ABSTRACT EULER SUMMABILITY OF ALLIED SERIES OF THE FOURIER SERIES

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In this paper, necessary and sufficient condition for the sequence of factors so as to ensure \(| E, q | (q > 0)\) summability of factored Fourier series are obtained. Two more theorems are proved which lead to interesting results that factored Fourier series and conjugate series are almost everywhere \(| E, q | (q > 0)\) summable whenever the sequence of factors satisfies certain conditions. Lastly, a theorem which improves an earlier result due to Tripathi (1973) is proved.

1. Definitions and Notations

Let \(\sum_{n=0}^{\infty} d_n\) be a given infinite series and let \(q\) be a real or complex number such that \(q \neq -1\). Then we define

\[ d_n^q = (1 + q)^{-n-1} \sum_{m=0}^{n} \binom{n}{m} q^{n-m}d_m; \quad d_n^0 = d_n. \quad ...(1.1) \]

Following Chandra (1975b), we write

\[ \sum_{n=0}^{\infty} d_n \in | E, q | \iff \sum_{n=0}^{\infty} | d_n^q | < \infty. \quad ...(1.2) \]

Let \(f\) be 2\(\pi\)-periodic and \(L\)-integrable over \((-\pi, \pi)\), and its Fourier series, at a point \(x\), be

\[ \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(x). \quad ...(1.3) \]

Then the series conjugate to (1.3), will be given by

\[ \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x). \quad ...(1.4) \]

Throughout the paper we assume \(a_0 = 0\). Now, we write

\[ \phi(t) = \frac{1}{2} \{ f(x + t) + f(x - t) \} \quad ...(1.5) \]
\[ \psi(t) = \frac{1}{2} \{ f(x + t) - f(x - t) \} \quad \text{(1.6)} \]

\[ P(t) = \phi(t) - \frac{1}{t} \int_{0}^{t} \phi(u) \, du \quad \text{(1.7)} \]

\[ P^*_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t - u)^{\alpha - 1} P(u) \, du \quad (\alpha > 0) \quad \text{(1.8)} \]

\[ P_\alpha(t) = \Gamma(\alpha + 1) \, t^{-\alpha} P^*_\alpha(t) \quad (\alpha \geq 0) \quad \text{(1.9)} \]

\[ K(n, t) = \cos nt - \frac{\sin nt}{nt} \quad \text{(1.10)} \]

\[ K^{(1)}(n, t) = \frac{\partial}{\partial t} K(n, t) \quad \text{(1.11)} \]

\[ \Gamma(1 - \alpha) g(n, u) = \int_{0}^{\pi} (t - u)^{-\alpha} K(n, t) \, dt \quad \text{(1.12)} \]

\[ v_\alpha^\beta(n) = (1 + q)^{-n-1} \binom{n}{m} g^{n-m} \quad (n \geq m \geq 0) \quad \text{(1.13)} \]

\[ \log_2 = \log \log, \ \log_3 = \log. \quad \text{(1.14)} \]

Throughout, \( T \) denotes the integral part of \((k/t)^2\), where \( 0 < t \leq \pi \) and \( k \) is suitable positive constant and not necessarily the same at each occurrence.

2. Introduction

It is known (Chandra 1972b, c) that \( P(t) (\log (k/t))^{1+\varepsilon} \in BV(0, \pi) \), where \( \varepsilon > 0 \), is a sufficient condition for \( \sum_{1}^{\infty} A_n(x) \in E, q \mid (q > 0) \) while neither (see Theorem 1 of this paper)

\[ P(t) \log \frac{k}{t} \in BV(0, \pi) \quad (k > \pi e^2) \quad \text{(2.1)} \]

nor (see Tripathi 1969)

\[ \phi(t) \in BV(0, \pi) \quad \text{(2.2)} \]

is a sufficient condition for \( \sum_{1}^{\infty} A_n(x) \in E, q \mid (q > 0) \). Also (2.1) and (2.2) are non-comparable since the later holds if and only if (see Chandra 1978b)

(i) \( P(t) \in BV(0, \pi) \);  
(ii) \( t^{-1} P(t) \in L(0, \pi) \).  

\[ \text{(2.3)} \]
Thus the natural question arises as to what \((y_n)\) should be to ensure
\[
\sum_{1}^{\infty} A_n(x) y_n \in \mathbb{E}, q \mid (q > 0)
\]...

whenever (2.1) and (2.2) hold. Recently, Chandra (1978a) and Tripathi (1974) independently have shown that (2.4) with \(y_n = (\log (n + 1))^{-1}\) holds whenever (2.2) holds. To supplement an answer to the above question, we prove the following:

**Theorem 1** — Let \(y_n = (\log (n + 1))^{-1}\). Then, in order that (2.1) \(\Rightarrow\) (2.4), it is necessary and sufficient that \(c > 0\).

