THE FREDHOLM INDEX OF A CLASS OF VECTOR-VALUED
SINGULAR INTEGRAL OPERATORS

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In this paper the operators of interest are the singular integral operators $T$
defined on $L^2_n(E)$ by

$$Tf(s) = sf(s) + \frac{B^*(s)}{\pi} \int_{E} \frac{B(t)f(t)}{s - t} \, dt.$$ 

In particular it is shown that a real number $x$ is in the essential resolvent of $T$
if and only if there is a neighbourhood $\Delta$ of $x$ such that the Lebesgue measure
of $\Delta - E$ equals zero, and $\inf_{t \in \Delta} |\det B(t)| > 0$. Moreover, in this case
the index of $T - x$ is $-n$.

INTRODUCTION

Let $E$ be a bounded measurable subset of the real line $R$, and $B \in L^\infty_{M_n}(E)$,
where the space $L^\infty_{M_n}(E)$ is the set of all $n \times n$ matrices $(\phi_{ij})$ ($1 \leq i, j \leq n$),
such that each of the functions $\phi_{ij} \in L^\infty(E)$. The notation $L^2_n(E)$ will denote the usual Lebesgue
space of $\mathbb{C}^n$ valued square integrable functions on $E$. The operators of interest are the
singular integral operators $T$ defined on $L^2_n(E)$ by

$$Tf(s) = sf(s) + \frac{B^*(s)}{\pi} \int_{E} \frac{B(t)f(t)}{s - t} \, dt.$$ 

The singular integral operator $T$ is hyponormal on $L^2_n(E)$, that is, the self-commutator
$[T^*, T] = T^*T - TT^*$ of $T$ is a non-negative operator, moreover $[T^*, T]$ is $n$-dimen-
sional. It should be remarked that a complete description of the Fredholm behaviour
of the operator $T - z$, for $z$ complex, has been given for the case $n = 1$ (see Clancey
1974a). In this paper a characterization is given for the Fredholm behaviour of
$T - x$, where $x \in R$ and $n > 1$.

Section 1 is concerned with some technical machinery needed for this paper. In
section 2 we prove that $T - x$ is Fredholm if and only if there exists a neighbour-
hood $\Delta$ of $x$ such that the Lebesgue measure of $\Delta - E$ equals zero. In section 3,
the definition of the principal function and some of its properties are given. In
section 4, we prove that index of \( T - x = -n \).

1. Preliminaries

Let \( z \) be any non-real complex number and let the function \( f \) be an element
of \( L^2_n(R) \). The Cauchy transform of \( f \), denoted by \( Cf \), is equal to \((Cf_j)_{j=1}^n \) where

\[
Cf_j(z) = \frac{1}{2\pi i} \int_R \frac{f(t)}{t - z} \, dt, \text{ for } 1 \leq j \leq n. \quad \ldots(1.1)
\]

Each function \( Cf_j (1 \leq j \leq n) \) is separately holomorphic in the upper half-plane and
lower half-plane. Moreover, the functions \( f_j^\pm \) defined by \( f_j^\pm(x) = \lim_{y \to 0^\pm} Cf_j(x \pm iy) \)
exist almost everywhere. The function \( f_j^+ (1 \leq j \leq n) \) is in \( H^2 \) and \( f_j^- \) is in \( \overline{H^2} \), where
the space \( H^2 \) is the usual Hardy space, and \( \overline{H^2} \) is its complex conjugate.

For \( f \in L^2_n(R) \), write \( f^\pm \) for \((f_j^\pm)_{j=1}^n \). The functions \( f^\pm \) satisfy the Plemelj
identities

\[
f^+ - f^- = f, \quad f^+ + f^- = (1/i) Hf. \quad \ldots(1.2)
\]

Here, \( Hf = (Hf_j)_{j=1}^n \) is the Hilbert transform of \( f \). The map \( f \to Hf \) is a bounded
linear operator on \( L^2_n(R) \) and this implies that the maps \( f \to f^\pm \) are bounded in
\( L^2_n(R) \). For the boundedness of the Hilbert transform see Neri (1971).

