ON NÖRLUND SUMMABILITY OF FOURIER SERIES AND ITS CONJUGATE SERIES

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In this paper the authors have obtained two theorems for the Nörlund summability of Fourier series and its conjugate series respectively under very general conditions.

§1. Let \( \sum_{n=0}^{\infty} a_n \) be a given infinite series with the sequence of partial sums \( \{S_n\} \).

Let \( \{p_n\} \) be a sequence with \( p_0 > 0 \) and \( p_n \geq 0 \) for \( n > 0 \), and let

\[
P_n = p_0 + p_1 + p_2 + \ldots + p_n, \quad (P_{-1} = p_{-1} = 0).
\]

Let

\[
t_n = \sum_{v=0}^{n} \frac{p_{n-v} S_v}{P_n}, \quad (P_n \neq 0)
\]

or

\[
t_n = \sum_{v=0}^{n} \frac{p_{n-v} S_v}{P_n}, \quad \ldots(1.1)
\]

If \( t_n \to S \) as \( n \to \infty \) we write

\[
\sum_{n=0}^{\infty} a_n = S(N, p_n)
\]

or

\[
S_n \to S(N, p_n).
\]

The conditions of regularity of the method of summability \( (N, p_n) \) defined by (1.1) is

\[
\lim_{n \to \infty} \frac{p_n}{P_n} = 0. \quad \ldots(1.2)
\]

§2. Let \( f(t) \) be \( 2\pi \)-periodic function and \( L \)-integrable over an interval \( (-\pi, \pi) \). Let the Fourier series of \( f(t) \) be given by

\[
\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t) \quad \ldots(2.1)
\]
and then the conjugate series of (2.1) is

\[ \sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) \equiv \sum_{n=1}^{\infty} B_n(t). \] ...

(2.2)

We shall use the following notations:

\[ \phi(t) = \phi(x, t) = f(x + t) + f(x - t) - 2f(x) \]

\[ \psi(t) = \psi(x, t) = f(x + t) - f(x - t) \]

\[ \Phi(t) = \int_{0}^{t} |\phi(u)| \, du \]

\[ \Psi(t) = \int_{0}^{t} |\psi(u)| \, du \]

\[ N_n(t) = \frac{1}{2\pi P_n} \sum_{\nu=0}^{n} P_{\nu} \frac{\sin (n - \nu + \frac{1}{2}) t}{\sin \frac{1}{2} t} \]

\[ \overline{N}_n(t) = \frac{1}{2\pi P_n} \sum_{\nu=0}^{n} P_{\nu} \frac{\cos (n - \nu + \frac{1}{2}) t}{\sin \frac{1}{2} t} \]

\[ p_{(1/t)} = p_\tau \quad \text{and} \quad P_{(1/t)} = P_\tau \]

where \( \tau \) denotes the integral part of \( 1/t \).

§3. Siddiqi (1948) proved the following theorems:

**Theorem A** — If

\[ \Phi(t) = \int_{0}^{t} |\phi(u)| \, du = o \left[ \frac{t}{\log \left( \frac{1}{t} \right)} \right] \]

...(3.1)

as \( t \to +0 \), then the series (2.1), at \( t = x \), is summable \((H)\) to \( f(x) \).

He also proved the corresponding theorem for the conjugate series (2.2). His theorem concerning the conjugate series is

**Theorem B** — If

\[ \Psi(t) = \int_{0}^{t} |\psi(u)| \, du = o \left[ \frac{t}{\log \left( \frac{1}{t} \right)} \right] \]

...(3.2)
as \( t \to +0 \), then the conjugate series (2.2) is summable \((H)\) to

\[
\frac{1}{2\pi} \int_{0}^{\pi} \psi(t) \cot \frac{1}{4} t \, dt,
\]

at points where this integral exists.

In this direction, Pati (1961) has proved the following theorem.