Inspired by the above theorem, we also prove

**Theorem 2** — Let \(y_n = (\log (n + 1))^{-1} (\log_2 (n + 2))^{-c}\). Then in order that
\[
P(t) \log_2 (k/t) \in BV(0, \pi)
\]...
implies (2.4), it is necessary and sufficient that \(c > 0\).

**Remark 1**: It may be observed that (2.5) is not a stronger condition than (2.2). However, the class of the functions satisfying (2.5) contains even those functions which are not of bounded variation in \((0, \pi)\). For example, let \(f\) be \(2\pi\)-periodic and even function so that, for \(x = 0\), \(\phi(t) = f(t)\) and let
\[
f(t) = \left(\log \frac{2\pi}{t}\right)^{1/2} \text{ in } (0, \pi).
\]
Then \(f\) satisfies (2.5) but \(f \not\in BV(0, \pi)\).

We further prove the following theorems concerning absolute summability factors for Fourier series and conjugate series:

**Theorem 3** — Let
\[
(i) \quad 0 < n^{\delta} y_n \uparrow \text{ with } n \geq 1, \text{ for } \delta > 0;
(ii) \quad \sum_{n=1}^{\infty} n^{-1/2} \log (n + 1) y_n < \infty.
\]
...

Then (2.4) holds for every value of \(x\) for which
\[
\int_{0}^{t} \left| f(x + u) - f(x) \right| \, du = O(t) \quad (t \to 0).
\]...

**Theorem 4** — Let (2.6) hold. Then \(\sum_{1}^{\infty} B_n(x) y_n \in \mathbb{E}, q \mid (q > 0)\) for every value of \(x\) for which (2.7) holds.

**Remark 2**: Since, for every \(L\)-integrable function \(f\),
\[
\int_0^t |f(x + u) - f(x)| \, du = o(t) \quad (t \to 0)
\]  
\[\ldots(2.8)\]

for almost all values of \(x\), therefore it follows from Theorems 3 and 4 that \(\sum_{n=1}^{\infty} A_n(x) y_n\) and \(\sum_{n=1}^{\infty} B_n(x) y_n\) are \(|E, q|\), where \(q > 0\), summable almost everywhere whenever \((2.6)\) holds.

Tripathi (1973) established the following:

**Theorem A** — Let \(\theta < \alpha < \frac{3}{2}\) and let \(\lambda > \max (\alpha - \frac{1}{2}, \frac{1}{2})\). Then

\[
\phi_n(t) \in BV(0, \pi) \Rightarrow \sum_{n=1}^{\infty} n^{-\lambda} A_n(x) \in |E, q| \quad (q > 0).
\]

It follows from Theorem 3 that \(\sum_{n=1}^{\infty} n^{-\lambda} A_n(x) \in |E, q| \quad (q > 0, \lambda > \frac{1}{2})\) almost everywhere. Thus the case \(\alpha > 1\) of Theorem A is not interesting. Further, it may be noted that \(\lambda > \frac{1}{2}\) for \(0 < \alpha \leq 1\), therefore Theorem A loses its significance when \(0 < \alpha < 1\). In view of these observations we prove the following:

**Theorem 5** — Let \(0 \leq \alpha \leq 1\) and let bounded sequence \((y_n)\) satisfy \((2.6)\) (i) and

\[
\sum_{n=1}^{\infty} n^{-1} y_n < \infty.
\]  
\[\ldots(2.9)\]

Then \(P_n(t) \in BV(0, \pi) \Rightarrow \sum_{1}^{\infty} n^{-\alpha/2} A_n(x) y_n \in |E, ap| \quad (p > 0)\).

