SOME RESULTS ON FIXED POINTS IN COMPACT METRIC SPACES

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A sufficient condition for the existence of a unique fixed point of a self-map of a compact metric space has been obtained.

Edelstein (1962) established the existence of a unique fixed point of a self-map $T$ of a compact metric space satisfying the inequality $\rho(Tx, Ty) < \rho(x, y)$. Fisher (1977) obtained a generalization of this result. We shall now prove the following generalization of Fisher's Theorem.

Theorem 1 — Let $T$ be a self-map of a compact metric space $(X, \rho)$ such that for some positive integer $m$, $T^m$ is continuous and for every $x, y \in X$ with $x \neq y$, $T^m x \neq T^m y$

$$\rho(T^m x, T^m y) < \frac{\alpha_1 \rho(x, T^m x) \rho(y, T^m y)}{\rho(x, y)} + \frac{\alpha_2 \rho(x, T^m x) \rho(y, T^m x)}{\rho(T^m x, T^m y)}$$

$$+ \frac{\alpha_3 \rho(x, T^m y) \rho(y, T^m y)}{\rho(T^m x, T^m y)} + \beta_1 \rho(x, y) + \beta_2 \rho(x, T^m x)$$

$$+ \beta_3 \rho(y, T^m y) + \beta_4 \rho(x, T^m y) + \beta_5 \rho(y, T^m x) \quad \ldots \quad (A)$$

where $\alpha_1 + \alpha_3 + \beta_3 + \beta_4 < 1$, $\alpha_2 > 0$, $\beta_4 > 0$, $\alpha_1 + 2\alpha_3 + \beta_1 + \beta_2 + \beta_3 + 2\beta_4 = 1$ and $\beta_1 + \beta_4 + \beta_5 < 1$.

Then $T$ has a unique fixed point.

Proof: Define $f : X \to [0, \infty)$ by $f(x) = \rho(x, T^m x)$ for every $x \in X$.

Continuity of $\rho$ and $T^m$ ensure the continuity of $f$. Then compactness of $X$ yields a point $z \in X$ such that $f(z) = \inf \{ f(x) ; x \in X \}$.

Then $f(z) \neq 0$ gives $z \neq T^m z$. If $T^m z = T^{2m} z$, then $T^m z$ is a fixed point of $T^m$. So we assume that $T^m z \neq T^{2m} z$.

Then $f(T^m z) = \rho(T^m z, T^{2m} z)$

$$\leq \frac{\alpha_1 \rho(z, T^m z) \rho(T^m z, T^{2m} z)}{\rho(z, T^m z)} + \frac{\alpha_2 \rho(z, T^m z) \rho(T^m z, T^{2m} z)}{\rho(T^m z, T^{2m} z)}$$

$$+ \frac{\alpha_3 \rho(z, T^{2m} z) \rho(T^m z, T^{2m} z)}{\rho(T^m z, T^{2m} z)} + \beta_1 \rho(z, T^m z) + \beta_2 \rho(z, T^m z)$$

$$+ \beta_3 \rho(T^m z, T^{2m} z) + \beta_4 \rho(z, T^{2m} z) + \beta_5 \rho(T^m z, T^{2m} z) \quad \ldots \quad (b y \ (A))$$

$$\leq (\alpha_1 + \alpha_3 + \beta_3 + \beta_4) f(T^m z) + (\alpha_2 + \beta_1 + \beta_2 + \beta_5) f(z) \quad (\because \alpha_3, \beta_4 > 0).$$
Therefore \[ f(T^m z) \leq \frac{\alpha_3 + \beta_2 + \beta_3 + \beta_4}{1 - \alpha_1 - \alpha_3 - \beta_3 - \beta_4} f(z) \quad (\because \alpha_1 + \alpha_3 + \beta_3 + \beta_4 < 1) \]

\[ = f(z) \quad (\therefore \alpha_1 + 2\alpha_3 + \beta_2 + \beta_3 + 2\beta_4 = 1) \]

But this contradicts the definition of \( f(z) \); thus \( f(z) = 0 \) and this gives \( z = T^m z \).

If possible, let \( T^m \) possess another fixed point \( z' \).

Then \( \rho(z, z') = \rho(T^m z, T^m z') \)
\[ < \beta_1 \rho(z, z') + \beta_4 \rho(z, z') + \beta_4 \rho(z, z') \quad \{\text{by (A)}\} \]
\[ \leq \rho(z, z'), \quad \text{(as } \beta_1 + \beta_4 + \beta_4 < 1) \]

which is a contradiction.

Thus \( z \) is the unique fixed point of \( T^m \). Then \( T z = T(T^m z) = T^m(Tz) \) and the unicity of the fixed point of \( T^m \) yield \( T z = z \). The unicity of the fixed point of \( T \) follows from that of \( T^m \) and the fact that any fixed point of \( T \) is a fixed point of \( T^m \).