**Theorem C** — If \((N, p_n)\) be a regular Nörlund method, defined by a real, non-negative, monotonic non-increasing sequence of coefficients \(\{p_n\}\), such that \(P_n \to \infty\), and \(\log n = O(P_n)\), as \(n \to \infty\), then, if

\[
\Phi(t) = \int_{0}^{t} |\phi(u)| \, du = o \left[ t/P_\lambda \right]
\]

as \( t \to +0 \), the Fourier series of \(f(t)\), at \(t = x\), is summable \((N, p_n)\) to \(f(x)\).

The object of the present paper is to obtain, for the Nörlund summability of Fourier series and its conjugate series a criterion of a different type replacing the conditions (3.1) and (3.2) by more general conditions. We prove the following theorems.

§4. **Theorem 1** — Let \(\lambda(t)\) and \(K(t)\) be two positive functions. If

\[
\Phi(t) = \int_{0}^{t} |\phi(u)| \, du = o \left[ \lambda(1/t) P_\lambda/K(P_\lambda) \right], \quad \text{as} \quad t \to 0
\]

and

\[
\lambda(n) P_n = O \left[ K(P_n) \right]
\]

as \( n \to \infty \), then Fourier series of \(f(t)\) at \(t = x\) is summable \((N, p_n)\) to \(f(x)\) where \(\{p_n\}\) is a real non-negative and non-increasing sequence such that \(P_n \to \infty\), as \(n \to \infty\).

**Theorem 2** — Let the sequence \(\{p_n\}\) and the functions \(\lambda(t)\) and \(K(t)\) be the same as in Theorem 1. Then if,

\[
\Psi(t) = \int_{0}^{t} |\psi(u)| \, du = o \left[ \lambda(1/t) P_\lambda/K(P_\lambda) \right]
\]

as \( t \to +0 \), then the conjugate series (2.2) is summable \((N, p_n)\) to

\[
\frac{1}{2\pi} \int_{0}^{\pi} \psi(t) \cot \frac{1}{4} t \, dt
\]

at every point where this integral exists. For the proof of our theorems we need the following lemmas.
Lemma 1 (McFadden 1942) — If \( \{p_n\} \) is non-negative and non-increasing, then for \( 0 \leq a < b \leq \infty \), \( 0 \leq t \leq \pi \), and any \( n \)

\[
\left| \sum_{v=a}^{b} p_v e^{i(n-v)t} \right| \leq A P \pi,
\]

where \( A \) is an absolute constant.

Lemma 2 — If \( 0 \leq t \leq \frac{1}{n} \), then

\[
N_n(t) = O(n),
\]

we have

\[
| N_n(t) | = \frac{1}{2\pi P_n} \left| \sum_{v=0}^{n} p_v \frac{\sin (n - v + \frac{1}{2}) t}{\sin \frac{1}{2} t} \right|
\]

\[
= O \left\{ P_n^{-1} \sum_{v=0}^{n} p_v \frac{(2n - 2v + 1) | \sin \frac{1}{2} t |}{| \sin \frac{1}{2} t |} \right\}
\]

\[
= O \left\{ (2n + 1) P_n^{-1} \sum_{v=0}^{n} p_v \right\}
\]

\[
= O(n), \quad \text{as} \quad n \to \infty.
\]

Lemma 3 — For \( \frac{1}{n} \leq t \leq \delta < \pi \),

\[
N_n(t) = O [P \pi / P_n t].
\]

Proof: We have

\[
| N_n(t) | = \frac{1}{2\pi P_n} \left| \sum_{v=0}^{n} p_v \frac{\sin (n - v + \frac{1}{2}) t}{\sin \frac{1}{2} t} \right|
\]

\[
= \frac{1}{2\pi P_n | \sin \frac{1}{2} t |} \left| \Im \sum_{v=0}^{n} p_v \exp (i(n - v + \frac{1}{2}) t) \right|
\]

\[
= \frac{1}{2\pi P_n | \sin \frac{1}{2} t |} \left| \Im \left( e^{it/2} \sum_{v=0}^{n} p_v \exp (i(n - v) t) \right) \right|
\]