The Riemann-Hilbert barrier operator with symbol \( G \in L^\infty_{M_n}(R) \) is the bounded
linear operator \( B_G \) defined on the space \( L^2_n(R) \) by

\[
B_G f = Gf^+ - f^- \quad \ldots(1.3)
\]

The Toeplitz operator with symbol \( G \) is the bounded operator \( T_G \) on \( H^2_n \) defined by

\[
T_G f = P(Gf). \quad \ldots(1.4)
\]

Here \( P \) stands for the orthogonal projection of \( L^2_n(R) \) onto \( H^2_n \). The operator \( P \) is
given by

\[
Pf = \frac{1}{i}(f - iHf). \quad \ldots(1.5)
\]

It is known that \( T_G \) is Fredholm if and only if \( B_G \) is Fredholm. Moreover, they have
the same Fredholm index.
Let $H$ be a Hilbert space and let $L(H)$ denote the algebra of all bounded operators on $L(H)$. An operator $T$ in $L(H)$ is called Fredholm in case $T$ has closed range, and $\dim (\ker T), \dim (\ker T^*)$ are finite. The index of a Fredholm operator $T$ is defined by

$$\text{ind} (T) = \dim (\ker T) - \dim (\ker T^*). \quad \ldots (1.6)$$

The essential spectrum of $T$, denoted by $\sigma_e(T)$, is the set of all $\lambda$ in the field of complex numbers $\mathbb{C}$ such that $T - \lambda$ is not Fredholm. The essential resolvent of $T$, denoted by $\rho_e(T)$, is the set of all $\lambda \in \mathbb{C}$, such that $\lambda \not\in \sigma_e(T)$.

Let $G \in L_{M_n}^\infty (R)$. For $\lambda$ in $R$ the cluster set of $G$ at $\lambda$, denoted by $\mathcal{C}(G : \lambda)$, is the set of all $n \times n$ matrices $M$ such that the set

$$\{t \in R : \|G(t) - M\| < \epsilon\} \cap N$$

has positive measure for every $\epsilon > 0$ and every neighbourhood $N$ of $\lambda$.

2. The Essential Spectrum

Let $E$ be a bounded measurable subset of the real line $R$, and $B \in L_{M_n}^\infty (E)$ such that

$$\text{ess inf} \inf_{t \in E} |\det B(t)| > 0. \quad \ldots (2.1)$$

The operators of interest are the singular integral operators $T$ defined on $L_n^2 (E)$ by

$$Tf(s) = sf(s) + \frac{B^*(s)}{\pi} \int_E \frac{B(t)f(t)}{s - t} \, dt. \quad \ldots (2.2)$$

Since $(T^*T - TT^*)f(s) = \frac{2}{\pi} B^*(s) \int_E B(t)f(t) \, dt$, it follows that the operator $T$ is hyponormal with $n$-dimensional self-commutator.

Note that with the hypothesis (2.1), the operator $T$ defined in (2.2) is similar to the operator $S$ defined on $L_n^2 (E)$ by

$$Sf(s) = sf(s) + \frac{B(s)B^*(s)}{\pi} \int_E \frac{f(t)}{s - t} \, dt. \quad \ldots (2.3)$$

Let $x$ be a real number. From (2.1) we can say that

$$\text{ess inf} \inf_{s \in E} \| (s - x) I + i(B(s)B^*(s)) \| > 0. \quad \ldots (2.4)$$
The symbol of the operator $S - x$ is the function

$$G_x(s) = \begin{cases} \frac{(s - x) I - i BB^*(s)}{(s - x) I + i BB^*(s)}, & s \in E \\ I, & s \notin E \end{cases} \quad \ldots(2.5)$$

where $I$ is the $n \times n$ identity matrix.

Before proving the main theorem of this section some technical lemmas are needed.

**Lemma 2.1** — If $G \in L^\infty_{M_n}(R)$ and for some $\epsilon > 0$, $\text{Re} \ G(t) \geq \epsilon I$ almost everywhere, then the Riemann-Hilbert barrier operator $B_G$ is invertible.

**Proof:** For proof of the lemma see Clancey (1974b).

There are localization techniques due to Simonenko that will be useful in establishing when certain Riemann-Hilbert barrier operators are Fredholm.