**Remark 3**: It may be observed that, for \(\alpha \geq 0\), \(P_n(t) \in BV(0, \pi)\) is a weaker condition than \(\phi_n(t) \in BV(0, \pi)\). Thus Theorem 5 yields the following interesting result:

**Corollary** — Let \(0 \leq \alpha \leq 1\). Then \(P_n(t) \in BV(0, \pi)\) implies

\[
\sum_{n=1}^{\infty} \frac{n^{-\alpha/2} A_n(x)}{(\log (n + 1))^{1+\epsilon}} \in |E, ap| \quad (p > 0, \epsilon > 0).
\]

**3. LEMMAS**

We shall use the following lemmas in the proofs of the theorems:

**Lemma 1** (Chandra 1975b) — Let \(q > p > -1\). Then \(\sum_{0}^{\infty} a_n \in |E, p|\) implies that \(\sum_{0}^{\infty} a_n \in |E, q|\).
Lemma 2 — Let $\Delta > -1$ and let $(y_n)$ be any non-negative sequence such that $(n^{1+\Delta}y_n) \uparrow$ with $n \gg 1$. Then uniformly in $0 < t \leq \pi$

$$\sum_{m=1}^{n+1} v_m^q (n+1) m^\Delta y_m \exp (imt) = O\{t^{-1}(n+1)^{\Delta-(1/2)}y_{n+1}\}.$$  

Proof: We have

$$\sum_{m=1}^{n+1} v_m^q (n+1) m^\Delta y_m \exp (imt)$$

$$= \frac{1+q}{n+2} \sum_{m=1}^{n+1} v_{m+1}^q (n+2) m^{1+\Delta} y_m \exp (imt)$$

$$+ \frac{1+q}{n+2} \sum_{m=1}^{n+1} v_{m+1}^q (n+2) m^\Delta y_m \exp (imt)$$

$$= \frac{1+q}{n+2} \sum_{m=0}^{n} v_{m+2}^q (n+2) (m+1)^{1+\Delta} y_{m+1} \exp \{i(m+1)t\}$$

$$+ \frac{(1+q)^2}{(n+2)(n+3)} \sum_{m=1}^{n+1} v_{m+2}^q (n+3) \left(1+\frac{2}{m}\right) m^{1+\Delta} y_m \exp (imt)$$

$$= \Sigma_1 + \Sigma_2, \text{ say}$$

where

$$\Sigma_2 = O\left\{\frac{1}{(n+1)^2} \sum_{m=1}^{n+1} v_{m+2}^q (n+3) m^{1+\Delta} y_m\right\}$$

$$= O\{(n+1)^{\Delta-1}y_{n+1}\}.$$  

Let $s$ denote the integral part of $(n-3q)/(1+q)$. Then it may be observed that $v_{m+3}^q (n+2)$ is maximum for $m = s$ and, whenever $(n-3q)/(1+q)$ is an integer,

$$v_{s+3}^q (n+2) = v_{s+3}^q (n+2).$$

Thus, by using the fact that $0 < m^{1+\Delta} y_m \uparrow$ for $m \gg 1$, it follows by Abel's lemma that

$$\Sigma_1 = O\{t^{-1}(n+1)^\Delta y_{n+1} v_{s+3}^q (n+2)\}$$
uniformly in \(0 < t \leq \pi\). Further

\[
v_{s+3}^g (n + 2) = \frac{q^{n-s-1} \Gamma(n + 3)}{(1 + q)^{n+3} \Gamma(s + 4) \Gamma(n - s)}.
\]

Therefore, by using Stirling's asymptotic values (see Hobson 1957, p. 70) for \(\Gamma(n + 3), \Gamma(s + 4)\) and \(\Gamma(n - s)\), it follows that

\[
v_{s+3}^g (n + 2) = O\left(\frac{1}{\sqrt{n + 1}}\right).
\]

Collecting the results, proof of the lemma follows.

\textbf{Lemma 3} — Let \(\beta > 0\) and \(s = 1, 2\). Then, uniformly in \(0 < t \leq \pi\),

\[
\int_0^t \frac{\sin nu}{nu (\log, k/u)^\beta} \, du = O\left\{\frac{1}{n (\log, n)^\beta}\right\}.
\]

For \(s = 1\) see Chandra (1972a, d), and for \(s = 2\) see Chandra (1975a); Lemma 4.

\textbf{Lemma 4} — Let \(s = 1, 2\). Then for all real \(\beta\)

\[
\int_0^\pi \left(\log, \frac{k}{t}\right)^\beta \frac{\sin nt}{t} \, dt \sim \frac{\pi}{2^{\beta}} \left(\log, n\right)^\beta
\]

where \(k \geq \pi e^2\).

