*Indian J. pure appl. Math.*, **25**(12): 1259-1265, December 1994

# THE SPLICING OF CYCLE PERMUTATION GRAPHS

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(*Received* 12 *November* 1992; *after revision* 14 *June* 1994; *accepted* 27 *June* 1994)

A cycle permutation graph $P(C_n, \alpha)$ consists of two copies of the $n$-cycle $C_n$ along with a set of permutation edges connecting vertex $i$ in one copy of $C_n$ to vertex $\alpha(i)$ in the other. Cycle permutation graphs may be considered to be a generalization of the Petersen graph since the Petersen graph is isomorphic to $P(C_5, \alpha)$ where $\alpha$ = [2 4 1 3 5] ($\alpha(1) = 2$, $\alpha(2) = 4$, etc.).

Klee (*Lecture Notes in Math.* **303** (1972), pp. 173-78) and Piazza (*Congr. Numer.* **45** (1984), pp. 83-90) were interested in determining which cycle permutation graphs are hamiltonian and for which $n$ there exists a nonhamiltonian cycle permutation graph. Klee calls the associated permutation bad if the cycle permutation graph is nonhamiltonian; otherwise it is good. He defines an operation called catenation that he employs to build bad permutations of large order.

For two permutations $\alpha$ and $\beta$ we generalize the operation of catenation to an operation called the $(i, j)$-splice of $\beta$ into $\alpha$ and examine some conditions for determining whether the resulting permutation is good or bad.

## INTRODUCTION

For a labelled graph $G$ with $V(G) = \{1, 2, ..., n\}$ and a permutation $\alpha$ in the symmetric group $S_n$ on $\{1, 2, ..., n\}$, the permutation graph $P(G, \alpha)$ consists of two copies of $G$, i.e., $G_x$ and $G_y$, with vertex sets $V(G_x) = \{x_1, x_2, ..., x_n\}$ and $V(G_y) = \{y_1, y_2, ..., y_n\}$ along with the permutation edges $(x_i, y_{(\alpha)\,i})$, for $1 \leq i \leq n$. Permutation graphs were introduced by Chartrand and Harary[1] who were interested in finding those which are planar. For a path $P_n$ on $n$ vertices with edge set $E(P_n) = \{(1, 2), (2, 3), ..., (n-1, n)\}$, the graph $P(P_n, \alpha)$ is called a path permutation graph. For an $n$-cycle $C_n$ with edge set $E(C_n) = \{(1, 2), (2, 3), ..., (n-1, n), (n, 1)\}$, the graph $P(C_n, \alpha)$ is called a cycle permutation graph (sometimes called a generalized prism or an $\alpha$-prism). Cycle permutation graphs may be considered to be a generalization of the Petersen graph, since the Petersen graph is isomorphic to $P(C_5, \alpha)$ where $\alpha$ = [2 4 1 3 5] ($\alpha(1) = 2$, $\alpha(2) = 4$, etc.).

Klee[2], Mohanty and Rao[3], and Piazza[4, 5] were interested in determining which cycle permutation graphs are hamiltonian and for which $n$ there exists a nonhamiltonian cycle permutation graph. Mohanty and Rao[6] also examined hypo-hamiltonian cycle permutation graphs. Recently, Alspach and Zhang[7] have made use of non-3-egde-colorable (and hence nonhamiltonian) cycle permutation graphs in the search for admissible (1, 2)-weighted cubic graphs that have no cycle $w$-cover, a problem related to the cycle double cover conjecture and the Chinese postman problem.

In this paper, we continue the examination of hamiltonian properties of cycle permutation graphs. We begin by stating some results of Klee[2] and Piazza[4].

*Proposition* 1[2] — For $n \leq 8$, the only nonhamiltonian cycle permutation graphs are those isomorphic to the Petersen graph. ♦

*Proposition* 2[4] — For even $n \leq 14$, all cycle permutation graphs are hamiltonian. ♦

*Proposition* 3[4] — Let $f(n)$ denote the number of nonisomorphic nonhamiltonian cycle permutation graphs of $C_n$. Then $f(3) = 0$, $f(5) = 1$, $f(7) = 0$, $f(9) = 2$, $f(11) = 1$, and $f(13) = 64$. ♦

Permutations corresponding to these nonhamiltonian cycle permutation graphs are found in Piazza[5]. Stueckle[8] and Kwan and Lee[9] have also examined the isomorphism of cycle permutation graphs.

