(n + 2) \) \( U = \bigcup_{h \in \text{SO}(n+2)} hU \). Suppose that \( f(p) = f(q) \), this implies that \( g(a) = g(b) \) if and only if \( a = b \). Thus a contradiction.

Proof of Theorem 2 — Since (i) and (ii) are consequences of (iii). Then our proof is devoted to showing (iii). We need only to prove the following:

(a) \( V_{n+2,2}(R) \subset \) the level surfaces of \( f \);

(b) \( V_{n+2,2}(R) \supset \) the level surfaces of \( f \).
But (a) follows from the following fact, let \( p \in P_{n+1}(C) \) and \( g \in SO(n + 2) \), then we can see easily that \( f(p) = f(p \cdot g) \). It remains only to prove (b) for which we need to prove two things:

(i) \( V_{n+2,2}(R) \) and the level surfaces of \( f \) have the same dimensions but this is evident, because \( \dim V_{n+2,2} = 2n + 1 \) and the level surfaces of \( f \) have codimensions one. This implies that any general orbit is one component of a level surface of \( f \).

(ii) The level surfaces of \( f \) are connected, but this follows from Lemma 2. Thus the theorem is proved.

Proof of Theorem 3 — Let \( N_{\epsilon} \) be an \( \epsilon \)-tubular neighbourhood of the compact oriented manifold \( Q_n(C) \) so small that the mapping \( \pi : N_{\epsilon} \rightarrow Q_n(C) \) defined by the orthogonal projection of points of \( N \) onto \( Q_n(C) \) is well defined, since \( Q_n(C) \) has a Riemannian structure induced from \( P_{n+1}(C) \), which is invariant under the action of \( SO(N + 2) \), and constitutes a fibre decomposition of \( N_{\epsilon} \), with \( Q_n(C) \) as a base space and the orthogonal 2-planes \( F_b(= R^2) \) for all \( b \in Q_n(C) \) as fibres. Thus the quadruple \( (N_{\epsilon}, Q_n(C), \pi, R^2) \) is a normal 2-plane of \( Q_n(C) \) in \( P_{n+1}(C) \). By Theorem 2 we can see easily that \( \partial N_{\epsilon} \) (the boundary of \( N_{\epsilon} \)) is a union of orbits of \( V_{n+2,2} \), but since \( N_{\epsilon} \) is connected then \( \partial N_{\epsilon} \) is an orbit. By using Theorem 1, it follows that

\[
Y = P_{n+1}(C) - P_{n+1}(R) \rightarrow N_{\epsilon}
\]

is diffeomorphic to \( V_{n+2,2} \times (\epsilon, 1) \). Therefore by the identification of the boundaries \( \partial Y \) and \( \partial N_{\epsilon}, P_{n+1}(C) - P_{n+1}(R) \) is diffeomorphic to the normal 2-plane bundle of \( Q_n(C) \) in \( P_{n+1}(C) \).

References

Bott, R. (1960). Morse theory and its application to homotopy theory. Lecture notes by A. Van de Ven (mimeographed), University of Bonn, West Germany.