\textbf{Proof:} We sketch a proof for \(s = 2\). The proof for \(s = 1\) is similar. For arbitrarily large \(\Delta > 0\), we write

\[
\int_0^\pi \left(\log, \frac{k}{t}\right)^\beta \frac{\sin nt}{t} \, dt = \int_0^{n\pi} \left(\log, \frac{kn}{\theta}\right)^\beta \frac{\sin \theta}{\theta} \, d\theta = \int_0^\Delta + \int_\Delta^{\sqrt{n}} + \int_{\sqrt{n}}^{n\pi} \left(\log, \frac{kn}{\theta}\right)^\beta \frac{\sin \theta}{\theta} \, d\theta
\]

\[
= I_1 + I_2 + I_3, \text{ say.}
\]

Now

\[
I_1 = \left(\log, n\right)^\beta \int_0^\Delta \left[1 + \left(\log, n\right)^{-1} \log \left\{1 + \frac{\log (k/\theta)}{\log n}\right\}\right]^\beta \frac{\sin \theta}{\theta} \, d\theta
\]

\textit{(equation continued on p. 221)}
\[
\rightarrow (\log n)^{\alpha} \int_0^\Delta \frac{\sin \theta}{\theta} \, d\theta \quad (n \to \infty)
\]

\[
= (\log n)^{\alpha} \frac{\pi}{2} \left\{ 1 + O\left(\frac{1}{\Delta}\right) \right\}. \quad \text{...(3.1)}
\]

By the second mean value theorem for \( \eta \) and \( \eta' \) in \((\Delta, \sqrt{n})\), we have

\[
I_2 = \begin{cases} 
(\log n^{1/2})^{\alpha} \int_\eta^{\sqrt{n}} \frac{\sin \theta}{\theta} \, d\theta \quad (\beta \leq 0) \\
(\log n^{1/2})^{\alpha} \int_\Delta^{\sqrt{n}} \frac{\sin \theta}{\theta} \, d\theta \quad (\beta > 0)
\end{cases}
\]

\[
= O\{ (\log n)^{\alpha} \}. \quad \text{...(3.2)}
\]

Again, by using the second mean value theorem for \( \xi \) and \( \xi' \) in \((\sqrt{n}, n\pi)\), we get

\[
I_3 = \begin{cases} 
(\log n^{1/2})^{\alpha} \int_{\sqrt{n}}^{\xi} \frac{\sin \theta}{\theta} \, d\theta \quad (\beta \leq 0) \\
(\log n^{1/2})^{\alpha} \int_{\sqrt{n}}^{\xi'} \frac{\sin \theta}{\theta} \, d\theta \quad (\beta > 0)
\end{cases}
\]

\[
= O\{ n^{-1/2} (\log n)^{\alpha} \}. \quad \text{...(3.3)}
\]

Combining (3.1), (3.2) and (3.3), we obtain that

\[
\int_0^\pi \left( \log \frac{k}{t} \right)^{\alpha} \frac{\sin nt}{t} \, dt = \frac{\pi}{2} (\log n)^{\alpha} \left[ 1 + O\left(\frac{\Delta}{n}\right) + O(n^{-1/2}) \right].
\]

Since \( \Delta > 0 \) is arbitrarily large, the desired result follows on letting \( n \to \infty \).

**Lemma 5** — Let \( 0 < u < \pi \) and \( 0 < z < 1 \). Then

\[
\Gamma(1 - \alpha) g(n, u) = \frac{n^{\alpha - 1}}{1 - \alpha} K(n, z)
\]

\[
+ n^{\alpha - 1} \left\{ \sin ny - \sin (nu + 1) - \int_{nu + 1}^y \frac{\sin \theta}{\theta} \, d\theta \right\}
\]

where \( u \leq z \leq u + \frac{1}{n} \) and \( u + \frac{1}{n} < y < \pi \).
PROOF: \( \Gamma(1 - \alpha) g(n, u) = \int_{u}^{\pi} (t - u)^{-\alpha} K(n, t) \, dt \)

\[
= \left( \int_{u}^{u+(1/n)} + \int_{u+(1/n)}^{\pi} \right) (t - u)^{-\alpha} K(n, t) \, dt
\]

\[= I_1 + I_2, \text{ say.} \]

By the first mean value theorem

\[ I_1 = K(n, z) \int_{u}^{u+(1/n)} (t - u)^{-\alpha} \, dt \quad (u \leq z \leq u + n^{-1}) \]

\[= \frac{n^{\alpha-1}}{1 - \alpha} K(n, z). \]