*Proposition* 4[2] — For all odd $n \geq 9$, there exists a nonhamiltonian cycle permutation graph. ♦

Proposition 4 was obtained by catenating bad permutations to form bad permutations of larger order. For two permutations $\alpha$ and $\beta$ we generalize the operation of catenation to an operation called the $(i, j)$-splice of $\beta$ into $\alpha$. Some conditions for determining whether the resulting permutation is 'good' or 'bad' are examined.

## PRELIMINARIES

For a permutation $\alpha$ in $S_n$, the extension of $\alpha$ is the permutation $\alpha'$ in $S_{n+1}$ given by $\alpha'(i) = \alpha(i)$, $1 \leq i \leq n$, and $\alpha'(n + 1) = n + 1$. For each permutation $\alpha$ we shall associate the path permutation graph $P(P_n, \alpha)$ and the cycle permutation graph $P(C_{n+1}, \alpha')$.

In $P(P_n, \alpha)$, an $E$-path is any path both of whose end vertices are in the set $E = \{x_1, x_n, y_1, y_n\}$. An $EH$-path is an $E$-path that contains all the vertices of $P(P_n, \alpha)$. An $EH$-pair is a pair of vertex-disjoint $E$-paths whose union contains all the vertices of $P(P_n, \alpha)$. An $EH$-path is level preserving if both of its ends are in $G_x$ or both in $G_y$; otherwise it is level reversing. An $EH$-pair is level preserving if one $E$-path has both of its ends in $G_x$ and the other $E$-path has both of its ends in $G_y$; otherwise it is level reversing. A permutation $\alpha$ is 'good' if $P(P_n, \alpha)$ contains a level reversing $EH$-path or $EH$-pair; otherwise $\alpha$ is 'bad'.

*Proposition* 5[2] — The permutation graph $P(C_{n+1}, \alpha')$ is hamiltonian if and only if $\alpha$ is good. ♦

For $\alpha$ in $S_n$ and $\beta$ in $S_m$, the catenation of $\alpha$ and $\beta$, denoted $(\alpha, \beta)$, is the permutation in $S_{n+m}$ given by

$$(\alpha, \beta)(k) = \begin{cases} \alpha(k) & , \quad 1 \le k \le n \\ \beta(k-n)+n & , \quad n+1 \le k \le n+m. \end{cases}$$

The operation of catenation can be extended in a natural way to include arbitrary finite sequences of permutations. The catenation of $\alpha_1, \alpha_2, \ldots, \alpha_k$ shall be denoted $(\alpha_1, \alpha_2, \ldots, \alpha_k)$.

Klee used Proposition 5, the idea of catenation and the fact that there exist bad permutations of orders 4 and 10 to obtain Proposition 4.

For the remainder of this paper we shall assume that $\alpha \in S_n$, $\beta \in S_m$, and that all subscripts are modulo the order of the appropriate cycle or path under consideration.

## SPLICING

For $0 \le i, j \le n$, the $(i, j)$-splice of $\beta$ into $\alpha$, denoted $(\alpha, \beta, i, j)$, is the permutation in $S_{n+m}$ given by

$$(\alpha, \beta, i, j)(k) = \begin{cases} \alpha(k) & , \quad 1 \le k \le i \text{ and } 1 \le \alpha(k) \le j \\ \alpha(k)+m & , \quad 1 \le k \le i \text{ and } j+1 \le \alpha(k) \le n \\ \beta(k-i)+j & , \quad i+1 \le k \le i+m \\ \alpha(k-m) & , \quad i+m+1 \le k \le n+m \text{ and } 1 \le \alpha(k-m) \le j \\ \alpha(k-m)+m & , \quad i+m+1 \le k \le n+m \text{ and } j+1 \le \alpha(k-m) \le n. \end{cases}$$

Note that $(\alpha, \beta, n, n) = (\beta, \alpha, 0, 0) = (\alpha, \beta)$. Thus splicing is a generalization of catenation. Unlike catenation, the operation of splicing cannot be extended to more than two permutations at one time since the resulting permutations are dependent upon $i$ and $j$.