Let $G_1$ and $G_2$ be symbols of the Riemann-Hilbert barrier operators $B_{G_1}$ and $B_{G_2}$, respectively. Let $x_0$ be a fixed real number. The operators $B_{G_1}$ and $B_{G_2}$ are said to be locally equal at $x_0$ in case there is a neighbourhood $U$ of $x_0$ such that $G_1(t) = G_2(t)$, for all $t \in U$. The operator $B_{G_1}$ is said to be locally Fredholm at $x_0 \in R$ in case $B_{G_1}$ is locally equal to a Fredholm barrier operator $B_{G_2}$ at $x_0$.

The following lemma is a special case of a result of Simonenko (1964).

**Lemma 2.2** — Let $G \in L^\infty_{M_n}(R)$. If the Riemann-Hilbert barrier operator $B_G$, acting on $L^2_n(R)$, is locally Fredholm at each $x \in R$, then $B_G$ is a Fredholm operator.

The Riemann-Hilbert barrier operators $B_{G_1}$ and $B_{G_2}$ are said to be locally equivalent at $x_0 \in R$, if for every $\epsilon > 0$, there exists a neighbourhood of the point $x_0$ such that

$$\| (B_{G_1} - B_{G_2}) P_U \| = \inf \| (B_{G_1} - B_{G_2}) P_U - K \| < \epsilon \quad \ldots(2.6)$$

where $K$ runs through the ideal of compact operators on $L^2_n(R)$, and the operator $P_U$ is defined on $L^2_n(R)$ by

$$P_U f(x) = \chi_U f(x) \quad \ldots(2.7)$$

where $\chi$ is the characteristic function of $U$. The following lemma also appears in Simonenko (1964).
Lemma 2.3 — Let $G_1$ and $G_2$ be in $L_{\mathcal{M}_a}^\infty (R)$ and suppose $B_{G_1}$ and $B_{G_2}$ are locally equivalent at $x_0$ in $R$. Then $B_{G_1}$ is locally Fredholm at $x_0$ if and only if $B_{G_2}$ is locally Fredholm at $x_0$.

The main theorem of this section is the following:

**Theorem 2.1** — If $x$ is a real number, then $x$ is an element of the essential resolvent of $S$ if and only if there exists a neighbourhood $\Delta$ of $x$ such that the Lebesgue measure of $\Delta - E$ equals zero.

**Proof:** Let $G_x$ be the symbol of the operator $S - x$ defined by (2.5). Since it is known that $S - x$ is Fredholm if and only if the Riemann-Hilbert barrier operator $B_{G_x}$ is Fredholm, it suffices to prove Theorem 2.1 for the operator $B_{G_x}$.

Suppose that there is no neighbourhood $\Delta$ of $x$ such that the Lebesgue measure of $\Delta - E$ equals zero. Then it follows that the cluster set $C(G_x : x)$ of $G_x$ at $x$ is $\{I, -I\}$. Since there exists a $0 \leq \lambda \leq 1$ ($\lambda = \frac{1}{2}$) such that

$$\det ((1 - \lambda) I + \lambda(-I)) = 0$$

it follows by a result of Clancey (1974b, Theorem 3.2) that the operator $B_{G_x}$ is not Fredholm.

Suppose for some neighbourhood $\Delta$ of $x$ the Lebesgue measure of $\Delta - E$ equals zero. Since $C(G_x : x) = \{-I\}$, it follows that $B_{G_x}$ and $B_{-I}$ are locally equivalent at $x$. By Lemma 2.3, $B_{G_x}$ is locally Fredholm at $x$. Let $t_0 \in R$ and suppose that $t_0 < x$ (a similar argument handles the case in which $t_0 \geq x$). It is clear that

$$C(G_x : t_0) \subseteq \{I\} \cup \left\{ \frac{(t_0 - x)I - iA}{(t_0 - x)I + iA} : A \in C(BB^* : t_0) \right\}. \quad \text{(2.8)}$$

Fix such an $A$, and let

$$C = \frac{(t_0 - x)I - iA}{(t_0 - x)I + iA}. \quad \text{(2.9)}$$

It can be concluded from (2.1) and (2.9) that there is a $\beta > 0$ (independent of the choice of $A$ in $C(BB^* : t_0)$) such that $\text{Im} (C) > \beta I$. Let $M \in C(-iG_x : t_0)$, then $M = -iI$ or $\text{Re} M > \alpha I$. For $\theta$ small, there is a $\alpha' = \omega(\theta)$ such that $\text{Re} Q > \alpha' I$, where $Q \in C(-ie^{i\theta}G_x : t_0)$. Fix such a $\theta$. Let $0 < \epsilon < \frac{1}{2}x'$, and choose a neighbourhood $N_\epsilon$ of $t_0$ such that for $t \in N_\epsilon (t_0)$,

$$\text{dist} (-ie^{i\theta}G_x(t), C(-ie^{i\theta}G_x : t_0)) < \epsilon. \quad \text{(2.10)}$$