And, by the second mean value theorem

\[
I_2 = n^\alpha \int_{u+(1/n)}^{y} K(n, t) \, dt \quad \left( u + \frac{1}{n} < y < \pi \right)
\]

\[= n^\alpha \left\{ \int_{u+(1/n)}^{y} \cos nt \, dt - \int_{u+(1/n)}^{y} \frac{\sin nt}{nt} \, dt \right\}
\]

\[= n^{\alpha-1} \left\{ \sin ny - \sin (nu + 1) - \int_{nu+1}^{ny} \frac{\sin \theta}{\theta} \, d\theta \right\}. \]

This completes the proof of the lemma.

4. PROOF OF THEOREMS 1 AND 2

For \( s = 1, 2 \), we write

\[ g(t, s) = P(t) \log_s (k/t) \]

and

\[ h(n, s) = (\log (n + 1))^{1-s} (\log_s (n + s))^{-s}. \]

Then (see Chandra 1974)

\[
A_n(x) = \frac{2}{\pi} \int_{0}^{\pi} P(t) K(n, t) \, dt
\]

(equation continued on p. 223)
\[
\begin{align*}
= \frac{2}{\pi} g(\pi, s) \int_0^\pi \left( \log_s \frac{k}{t} \right)^{-1} K(n, t) \, dt \\
- \frac{2}{\pi} \int_0^\pi dg(t, s) \int_0^t \left( \log_s \frac{k}{u} \right)^{-1} K(n, u) \, du.
\end{align*}
\]

And, for \(0 < t \leq \pi\),
\[
\int_0^t \left( \log_s \frac{k}{u} \right)^{-1} K(n, u) \, du
= \frac{\sin nt}{n} \left( \log_s \frac{k}{t} \right)^{-1} + O\left\{ \frac{1}{n \log_s (n + s)} \right\},
\]
by Lemma 2. Therefore
\[
A_n(x) = O\left\{ \frac{1}{n \log_s (n + s)} \right\} - \frac{2}{\pi} \int_0^\pi dg(t, s) \frac{\sin nt}{n \log_s (k/t)}.
\]

4. SUFFICIENT AND NECESSARY CONDITIONS

(i) The condition is sufficient — By Lemma 1,
\[
\sum_{n=1}^\infty \frac{h(n, s)}{n \log_s (n + s)} \in \{|E, q| (q > 0)\}.
\]

Therefore it is sufficient to show that
\[
\sum = \sum_{n=0}^{\infty} \frac{1}{n + 1} \left| \sum_{m=1}^{n+1} v^q_m (n + 1) \sin m\theta(n, s) \right| = O \{\log_s (k/t)\}
\]
uniformly in \(0 < t < \pi\), since
\[
v^q_m (n) = (1 + q) \frac{m + 1}{n + 1} v^q_{m+1} (n + 1) \quad (m \geq 0).
\]

Now, we write
\[
\Sigma = \sum_{n < T} + \sum_{n > T}.
\]

The inner sum in \(\sum_{n < T}\) does not, in modulus, exceed \(O \{h(n, s)\}\), therefore uniformly in \(0 < t < \pi\).
\[ \sum_{n \leq T} = O \{ \log \log (k/t) \}. \]

And, by Lemma 2, it follows that
\[ \sum_{n > T} = O(t^{-1}) \sum_{n > T} n^{-3/2} = O(1) \]
uniformly in \( 0 < t < \pi \). Thus condition is sufficient.

(ii) The condition is necessary — Let
\[ \sum_{n=1}^{\infty} A_n(x) h(n, s) \in E, q \mid (q > 0), \]
whenever \( g(t, s) \in BV(0, \pi) \). Then, for \( g(t, s) = 1 \) in \( (0, \pi) \),
\[
A_n(x) = \frac{2}{\pi} \int_{0}^{\pi} \frac{\cos nt}{\log \log (k/t)} \, dt - \frac{2}{\pi} \int_{0}^{\pi} \frac{\sin nt}{nt \log \log (k/t)} \, dt
\]
\[ = -\frac{2}{\pi} \int_{0}^{\pi} \frac{\sin nt}{tn} \left[ \frac{1}{\log \log (k/t)} + t \frac{d}{dt} \left( \frac{1}{\log \log (k/t)} \right) \right] \, dt \]
\[ \sim -\frac{1}{n \log n}\]
by Lemma 4, and in order that \( \sum A_n(x) h(n, s) \in E, q \mid (q > 0) \), it is necessary that
\[ \sum_{n=1}^{\infty} \frac{h(n, s)}{n \log (n + s)} < \infty. \]

Hence the condition is necessary.