\[
\leq \frac{1}{2\pi P_n t} \left| \sum_{v=0}^{n} p_v \exp (i(n - v) t) \right|
\]

\[
= O [P \pi / P_n t], \quad \text{by Lemma 1}.
\]
Lemma 4 — If \( \frac{1}{n} \leq t \leq \delta < \pi \), then

\[
\bar{N}_n(t) = \frac{1}{2\pi P_n} \sum_{\nu=0}^{n} p_{\nu} \frac{\cos(n - \nu + \frac{1}{2}) t}{\sin \frac{1}{2} t}
\]

\[
= O \left[ P_{\nu}/P_n t \right].
\]

The proof is similar to that of Lemma 3.

§5. Proof of the Theorem 1 — Let

\[
S_n(x) = \sum_{\nu=1}^{n} A_{\nu}(x)
\]

then, we have

\[
S_n(x) - f(x) = \frac{1}{2\pi} \int_{0}^{\pi} \phi(t) \frac{\sin(n + \frac{1}{2}) t}{\sin \frac{1}{2} t} dt.
\]

Using (1.1), we get

\[
t_n - f(x) = P_n^{-1} \sum_{\nu=0}^{n} p_{\nu} [S_{n-\nu}(x) - f(x)]
\]

\[
= P_n^{-1} \sum_{\nu=0}^{n} p_{\nu} \frac{1}{2\pi} \int_{0}^{\pi} \phi(t) \frac{\sin(n - \nu + \frac{1}{2}) t}{\sin \frac{1}{2} t} dt
\]

\[
= \int_{0}^{\pi} \phi(t) \left\{ \frac{1}{2\pi P_n} \sum_{\nu=0}^{n} p_{\nu} \frac{\sin(n - \nu + \frac{1}{2}) t}{\sin \frac{1}{2} t} \right\} dt
\]

\[
= \int_{0}^{\pi} \phi(t) N_n(t) dt \text{ (} = M \text{ say).}
\]

In order to prove the theorem, we have to show that, under our assumptions,

\[
\int_{0}^{\pi} \phi(t) N_n(t) dt = o(1), \text{ as } n \to \infty.
\]

We write, for \( 0 < \delta < \pi \)

\[
\int_{0}^{\pi} \phi(t) N_n(t) dt = [ \int_{0}^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi} ] \phi(t) N_n(t) dt.
\]

\[
= M_1 + M_2 + M_3, \text{ say.} \quad \ldots(5.1)
\]
Now, by Lemma 2,
\[ M_1 = O \left[ n \int_0^{1/n} | \phi(t) | \, dt \right] \]
\[ = o \left[ n \lambda(n) p_n / K(P_n) \right]. \]

By assumption that \( \{ p_n \} \) is non-negative, monotonic non-increasing, we have obviously \( (n + 1) p_n \ll P_n \), therefore
\[ M_1 = o \left[ \lambda(n) P_n / K(P_n) \right] \]
\[ M_1 = o(1), \quad \text{as} \quad n \rightarrow \infty. \] \( \ldots(5.2) \)

Again by Lemma 3,
\[ M_2 = \int_{1/n}^{\delta} \phi(t) N_n(t) \, dt \]
\[ = O \left[ P_n^{-1} \int_{1/n}^{\delta} | \Phi(t) | \frac{P_t}{t} \, dt \right] \]
\[ = O \left[ P_n^{-1} \left( \Phi(t) \frac{P_t}{t} \right)_{1/n}^{\delta} \right] \]
\[ + O \left[ \int_{1/n}^{\delta} P_n^{-1} \Phi(t) \frac{P_t}{t^2} \, dt \right] + O \left[ \int_{1/n}^{\delta} P_n^{-1} \Phi(t) \frac{1}{t} \, dP_t \right] \]
\[ = M_{2,1} + M_{2,2} + M_{2,3}, \quad \text{say.} \]