From above it follows that $\text{Re} (-ie^{i\theta}G_x(t)) \geq \frac{1}{2}x'I$ for $t \in N_\epsilon (t_0)$. Define the function $H(t)$ as follows:

$$H(t) = \begin{cases} -ie^{i\theta}G_x(t), \ t \in N_\epsilon (t_0) \\ Q_0, \ t \notin N_\epsilon (t_0) \end{cases} \quad \text{(2.11)}$$
where $Q_0$ is any fixed element in $C(-ie^{i\theta}G_x : t_0)$. Since $H \in L^\infty_{M,M} (R)$ and
$\Re H(t) > \frac{1}{2} a' I$, by Lemma 2.1 the Riemann-Hilbert barrier operator $B_H$ on
$L^2_a (R)$ is Fredholm. Since $H(t)$ is equal to $-ie^{i\theta}G_x(t)$ for $t \in N_a(t_0)$ it follows
that $B_G_x$ is locally Fredholm at $t_0$. Therefore, $B_G_x$ is locally Fredholm at every $t \in R$. Hence,
by Lemma 2.2, the operator $B_G_x$ is Fredholm, and this ends the proof of
the theorem.

**Corollary 2.1** — Let $T$ be the singular integral operator defined by (2.2). Then
a real number $x$ is in the essential resolvent of $T$ if and only if there exists a neighbour-
hood $\Delta$ of $x$ such that the Lebesgue measure of $\Delta - E$ equals zero.

**PROOF** : The operators $S$ and $T$ are similar.

3. **THE PRINCIPAL FUNCTION**

In order to show that the Fredholm index of $T - x$ is $-n$, some properties
of the principal function of $T$ are needed. For the sake of completeness the
definition of the principal function of $T$ and some of its basic properties are presented
in this section.

Let $H$ be a separable complex Hilbert space. An operator $J$ on $H$ is said to
be completely non-normal if there exists no reducing sub-space of $J$ on which $J$ is
normal. The operator $J$ is a trace class operator if $\sum_j (J*J)^{1/2} \phi_i, \phi_j < \infty$ for an
orthonormal basis $\{\phi_k\}$ of $H$. The trace of the operator $J$, denoted by

$$tr(J) = \sum_j (J\phi_j, \phi_j).$$

Let $A = X + iY$ be the Cartesian decomposition of a bounded operator on
$H$, such that $A$ is completely non-normal hyponormal operator with trace class
self-commutator. Helton and Howe (1973) have associated with the operator $A$, a set function $\tilde{\mu}$ defined on the collection $\sigma$ of semi-closed rectangles in the following
manner:

Let $\alpha = [a, b)$ and $\beta = [c, d)$ be half-open intervals such that $\alpha \times \beta \in \sigma$. Denote by $\int_R \lambda E(\lambda)$, and $\int_R \lambda F(\lambda)$ the spectral resolutions of the operators $X$ and
$E(\alpha)Y E(\alpha)$, respectively. The set function $\tilde{\mu}$ is defined on $\alpha \times \beta$ by

$$\tilde{\mu}(\alpha \times \beta) = tr(E(\alpha) F(\beta) E(\alpha) [A^*, A]).$$

These authors established that $\tilde{\mu}$ extends to a non-negative regular Borel
measure $\mu$ of bounded total variation on the plane. Pincus (1979) (see also Carey
and Pincus 1974) has established that \( \mu \) is absolutely continuous with respect to planar Lebesgue measure. The derivative

\[
g = \frac{\pi}{d} \frac{d\mu}{d\gamma}
\]

...(3.2)

is called the principal function of the operator \( A \). The following is a summary of some of the principal functions associated with the operator \( A \).

(i) On a component of the complement of the essential spectrum of \( A \), \( g(\lambda) = \text{ind} (A - \lambda) \). For a proof, see Helton and Howe (1973).