This proves Theorems 1 and 2 completely.

5. PROOF OF THEOREMS 3 AND 4

Proof of Theorem 3 — We have
\[ A_n(x) = \frac{2}{\pi} \int_{0}^{\pi} \phi^*(t) \cos nt \, dt \]
where \( \phi^*(t) = \phi(t) - f(x) \). Then \( \sum_{n=0}^{\infty} A_{n+1}(x) y_{n+1} \in E, q \mid (q > 0) \) if
\[ \sum_{n=0}^{\infty} \left| \sum_{m=0}^{n} y_{m}^q (n) y_{m+1} A_{m+1}(x) \right| < \infty. \]... (5.1)
However

\[
\sum = \frac{2}{\pi} \sum_{n=0}^{\infty} \left( \sum_{m=1}^{n+1} \int_0^{\pi/n} \phi^*(t) \left( \int_0^{\pi/n} v_m^q (n + 1) \, my_m \cos mt \, dt \right) \right).
\]

... (5.2)

Splitting up the integral into \( \int_0^{\pi/n} \) and \( \int_0^{\pi/n} \) and denoting them by \( I_1 \) and \( I_2 \), respectively, we have

\[
| I_1 | \leq \int_0^{\pi/n} | \phi^*(t) | \left( \sum_{m=1}^{n+1} v_m^q (n + 1) \, my_m \right) \, dt
\]

\[
\leq (n + 1) \, y_{n+1} \int_0^{\pi/n} | \phi^*(t) | \, dt \quad \text{by (2.6i)}
\]

\[
= O \{ y_{n+1} \}
\]

by (2.7). And, by Lemma 2,

\[
I_2 = \int_0^{\pi/n} | \phi^*(t) | \, O(t^{-1} \sqrt{n + 1}) \, dt
\]

\[
= O \left\{ (n + 1)^{1/2} \, y_{n+1} \left\{ \int_0^{\pi/n} \frac{| \phi^*(t) |}{t} \, dt \right\} \right\}
\]

\[
= O \{ (n + 1)^{1/2} \, \log (n + 1) \, y_{n+1} \},
\]

integrating by parts and using (2.7).

Combining \( I_1 \) and \( I_2 \) and using (2.6ii), we observe that (5.1) holds. Consequently, this completes the proof of Theorem 3.

Proof of Theorem 4 — We have

\[
B_n(x) = \frac{2}{\pi} \int_0^{\pi} \psi(t) \sin nt \, dt
\]

where, by (2.7),

\[
\int_0^t | \psi(u) | \, du = O(t).
\]

The series \( \sum_{n=1}^{\infty} B_n(x) y_n \in |E, q| (q > 0) \) if (5.1) with \( B_{m+1}(x) \) for \( A_{m+1}(x) \) holds. On replacing \( \phi^*(t) \) and \( \cos nt \) by, respectively, \( \psi(t) \) and \( \sin nt \) in (5.2) and proceeding as in Theorem 3, the proof of the theorem follows.
6. Proof of Theorem 5

The proof of the theorem for \( \alpha = 0 \) follows from Chandra (1975c). Thus we prove the theorem for \( 0 < \alpha < 1 \).

Writing \( q = \alpha p \) \((0 < \alpha < 1)\); we first consider the following:

**Case I**: When \( \alpha = 1 \) — Integrating by parts and using \( P_1(t) \in BV(0, \pi) \), we obtain that

\[
A_n(x) = 2P_1(\pi) \cos n\pi - \frac{2}{\pi} \int_0^\pi P_1(t) tK^{(1)}(n, t) \, dt
\]

\[
= 2P_1(\pi) \cos n\pi - \frac{2}{\pi} P_1(\pi) \int_0^\pi tK^{(1)}(n, t) \, dt
\]

\[
+ \frac{2}{\pi} \int_0^\pi dP_1(t) \int_0^t uK^{(1)}(n, u) \, du
\]

\[
= O\left(\frac{1}{n}\right) + \frac{2}{\pi} \int_0^\pi t \cos nt \, dP_1(t).
\]

Following (2.9) and Lemma 1, it is sufficient to show, uniformly in \( 0 < t < \pi \), that

\[
\sum_{n=0}^{\infty} (1 + n)^{-1} \left| \sum_{m=1}^{n+1} \sqrt{m} \, y_m (n + 1) \sqrt{m} \, y_m \cos mt \right| = O(t^{-1})
\]

which follows from Lemma 2 \((\Delta = \frac{1}{2})\).

