A NOTE ON FIXED POINT IN COMPACT METRIC SPACES

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In this paper the results of Fisher (1978) and Khan (1978) have been extended
to a more general case.

The results of this paper are inspired by two recent papers of Fisher (1978) and
Khan (1978). They proved that a continuous mapping $T$ of a compact metric space
$(X, d)$ has a unique fixed point if $T$ satisfies

$$d(Tx, Ty) < \frac{1}{2} (d(x, Ty) + d(y, Tx))$$

or

$$d(Tx, Ty) < (d(x, Tx) d(y, Ty))^{1/2}$$

for all $x, y$ in $X$ with $x \neq y$.

The main purpose of this paper is to extend their results to a more general case.
For related results, we refer to Ciric (1976) and Yeh (1978).

**Theorem** — Let $T$ be a continuous mapping of a nonempty compact metric
space $(X, d)$ satisfying

(C) \hspace{1cm} d(Tx, Ty) < h(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx),
\hspace{2cm} (d(x, y))^{-1} d(x, Tx) d(y, Ty), a(x, y) d(x, Ty) d(y, Tx),
\hspace{2cm} b(x, y) (d(y, Tx) d(x, Ty))^{1/2})$

for all $x, y$ in $X$ with $x \neq y$, where

(i) $a(x, y)$ and $b(x, y)$ are nonnegative real functions;

(ii) $h : (R_+)^8 \to R_+ = [0, \infty)$ is nondecreasing in each coordinate variable; and

$g(t) = h(t, t, t, c_1 t, c_2 t, t, t, t) \leq t$ for each $t > 0$, where $c_1 + c_2 < 2$.

Then $T$ has a fixed point. If, in addition, $a(x, y) \leq (d(x, y))^{-1}$ and $b(x, y) \leq 1$, then
$T$ has a unique fixed point.

**Proof** : Define a real value function $f$ on $X$ by $f(x) = d(x, Tx)$. Since $d$ and
$T$ are continuous functions, it follows that $f$ is a continuous function on $X$. Since $X$
is compact, it attains its minimum value and so there is a point $u$ in $X$ such that 
\[ f(u) = \inf \{ f(x) : x \in X \}. \]
If $u \neq Tu$, then it follows from (C) that
\[
\begin{align*}
&d(Tu, T^2u) < h(d(u, Tu) d(u, Tu), d(Tu, T^2u), d(u, T^2u), d(Tu, Tu), \\
&(d(u, Tu))^{-1} d(u, Tu) d(Tu, T^2u), 0, \\
&b(u, Tu) (d(Tu, Tu) d(u, T^2u))^{1/2})
\end{align*}
\]
\[
\leq h(d(Tu, T^2u), d(Tu, T^2u), d(Tu, T^2u), 2d(Tu, T^3u), 0, \\
(d(Tu, T^2u), 0, 0) \leq g(d(Tu, T^2u)) \leq d(Tu, T^2u)
\]
a contradiction. This contradiction proves that $Tu = u$.

Next we prove that $u$ is unique for $a(x, y) \leq (d(x, y))^{-1}$ and $b(x, y) \leq 1$. Suppose that $v( \neq u)$ is a fixed point of $T$. Then
\[
\begin{align*}
d(u, v) = d(Tu, Tv) < h(d(u, v), d(u, u), d(v, v), d(u, v), d(v, u), \\
0, d(u, v) d(u, v)) \leq g(d(u, v)) \leq d(u, v)
\end{align*}
\]
a contradiction. This contradiction proves the uniqueness of $u$. Thus our proof is complete.

**Corollary** — Let $T$ be a continuous mapping of a nonempty compact metric space $(x, d)$ satisfying.
\[
\begin{align*}
d(Tx, Ty) &< \max \{ d(x, y), \frac{1}{2}(d(x, Tx) + d(y, Ty)), \\
&\frac{1}{4}(d(x, Ty) + d(y, Tx)), (d(x, y))^{-1} d(x, Tx) d(y, Ty), \\
&a(x, y) d(x, Ty) d(y, Tx), (d(x, Tx) d(y, Ty))^{1/2}, \\
b(x, y) &d(y, Tx) d(x, Ty))^{1/2}
\end{align*}
\]
for all $x, y$ in $X$ with $x \neq y$, where $a(x, y)$ and $b(x, y)$ are nonnegative real functions, then $T$ has a fixed point. If, in addition, $a(x, y) \leq (d(x, y))^{-1}$ and $b(x, y) \leq 1$, then $T$ has a unique fixed point.

**References**


