A NOTE ON PARTIAL ISOMETRIES—II

B. C. GUPTA

Department of Mathematics, Sardar Patel University,
Vallabh Vidyanagar 388120, Gujarat

(Received 3 March 1979)

A sufficient condition is obtained for square of an operator to be a partial isometry. Also, several conditions are discussed which imply that an operator is the direct sum of an isometry and zero.

A bounded linear operator $T$ on a Hilbert space $H$ is called a partial isometry if $T$ is isometric on the orthogonal complement of its kernel $N(T)$. The operators whose first $N$ powers and all powers are partial isometries have been characterized respectively by Guyker (1976) and Halmos and Wallen (1970). In this note we establish a condition for square of an operator to be a partial isometry and we also discuss conditions which imply that an operator is the direct sum of an isometry and zero.

An operator $T$ is said to be $n$-paranormal if $\|Tx\|^n \leq \|T^n x\|$ for every unit vector $x$ in $H$, $n$ being a positive integer; and $T$ is said to be binormal if $T^*T$ commutes with $TT^*$ or equivalently if $TT^{**}T \geq 0$ (Campbell 1972, 1974).

Gupta (1979) proved the following result.

Theorem A — Let $T$ be a contraction and let $N(T) = N(T^2)$. If $T^k$ is a partial isometry for some $k \geq 2$, then $T$ is a partial isometry.

From this theorem it was deduced that if $T^k$ is a partial isometry then $T$ is the direct sum of an isometry and zero if any one of the following holds:

(i) $T$ is $n$-paranormal

(ii) $T^*T$ commutes with $T^* + T$.

The following result implicitly contained in Lemma 2 of Halmos and Wallen (1970) gives a characterization of binormal partial isometries.

Theorem B — Suppose $T$ is a partial isometry. Then $T^*$ is a partial isometry if and only if $T$ is binormal.

In fact in one direction, we have a stronger result:

Theorem C (Guyker 1976) — If $T$ is a contraction and $T^2$ is a partial isometry then $TT^{**}T$ is a projection.
In the other direction we have the following interesting result.

**Theorem 1** — If $TT^{*}T$ is a projection, then $T^2$ is a partial isometry.

**Proof:** Let $T = UP$ and $T^* = VQ$ be polar decompositions of $T$ and $T^*$, where $U$ and $V$ are partial isometries with their initial spaces $\overline{R(T^*)}$ and $\overline{R(T)}$ respectively.

If $TT^{*}T = S$ then $S$ is a projection and

$$(PQ)^2 = P^2Q^2 = T^*T^2T^* = S$$

so that

$$PQ = S.$$ 

Therefore

$$T^2 = (UP)(VQ)^* = USV^*.$$ 

Since $R(S) \subset \overline{R(T)} \cap \overline{R(T^*)}$, $U$ and $V$ are both isometric on $R(S)$. So

$$SU^*US = S$$

and

$$SV^*VS = S.$$ 

Hence

$$T^2T^{*}T^2 = USV^*VSU^*USV^* = USV^* = T^2$$

and it follows that $T^2$ is a partial isometry.

**Corollary 1** — (i) $TT^{*}T = 0$ if and only if $T^2 = 0$.

(ii) $TT^{*}T = I$ if and only if $T^2$ is unitary.

**Proof:** (i) is obvious.

(ii) If $T^2$ is unitary then $T^{*} = T^{-1}$ and so $TT^{*}T = I$.

Conversely, if $TT^{*}T = I$ then $T^2x = 0$ implies that

$$Tx = TT^{*}T^2x = 0.$$ 

So

$$x = TT^{*}Tx = 0.$$ 

Thus $N(T^2) = \{0\}$.

Similarly $N(T^{*2}) = \{0\}$ and the result follows from Theorem 1.

**Corollary 2** — Let $TT^{*}T$ be a projection. If $T$ is invertible and

$$\sigma(T) \cap \sigma(-T) = \emptyset$$

then $T$ is unitary, $\sigma(T)$ being the spectrum of $T$.  

PROOF: By Corollary 1(ii) $T^2$ is unitary and since $\sigma(T) \cap \sigma(-T) = \emptyset$ the result follows from Embry (1968).

**Corollary 3** — If $TT^{*2}T$ is a projection and $N(T) = N(T^2)$, then $T^2$ is the direct sum of an isometry and zero.

PROOF: Since $TT^{*2}T = T^*T^2T^*$ and $N(T) = N(T^2)$, $Tx = 0$ implies $T^2T^*x = 0$ and so $T^*x = 0$. Therefore $N(T) (= N(T^2))$ reduces $T$. So $T^2$, being a partial isometry by Theorem 1, is the direct sum of an isometry and zero.

**Corollary 4** — If $T$ is a contraction operator with $N(T) = N(T^2)$ and $TT^{*2}T$ is a projection, then $T$ is the direct sum of an isometry and zero.

PROOF: As in Corollary 3, $N(T)$ is reducing for $T$. Since $T^2$ is a partial isometry the result follows from Theorem A.

By the remark following Theorem A, we have the following.

**Corollary 5** — If $T$ is $n$-paranormal and $TT^{*2}T$ is a projection, then $T$ is the direct sum of an isometry and zero.

**Example**: The condition that $T^2$ is a partial isometry is not sufficient for $TT^{*2}T$ to become a projection. This can be seen by taking

$$T = \begin{pmatrix} \alpha^{1/2} & \alpha^{-1/2}(1 - \alpha^2)^{1/2} \\ 0 & 0 \end{pmatrix}$$

where $0 < \alpha < 1$.

It has been shown by Gupta (1979) that if $T$ and $T^2$ both are partial isometries with the same kernel, then $T$ is the direct sum of an isometry and zero.

We improve this result in the following.

**Theorem 2** — If $T$ and $T^{k+1}$ ($k \geq 1$) are partial isometries and $N(T) = N(T^2)$ then $T$ is the direct sum of an isometry and zero.

PROOF: Since $S = T^*T^{k+1}T^{**}$ is contraction and idempotent, $S$ is self-adjoint. It follows from this that $T^*T$ commutes with $T^{k}T^{**}$. Therefore $N(T)$ reduces $T^kT^{**}$.

But $N(T) = N(T^2)$ implies $N(T) = N(T^n)$ for every positive integer $n$. Therefore if $x \in N(T)$ then $T^{*2}x \in N(T)$ and thus $N(T)$ reduces $T^k$.

Since $T$ and $T^{k+1}$ are isometric on $N(T)^\perp$ which reduces $T^k$, it follows that $T^k$ is isometric on $N(T)^\perp$. But $N(T)^\perp = N(T^*)^\perp$ and so $T^k$ is a partial isometry. Repeating the same argument we see that $T^2$ is a partial isometry and $N(T)$ reduces $T$. Therefore, $T$ is the direct sum of an isometry and zero.
REFERENCES