Now
\[ M_{2,1} = O \left[ P_n^{-1} \left( \Phi(t) \frac{P_t}{t} \right)_{1/n}^{\delta} \right] \]
\[ = O \left[ P_n^{-1} \right] + o \left[ \frac{\lambda(n) p_n}{K(P_n)} n P_n \right] \]
\[ M_{2,2} = o(1), \quad \text{as} \quad n \rightarrow \infty. \] \( \ldots(5.3) \)

\[ M_{2,2} = O \left[ P_n^{-1} \int_{1/n}^{\delta} \Phi(t) \frac{P_t}{t^2} \, dt \right] \]
\[ = o(1) + P_n^{-1} \sum_{m=1}^{n-1} \int_m^{m+1} \Phi(1/v) P_{[v]} \, dv \]
\[ \int \Phi(1/v) P_{[v]} \, dv \leq \Phi(1/m) \, P_m \]
\[ = o \left( \frac{\lambda(m) \, p_m}{K(P_m)} \right) \]
\[ = o \left[ P_m \right], \quad \text{as} \quad m \to \infty \]

so
\[ M_{2.3} = o(1) + o \left[ P_n^{-1} \, \sum_{m=1}^{n-1} P_m \right] \]

\[ M_{2.3} = o(1) \] ... (5.4)

and finally by the hypothesis of the theorem, we have

\[ M_{2.3} = \int_{1/n}^{\infty} \Phi(t) \frac{1}{t} \, dP_t \]
\[ = P_n^{-1} \int_{1/n}^{\infty} \Phi(1/v) \, v \, dP_{[v]} \]
\[ = o(1) + O \left( P_n^{-1} \sum_{m=1}^{n-1} m p_m \Phi(1/m) \right) \]
\[ = o(1) + o \left\{ P_n^{-1} \sum_{m=1}^{n-1} P_m \Phi(1/m) \right\} \]
\[ = o(1) + o \left[ P_n^{-1} \sum_{m=1}^{n-1} P_m \frac{\lambda(m) \, p_m}{K(P_m)} \right] \]

\[ M_{2.3} = o(1), \quad \text{as} \quad n \to \infty. \] ... (5.5)

Collecting (5.3), (5.4) and (5.5) we get

\[ M_2 = o(1). \] ... (5.6)

Lastly, by virtue of Riemann-Lebesgue theorem and regularity of the method of summation, we have

\[ M_3 = \int_{\frac{1}{2}}^{\pi} \phi(t) \, N_\pi(t) \, dt \]
\[ = o \left[ P_n^{-1} \sum_{v=0}^{n} p_v \right] \]

\[ M_3 = o(1), \quad n \to \infty. \] ... (5.7)
Hence on collecting (5.2), (5.6) and (5.7), we get

\[ M = o(1) \]

which completes the proof of Theorem 1.

§6. Proof of the Theorem 2 — Let \( \tilde{S}_n(x) \) denote the nth partial sum of the series \( \Sigma B_n(x) \). Then we have

\[ \tilde{S}_n(x) = \frac{1}{2\pi} \int_0^\pi \psi(t) \frac{\cos \frac{1}{2} t - \cos (n + \frac{1}{2}) t}{\sin \frac{1}{2} t} \, dt. \]

For \( \Sigma B_n(x) \), making use of the formula (1.1), we get

\[ \tilde{r}_n - \frac{1}{2\pi} \int_0^\pi \psi(t) \cot \frac{1}{2} t \, dt = P_n^{-1} \sum_{\nu=0}^n p_\nu \tilde{S}_{n-\nu}(x) \]

\[ - \frac{1}{2\pi} \int_0^\pi \psi(t) \cot \frac{1}{2} t \, dt \]

\[ = P_n^{-1} \sum_{\nu=0}^n p_\nu \frac{1}{2\pi} \int_0^\pi \psi(t) \frac{\cos \frac{1}{2} t - \cos (n - \nu + \frac{1}{2}) t}{\sin \frac{1}{2} t} \, dt \]

\[ - \frac{1}{2\pi} \int_0^\pi \psi(t) \cot \frac{1}{2} t \, dt \]

\[ = - \int_0^\pi \psi(t) \left\{ \frac{1}{2\pi P_n} \sum_{\nu=0}^n p_\nu \frac{\cos (n - \nu + \frac{1}{2}) t}{\sin \frac{1}{2} t} \right\} \, dt \]

\[ = - \int_0^\pi \psi(t) \tilde{N}_n(t) \, dt \ (= R \text{ say}). \]

In order to prove the theorem, we have to show that, under our assumptions

\[ \int_0^\pi \psi(t) \tilde{N}_n(t) \, dt = o(1), \]

as \( n \to \infty \).