Finally, we consider the following:

**Case II**: When \( 0 < \alpha < 1 \) — We have

\[
A_n(x) = \frac{2}{\pi(1 - \alpha)} \int_0^\pi K(n, t) \, dt \int_0^t (t - u)^{-\alpha} \, dP^*_\alpha(u)
\]

\[
= \frac{2}{\pi(1 - \alpha)} \int_0^\pi dP^*_\alpha(u) \int_u^\pi (\theta - u)^{-\alpha} \, K(n, \theta) \, d\theta
\]

(equation continued on p. 227)
\[ = \frac{2\alpha P_\alpha(\pi)}{\pi \Gamma(\alpha + 1)} \int_0^\pi u^{\alpha-1} g(n, u) \, du \]
\[ + \frac{2}{\pi \Gamma(\alpha + 1)} \int_0^\pi dP_\alpha(t) \int_0^t u^\alpha \frac{d}{du} g(n, u) \, du. \]

Writing
\[ A_n^{(1)}(x) = \int_0^\pi u^{\alpha-1} g(n, u) \, du \] ...

it follows that
\[ A_n^{(1)}(x) = \frac{1}{\Gamma(1 - \alpha)} \int_0^\pi K(n, t) \, dt \int_0^t u^{\alpha-1} (t - u)^{-\alpha} \, du \]
\[ = \Gamma(\alpha) \int_0^\pi K(n, t) \, dt \]
\[ = O\left(\frac{1}{n}\right). \] ...

Thus following Lemma 1, it is enough to show the uniform boundedness in 0 < t < \pi of the following:

\[ \sum_{n \geq 0} \left| \sum_{m=1}^{n+1} \nu_m^\varphi (n + 1) m^{1-(\alpha/2)} y_m \int_0^t u^\alpha \frac{d}{du} g(m, u) \, du \right|. \]

Now, we first consider those values of \( n \) for which \( n \leq T \). Integrating by parts

\[ \int_0^t u^\alpha \frac{d}{du} g(m, u) \, du = t^\alpha g(m, t) - \alpha \int_0^t u^{\alpha-1} g(m, u) \, du \] ...

since \( g(n, u) = O(n^{\alpha-1}) \). Therefore

\[ \sum_{n > T} = O(t^\alpha) \sum_{n > T} \int_0^{n+1} \int_0^t u^\alpha \frac{d}{du} g(m, u) \, du \]
\[ = O\left(t^\alpha \sum_{n > T} (n + 1)^{(\alpha/2)-1} y_{n+1}\right) \]
\[ = O(1) \]
uniformly in $0 < t < \pi$. And, following (6.1) and (6.3),

$$
\int_0^t u^n \frac{d}{du} g(m, u) \, du = t^n g(m, t) - aA_m^{(1)}(x) + a \int_t^{\pi} u^{n-1} g(n, u) \, du
$$

Therefore, following (6.2), (2.9) and Lemma 1, it is enough, for $n > T$, to show that

$$
\sum_{n>T} (n+1)^{-1} \left\{ \sum_{m=1}^{n+1} y_m^n (n+1) m^{1-(n/2)} y_m \{t^n g(m, t) \right. \\
\left. + a \int_t^{\pi} u^{n-1} g(m, u) \, du \} \right\}
$$

$$
= O(1), \text{ uniformly in } 0 < t < \pi.
$$

However by Lemmas 5 and 2,

$$
\sum_{m=1}^{n+1} y_m^n (n+1) m^{1-(n/2)} y_m \{t^n g(m, t) + a \int_t^{\pi} u^{n-1} g(m, u) \, du
$$

$$
= O(t^{n-1}) \sum_{m=1}^{n+1} y_m^n (n+1) m^{(n/2)-1} y_m + O(t^{n-1}) y_{n+1}(n+1)^{(n-1)/2}
$$

$$
= O(t^{n-1}(n+1)^{(n-1)/2}).
$$

Therefore

$$
\sum_{n>T} (n+1)^{(n-1)/2-1}
$$

$$
= O(1)
$$

uniformly in $0 < t < \pi$.

This completes the proof of the theorem.

**References**


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