We will need some further notation before describing the next property of the principal function. Let \( \int_{\mathbb{R}} \lambda \, dE(\lambda) \) be the spectral resolution of \( X \), and let \( \Delta \) be a Borel set in the real line \( \mathbb{R} \). Denote the Hilbert space \( E(\Delta)H \) by \( H_{\Delta} \). The operator \( A_{\Delta} \) on \( H_{\Delta} \) is defined by \( A_{\Delta}f = E(\Delta)Af \). It is known that \( A_{\Delta} \) is completely non-normal hyponormal operator with trace class self-commutator. Let \( g_{\Delta} \) be the principal function of \( A_{\Delta} \).

(ii) The principal functions \( g \) and \( g_{\Delta} \) of the operators \( A \) and \( A_{\Delta} \) are related by

\[
g_{\Delta} = g_{\chi_{\Delta \times \mathbb{R}}}
\]

...(3.3)

here, \( \chi_{\Delta \times \mathbb{R}} \) denotes the characteristic function of \( \Delta \times \mathbb{R} \). For a proof, see Carey and Pincus (1977).

4. The Fredholm Index

Let \( x \in \mathbb{R} \) be in the essential resolvent of \( T \), and in the spectrum of \( T \), where the operator \( T \) has been defined by (2.2). In this section it is shown that

\[
\text{ind} (T - x) = -n.
\]

Note that \( C(C_x : x) = \{-I\} \). Choose a small open ball \( D \) centered at \(-I\) so that \( D \) is contained in \( G(n : \emptyset) \). From this, it follows that there exists a small neighbourhood \( \Delta = (x - \xi, x + \xi) \) contained in \( E \) such that the closed convex hull of the essential range of \( G_x \) restricted to \( \Delta \) is contained in \( D \). Define the function \( G_{\Delta}^x \) as follows:

\[
G_{\Delta}^x(s) = \begin{cases} 
\frac{(s - x)I - iB(s)B^*(s)}{(s - x)I + iB(s)B^*(s)}, & s \in \Delta \\
I, & s \notin \Delta.
\end{cases}
\]

...(4.1)

This is the symbol of the singular integral operator \( T_{\Delta} - x \) defined on \( L^2_n(\Delta) \).
Theorem 4.1 — The operator $T - x$ is Fredholm if and only if the operator $T_\Delta - x$ is Fredholm. Moreover,
\[ \text{ind } (T_\Delta - x) = \text{ind } (T - x). \]

Proof: Putnam (1970) has shown that $x$ is in the essential resolvent of $T$ if and only if $x$ is in the essential resolvent of $T_\Delta$. Also, using properties (i) and (ii) of the principal function, it follows easily that $\text{ind } (T - x) = \text{ind } (T_\Delta - x)$, and this ends the proof.

From Theorem 4.1, it suffices to show that $\text{ind } (T_\Delta - x) = -n$. Since $T_\Delta - x$ is Fredholm if and only if $B_{G_\Delta}^\Delta$ is Fredholm, and $\text{ind } (T_\Delta - x) = \text{ind } (B_{G_\Delta}^\Delta)$, it suffices to show that $\text{ind } (B_{G_\Delta}^\Delta) = -n$. To establish our result, we proceed as follows:

Let $\eta$ be a function from $[x - \xi, x + \xi]$ into $[0, 1]$ such that $\eta$ is continuous, $\eta(x - \xi) = 0$ and $\eta(x + \xi) = 1$. Define the function $\tilde{G}$ as follows:
\[ \tilde{G}(t) = \begin{cases} 
(1 - \eta(t)) U^+ + \eta(t) U^-, & t \in \Delta \\
I, & t \notin \Delta 
\end{cases} \tag{4.2} \]
where $U^\pm$ are different from the identity $n \times n$ matrix $I$, and $U^\pm \in C(G_\Delta^\Delta : x \mp \xi)$. Any element in $C(G_\Delta^\Delta : x \mp \xi)$ is either the matrix $I$, or a matrix of the form
\[ ((s - x) I + iA^\pm)^{-1} ((s - x) I - iA^\pm) \tag{4.3} \]
where $A^+$ is a cluster value from the right of $BB^*$ at $x - \xi$, and $A^-$ is a cluster value from the left of $BB^*$ at $x + \xi$. Note that $U^\pm$ are unitary. The function $\tilde{G}$ is piecewise continuous, and it is clear from (Theorem 2.1, see Clancey 1974b), that the Riemann-Hilbert barrier operator $B_{\tilde{G}}$ is a Fredholm operator on $L_n^2(R)$.