**Corollary —** Let \( T_1 \) and \( T_2 \) be two commuting self-maps of a compact metric space \( (X, \rho) \) satisfying the conditions of Theorem 1 with \( T \) replaced by \( T_1 . T_2 \). Then \( T_1 \) and \( T_2 \) have a unique common fixed point.

**Proof:** By Theorem 1, \( T_1 . T_2 \) (\( = T_2 . T_1 \)) has a unique fixed point \( u \) (say). Then \( T_1 . T_2 (T_1 u) = T_1 u \) gives \( T_1 u = u \). Similarly \( T_2 u = u \). Since any common fixed point of \( T_1 \) and \( T_2 \) is a fixed point of \( T_1 . T_2 \), the unicity of the common fixed point of \( T_1 \) and \( T_2 \) follows from that of \( T_1 . T_2 \).

**Theorem 2 —** Let \( T \) be as in Theorem 1. Also, in addition, let \( \alpha_2 > 0, \beta_2 > 0, \alpha_2 + \beta_2 + \beta_6 < 1, 2\alpha_2 + \beta_2 + 2\beta_2 + \beta_4 + \beta_5 = 1 \). If \( \rho(T^m x, u) < \rho(x, u) \) for every \( x \in X \) with \( x \neq u \) where \( u \) is the unique fixed point of \( T \) (which exists by Theorem 1), then for every \( x \in X \), \( \lim_{n \to \infty} T^{m n} x = u \).

**Proof:** Let \( x \in X \). Since \( X \) is compact, there exists a subsequence \( \{T^{m n} x\} \) of \( \{T^{m n} x\} \) which converges to some point \( z \) (say). If \( T^{m n} x = u \) for any \( n \), then \( \lim_{n \to \infty} T^{m n} x = u \). So we assume that \( T^{m n} x \neq u \) for any \( n \). Therefore

\[ \rho(T^{m n} x, u) = \rho(T^m T^{m(n-1)} x, T^m u) \]
\[ < \frac{\alpha_2 \rho(T^{m(n-1)} x, T^{m n} x) \rho(u, T^m u)}{\rho(T^{m(n-1)} x, u)} + \frac{\alpha_2 \rho(T^{m(n-1)} x, T^{m n} x) \rho(u, T^{m n} x)}{\rho(T^{m n} x, u)} \]
\[ + \frac{\alpha_2 \rho(T^{m(n-1)} x, u) \rho(u, T^m u)}{\rho(T^{m n} x, u)} + \beta_1 \rho(T^{m(n-1)} x, u) + \beta_4 \rho(T^{m(n-1)} x, T^{m n} x) \]
\[ + \beta_4 \rho(u, T^m u) + \beta_4 \rho(T^{m(n-1)} x, T^{m n} x) + \beta_4 \rho(u, T^{m n} x) \quad \{\text{by (A)}\} \]
\( \leq (\alpha_2 + \beta_2 + \beta_3) \rho(T^{mn}x, u) + (\alpha_2 + \beta_1 + \beta_2 + \beta_4) \rho(T^{m(n-1)}x, u) \)

(as \( \alpha_2, \beta_2 \geq 0 \))

Thus \( \rho(T^{mn}x, u) \leq \frac{\alpha_2 + \beta_1 + \beta_2 + \beta_4}{1 - \alpha_2 - \beta_2 - \beta_5} \rho(T^{m(n-1)}x, u) \) (as \( \alpha_2 + \beta_2 + \beta_5 < 1 \))

\[ = \rho(T^{m(n-1)}x, u) \] (as \( 2\alpha_2 + \beta_1 + 2\beta_2 + \beta_4 + \beta_5 = 1 \))

Therefore \( \{\rho(T^{mn}x, u)\} \) is convergent.

Since \( \lim_{i \to \infty} T^{mn_i}x = z, \lim_{i \to \infty} \rho(T^{mn_i}x, u) = \rho(z, u), \lim_{n \to \infty} \rho(T^{mn}x, u) = \rho(z, u) \).

Now \( \rho(T^{m(n_i+1)}x, u) \leq \rho(T^{m(n_i+1)}x, T^mz) + \rho(T^mz, u) \)

Therefore \( \lim_{i \to \infty} \rho(T^{m(n_i+1)}x, u) \leq \rho(T^mz, u) \). This gives \( \rho(z, u) \leq \rho(T^mz, u) \) which contradicts the inequality \( \rho(T^mx, u) < \rho(x, u) \) unless \( z = u \). Thus

\[ z = u, \Rightarrow \lim_{n \to \infty} \rho(T^{mn}x, u) = 0. \Rightarrow \lim_{n \to \infty} T^{mn}x = u. \]

**REFERENCES**