For \( 0 < \delta < \pi \), we have

\[ \int_0^\pi \psi(t) \tilde{N}_n(t) \, dt = \left[ \int_0^{1/n} + \int_{1/n}^\delta + \int_0^\pi \right] \psi(t) \tilde{N}_n(t) \, dt \]

\[ R = R_1 + R_2 + R_3, \text{ say.} \]  

...(6.1)
Since the conjugate function exists, therefore

\[ \frac{1}{2\pi} \int_0^{1/n} \psi(t) \cot \frac{1}{2} t \, dt = o(1). \]

Also

\[ \frac{1}{2\pi P_n} \sum_{\nu=0}^{n} P_{\nu} \frac{\cos \frac{1}{2} t - \cos (n - \nu + \frac{1}{2}) t}{\sin \frac{1}{2} t} \]

\[ = \frac{1}{2\pi P_n} \sum_{\nu=0}^{n} P_{\nu} \sum_{k=0}^{n-\nu} 2 \sin kt \]

\[ = O \left[ \sum_{\nu=0}^{n} P_{\nu} \sum_{k=0}^{n-\nu} \left| \sin kt \right| \right] \]

\[ = O \left[ \sum_{\nu=0}^{n} P_{\nu} (n - \nu) \right] \]

\[ = O(n), \quad \text{for} \quad 0 \leq t \leq \pi. \]

Therefore

\[ R_1 = \frac{1}{n} \int_0^{1/n} \psi(t) \, d\bar{N}_n(t) \, dt \]

\[ = \int_0^{1/n} \frac{\psi(t)}{2\pi P_n} \sum_{\nu=0}^{n} P_{\nu} \frac{\cos (n - \nu + \frac{1}{2}) t}{\sin \frac{1}{2} t} \, dt \]

\[ = - \int_0^{1/n} \frac{\psi(t)}{2\pi P_n} \sum_{\nu=0}^{n} P_{\nu} \frac{\cos \frac{1}{2} t - \cos (n - \nu + \frac{1}{2}) t}{\sin \frac{1}{2} t} \, dt \]

\[ + \frac{1}{2\pi P_n} \sum_{\nu=0}^{n} P_{\nu} \int_0^{1/n} \psi(t) \, \cot \frac{1}{2} t \, dt \]

\[ = O(n \int_0^{1/n} |\psi(t)| \, dt) + o(1) \]

\[ = O[n \Psi(1/n)] + o(1) \]

\[ = o[n \lambda(n) p_n/K(P_n)] \]

\[ = o[\lambda(n) P_n/K(P_n)] \]

\[ R_1 = o(1), \quad \text{since} \quad np_n \leq P_n. \]  

...(6.2)
Now, for $1/n \leq t \leq \delta$

$$R_2 = O\left[ \int_{1/n}^{\delta} |\psi(t)| \, |\bar{N}_n(t)| \, dt \right]$$

$$= O\left[ \int_{1/n}^{\delta} |\psi(t)| \, \frac{P_x}{P_n} \, dt \right]$$

$$R_2 = o(1), \text{ as in } M_2. \quad \ldots(6.3)$$

Also

$$R_2 = o(1) \quad \ldots(6.4)$$

by virtue of Riemann-Lebesgue theorem and the regularity of the method of summation.

Hence on collecting (6.2), (6.3) and (6.4), we get

$$R = o(1)$$

which completes the proof of Theorem 2.

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REFERENCES