For $0 \leq s \leq 1$ and $t$ a real number, the function $H$ will be defined in the following way:
\[ H(s, t) = \begin{cases} 
(1 - s) G_\Delta^\Delta(t) + s \tilde{G}(t), & t \in \Delta \\
I, & t \notin \Delta 
\end{cases} \tag{4.4} \]

Theorem 4.2 — The Riemann-Hilbert barrier operator $B_{H(s, \cdot)}$ is a Fredholm operator on $L_n^2(R)$, and
\[ \text{ind } (B_{G_\Delta}^\Delta) = \text{ind } (B_{\tilde{G}}). \]
PROOF: It has already been observed that the operators $B_{H(0,\cdot)} = B_{G^\Delta}$, and $B_{H(1,\cdot)} = B_G$ are Fredholm operators.

Let $s$ be a fixed real number between 0 and 1. If $t_0 \not\in \Delta$, then the Riemann Hilbert barrier operator $B_{H(s,\cdot)}$ is locally equal to the Fredholm Riemann-Hilbert barrier operator $B_t$ at $t_0$. All that has to be shown now is that $B_{H(s,\cdot)}$ is locally Fredholm at $t_0$ in $\Delta$.

For $t = x$, the Riemann-Hilbert barrier operator $B_{H(s,\cdot)}$ is locally equivalent to the Riemann-Hilbert barrier operator with symbol $(1 - s)(-I) + s \hat{G}(x)$, which is an element of $D$. It follows by Lemma 2.3 that $B_{H(s,\cdot)}$ is locally Fredholm at $t = x$. If $t \in \Delta$, $t$ is different from $x$, then to show that the operator $B_{H(s,\cdot)}$ is locally Fredholm at $t$, we follow the argument of Theorem 2.1. For $s$ fixed the operator $B_{H(s,\cdot)}$ is locally Fredholm at every $t \in R$. Hence, by Lemma 2.2, the operator $B_{H(s,\cdot)}$ is Fredholm.

Since the function $s \mapsto H(s,\cdot)$ from $[0, 1]$ to $L^\infty_{M_n}(R)$ is continuous, it follows that $\text{ind} \ (B_{H(0,\cdot)}) = \text{ind} \ (B_{H(1,\cdot)})$, in other words $\text{ind} \ (B_{G^\Delta}) = \text{ind} \ (B_G)$, and this ends the proof.

It is known that the Toeplitz operator $T_G$ is Fredholm if and only if the Riemann-Hilbert barrier operator $B_G$ is Fredholm, moreover they have the same Fredholm index. Gohberg and Krupnik (1968) have shown that the Fredholm index of $T_G$ is equal to the negative of the winding number of $\det (\hat{G}^*)$ where $\hat{G}^*$ is the curve obtained from the piecewise continuous function $\hat{G}$ by joining the left and right hand limits by a line segment at points of discontinuity. From above it suffices to show that the winding number of $\det (\hat{G}^*)$ around zero is equal to $n$. In order to show that the winding number of $\det (\hat{G}^*)$ is equal to $n$, it is needed to be shown that the jump in the argument of $\det (\hat{G}^*)$ on sufficiently small $\Delta$ is close to $n$.

Let $U^\pm$ be the unitary $n \times n$ matrices defined by (4.3). Since $A^\pm$ is positive, the eigenvalues $e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}, 0 \leq \theta_j < 2\pi, 1 \leq j \leq n$ of $U^+$ are in the upper half-plane and close to $-1$. Similarly, the eigenvalues $e^{i\varphi_1}, \ldots, e^{i\varphi_n}, 1 \leq j \leq n$, of $U^-$ are in the lower half-plane and close to $-1$. Since the matrices $U^\pm$ are unitary, it follows that there exist unitary matrices $V_\pm$, such that
\[ D^+ = V_+ U^+ V^*_+ = \begin{bmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_n} \end{bmatrix} \]

and

\[ D^- = V_- U^- V^*_- = \begin{bmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_n} \end{bmatrix} \]

Since \( \tilde{G}(t) \) was defined in (4.2) by

\[ \tilde{G}(t) = (1 - \eta(t)) U^+ + \eta(t) U^-, \text{ for } t \in \Delta \]

it follows that

\[
\det \tilde{G}(t) = \det \left( (1 - \eta(t)) V^*_+ D^+ V_+ + \eta(t) V^*_- D^- V_- \right)
\]

\[ = \det (V^*_+ V_-) \cdot \det \left( (1 - \eta(t)) D^+ W + \eta(t) W D^- \right) \]

where \( W \) is the unitary matrix \( V_+ V^*_- \).