There is a nice interpretation of splicing in terms of the path permutation graphs $P(P_n, \alpha)$ and $P(P_m, \beta)$. For $(\alpha, \beta, i, j)$, the path permutation graph $P(P_m, \beta)$ is spliced into the path permutation graph $P(P_n, \alpha)$ after $x_i$ in the $G_x$ copy and after $y_j$ in the $G_y$ copy of $P_n$ to form $P(P_{n+m}, (\alpha, \beta, i, j))$. Note that if $i = 0$ or $j = 0$, $P(P_m, \beta)$ is inserted before $x_1$ or $y_1$, respectively. For example, if $\alpha = [2\ 4\ 1\ 3]$ and $\beta = [2\ 1\ 3]$ then $(\alpha, \beta, 0, 1) = [3\ 2\ 4\ 5\ 7\ 1\ 6]$ may be obtained via the path permutation graphs as shown in Fig. 1.

Let $\gamma = (\alpha, \beta, i, j)$ for some $i$ and $j$, $0 \le i, j \le n$. The restriction of $P(P_{m+n}, \gamma)$ to $\beta$, denoted $P(P_{m+n}, \gamma)|_\beta$, is the subgraph induced by $V_\beta = \{x_{i+1}, x_{i+2}, \ldots, x_{i+m}, y_{j+1}, y_{j+2}, \ldots, y_{j+m}\}$. The set $V_\beta$ corresponds to those vertices representing $P(P_m, \beta)$ in $P(P_{m+n}, \gamma)$. Similarly, we define the restriction of $P(C_{m+n+1}, \gamma')$ to $\beta$, denoted $P(C_{m+n+1}, \gamma')|_\beta$. Both $P(P_{m+n}, \gamma)|_\beta$ and $P(C_{m+n+1}, \gamma')|_\beta$ are isomorphic to $P(P_m, \beta)$.

If $T$ is an $EH$-path or $EH$-pair in $P(P_{m+n}, \gamma')$, then the set of vertices of $T$ which are in $V_\beta$ (the vertices of $P(P_{m+n}, \gamma)|_\beta$) is an $EH$-path or $EH$-pair with respect to $P(P_m, \beta)$ since $x_{i+1}$, $x_{i+m}$, $y_{j+1}$ and $y_{j+m}$ are the only vertices of

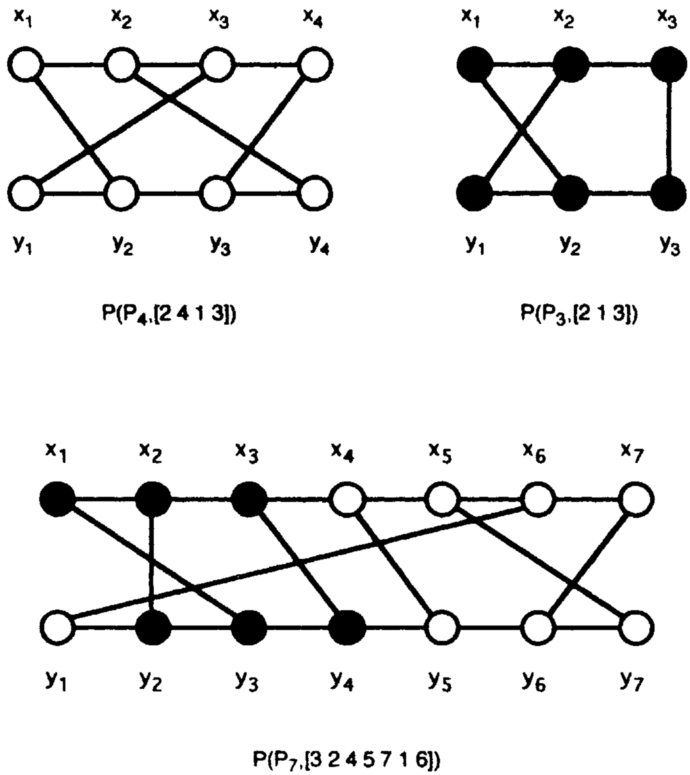


FIG. 1.

$P(P_{m+n}, \gamma)\,|_\beta$ adjacent to vertices not in $V_\beta$. Similarly, if $H$ is a hamiltonian cycle in $P(C_{m+n+1}, \gamma')$, then the set of vertices of $T$ which are in $V_\beta$ is an $EH$-path or $EH$-pair with respect to $P(P_m, \beta)$.

## SUFFICIENT CONDITIONS

In this section we present two sufficient conditions for the permutation $(\alpha, \beta, i, j)$ to be bad and two sufficient conditions for $(\alpha, \beta, i, j)$ to be good. Moreover, since every permutation is either good or bad, the contrapositive of each of these results gives necessary conditions for $(\alpha, \beta, i, j)$ to be either good or bad.

*Theorem* 1 — If $\beta$ is bad and $P(C_{n+1}, \alpha')$ possesses no hamiltonian cycle containing the edge $(x_i, x_{i+1})$ or $(y_j, y_{j+1})$, then $\gamma = (\alpha, \beta, i, j)$ is bad.

PROOF : Suppose $\gamma$ is good. Then $P(C_{m+n+1}, \gamma')$ contains some hamiltonian cycle $H$. If $R$ is the segment of $H$ containing vertices of $V_\beta$, then $R$ is an $EH$-path or $EH$-pair in $P(C_{m+n+1}, \gamma')\,|_\beta$. Furthermore, since $\beta$ is bad, $R$ must be level preserving.

If $R$ is a level preserving $EH$-pair, one $E$-path of $R$, say $R_1$, has end vertices $x_{i+1}$ and $x_{i+m}$ and the other $E$-path, say $R_2$, has end vertices $y_{j+1}$ and $y_{j+m}$. Since these four vertices are the only ones in $V_\beta$ adjacent to vertices not in $V_\beta$, the edges $(x_i, x_{i+1})$, $(x_{i+m}, x_{i+m+1})$, $(y_j, y_{j+1})$ and $(y_{j+m}, y_{j+m+1})$ are in $H$. Thus the segments $x_i - R_1 - x_{i+m+1}$ and $y_j - R_2 - x_{j+m+1}$ of $H$ contain all the vertices of $V_\beta$. Furthermore, since $P(C_{m+n+1}, \gamma')$ is hamiltonian, the graph $G_1 = P(C_{m+n+1}, \gamma') - V_\beta \cup \{(x_i, x_{i+m+1}), (y_j, y_{j+m+1})\}$ possesses a hamiltonian cycle that contains the edges $(x_i, x_{i+m+1})$ and $(y_j, y_{j+m+1})$. But, $G_1$ is isomorphic to $P(C_{n+1}, \alpha')$ with edges $(x_i, x_{i+m+1})$ and $(y_j, y_{j+m+1})$ in $G_1$ corresponding to $(x_i, x_{i+1})$ and $(y_j, y_{j+1})$, respectively, in $P(C_{n+1}, \alpha')$. Thus $P(C_{n+1}, \alpha')$ possesses a hamiltonian cycle containing $(x_i, x_{i+1})$ and $(y_j, y_{j+1})$, a contradiction.

If $R$ is a level preserving $EH$-path, we may assume, without loss of generality, that $R$ has end vertices $x_{i+1}$ and $x_{i+m}$ and so $H$ must contain the edges $(x_i, x_{i+1})$ and $(x_{i+m}, x_{i+m+1})$. Similar to the above argument, we construct $G_2 = P(C_{m+n+1}, \gamma') - V_\beta \cup \{(x_i, x_{i+m+1})\}$ possessing a hamiltonian cycle containing $(x_i, x_{i+m+1})$. Note that $y_j$ and $y_{j+m+1}$ are vertices of degree two in $G_2$. But, $G_2$ is isomorphic to $P(C_{n+1}, \alpha') - (y_j, y_{j+1})$ with the edge $(x_i, x_{i+m+1})$ and the vertices $y_j$ and $y_{j+m+1}$ in $G_2$ corresponding to the edge $(x_i, x_{i+1})$ and the vertices $y_j$ and $y_{j+1}$, respectively, in $P(C_{n+1}, \alpha') - (y_j, y_{j+1})$. Since $P(C_{n+1}, \alpha') - (y_j, y_{j+1})$, a subgraph of $P(C_{n+1}, \alpha')$, possesses a hamiltonian cycle containing $(x_i, x_{i+1})$, so does $P(C_{n+1}, \alpha')$, a contradiction. ♦

*Corollary* 1 — If $\alpha$ and $\beta$ are bad, then $\gamma = (\alpha, \beta, i, j)$ is bad for all $i$ and $j$, $0 \le i, j \le n$. ♦

Since trying to decide if a cycle permutation graph possesses a hamiltonian cycle containing a specific edge is difficult to do in general, using Theorem 1 to recursively build up large bad permutations would not be practical. However, Corollary 1 could be used fairly easily to build many large permutations since we could use the known small permutations as well as any that are produced during the recursion.