From the definition of the determinant, we see that \( \det \left( (1 - \eta(t)) D^+ W + \eta(t) W D^- \right) \) can be written as follows:

\[
\sum_{\sigma \in P} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} \left( (1 - \eta(t)) \ w_{ii} \ e^{i\theta_i} + \eta(t) \ w_{ii} \ e^{i\varphi_i} \right) \quad \ldots(4.5)
\]

where \( P \) is the set of all permutations on \( \{1, 2, \ldots, n\} \), and \( \epsilon(\sigma) \) is the sign of the permutation. The last expression (4.5) can be written in the following form

\[
\prod_{i=1}^{n} \left( (1 - \eta(t)) \ w_{ii} \ e^{i\theta_i} + \eta(t) \ w_{ii} \ e^{i\varphi_i} \right)
\]

\[ + \sum_{P-(\tau)} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} \left( (1 - \eta(t)) \ w_{i\sigma(i)} \ e^{i\theta_i} + \eta(t) \ w_{i\sigma(i)} \ e^{i\varphi_i} \right) \]

\[ + \sum_{P-(\tau)} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} \left( (1 - \eta(t)) \ w_{i\sigma(i)} \ e^{i\varphi_i} + \eta(t) \ w_{i\sigma(i)} \ e^{i\varphi_i} \right) \]

\[ - \sum_{P-(\tau)} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} \left( (1 - \eta(t)) \ w_{i\sigma(i)} \ e^{i\theta_i} + \eta(t) \ w_{i\sigma(i)} \ e^{i\varphi_i} \right) \]
where $\tau$ is the identity permutation. By adding the first term to the third, the following expression is obtained,

$$
\det W \prod_{i=1}^{n} ((1 - \eta(t)) e^{i\theta_i} + \eta(t) e^{i\psi_i})
$$

$$
+ \sum_{P-\{\tau\}} (-1)^{s(P)} \prod_{i=1}^{n} ((1 - \eta(t)) w_{i\sigma(i)} e^{i\theta_i} + \eta(t) w_{i\sigma(i)} e^{i\psi_i})
$$

$$
- \sum_{P-\{\tau\}} (-1)^{s(P)} \prod_{i=1}^{n} ((1 - \eta(t)) w_{i\sigma(i)} e^{i\theta_i} + \eta(t) w_{i\sigma(i)} e^{i\psi_i}).
$$

Since $e^{i\psi_i}$ can be chosen arbitrarily close to $-1$, the difference of the second and third terms in the last expression does not contribute significantly to the jump in the argument of $\det ((1 - \eta(t)) D^+ W + \eta(t) WD^-)$ on $\Delta$. The jump of the argument of

$$
\det W \prod_{i=1}^{n} ((1 - \eta(t)) e^{i\theta_i} + \eta(t) e^{i\psi_i})
$$

can be made arbitrarily close to $n$ by taking $\Delta$ sufficiently small. So we have established that the winding number of $\det (G^*)$ is $n$. The main result of this section is the following theorem.

**Theorem 4.3** — Let $T$ be the singular integral operator defined by (2.2), and assume that $x$ is a real number such that some interval $\Delta$ containing $x$ satisfies

(i) $\Delta \subset E$ and (ii) $\text{ess inf}_{t \in \Delta} |\det B(t)| > 0$.

The operator $T - x$ is Fredholm, and $\text{ind} (T - x) = -n$.

**Proof**: The fact that $T - x$ is Fredholm is a direct consequence of Theorem 2.1. The arguments given above established that if $\Delta$ is a sufficiently small neighbourhood of $x$, then the operator $T_\Delta - x$ has index $-n$. By Theorem 4.1, we conclude that

$$
\text{ind} (T - x) = \text{ind} (T_\Delta - x) = -n
$$

and that ends the proof.

**References**