As an aside we note that if $P(C_{n+1}, \alpha')$ and $P(C_{m+1}, \beta')$ are both non-3-edge-colorable, then $P(C_{n+m+1}, \gamma')$, where $\gamma = (\alpha, \beta, i, j)$ will also be non-3-edge-colorable for all $i$ and $j$, $0 \le i, j \le n$, since, in this case, the construction used to form $P(C_{n+m+1}, \gamma')$ is equivalent to one of the constructions given by Isaacs[10].

*Theorem* 2 — If $P(C_{n+1}, \alpha')$ possesses a hamiltonian cycle containing both $(x_i, x_{i+1})$ and $(y_j, y_{j+1})$, then $\gamma = (\alpha, \beta, i, j)$ is good.

PROOF : Assume that $P(C_{n+1}, \alpha')$ possesses a hamiltonian cycle containing both $(x_i, x_{i+1})$ and $(y_j, y_{j+1})$. Form a subdivision $G$ of $P(C_{n+1}, \alpha')$ with new vertices $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m$ such that $x_i - a_1 - a_2 - \ldots - a_m - x_{i+1}$ and $y_j - b_1 - b_2 - \ldots - b_m - y_{j+1}$ are subgraphs of $G$. Then $G$ is hamiltonian. But, $G$ is isomorphic to $P(C_{n+m+1}, \gamma') - \{(x_{i+k}, y_{\gamma'(i+k)}) \mid 1 \le k \le m\}$. Thus $P(C_{n+m+1}, \gamma')$ is hamiltonian and $\gamma = (\alpha, \beta, i, j)$ is good. ♦

*Theorem* 3 — If $P(P_m, \beta)$ contains a level reversing *EH*-pair, then for all $\alpha$ and all $i$ and $j$, $0 \leq i, j \leq n$, $\gamma = (\alpha, \beta, i, j)$ is good.

PROOF : Since $P(P_m, \beta)$ contains a level reversing *EH*-pair, $P(P_{n+m}, \gamma)|_\beta$ contains a level reversing *EH*-pair, say $R_1$ and $R_2$, which can be extended to a level reversing *EH*-pair for $P(P_{n+m}, \gamma)$ by adding to $R_1$ and $R_2$ those edges in the $G_x$ and $G_y$ copies of $P(P_{n+m}, \gamma)$ which are not in $P(P_{n+m}, \gamma)|_\beta$. Thus $\gamma = (\alpha, \beta, i, j)$ is good. ♦

## COMPUTATIONAL RESULTS

Catenation and splicing provide ways to generate bad permutations of large order from smaller bad permutations. An interesting problem is determining the number of bad permutations that can be generated by recursively performing these operations and determining the number of nonisomorphic nonhamiltonian cycle permutations corresponding to the extensions of these bad permutations.

When $n = 4$, the bad permutations are $\alpha_1 = [2\ 4\ 1\ 3]$ and $\alpha_2 = [3\ 4\ 1\ 2]$. Four bad permutations are generated by all possible catenations of $\alpha_1$ and $\alpha_2$, all of which have extensions associated with the same cycle permutation graph. Ninety-two bad permutations are generated by all possible splicings of $\alpha_1$ and $\alpha_2$. Their extensions correspond to both the nonhamiltonian cycle permutation graphs of order 9. There is total of 108 bad permutations of order 8.

Table I gives the number of bad permutations of order 12 formed when $\alpha_1$ and $\alpha_2$ are : (1) catenated with the 4 of order 8 previously generated, (2) catenated with all 108, (3) spliced with the 92 previously generated, and (4) spliced with all 108. Table I also gives the number of nonisomorphic nonhamiltonian cycle permutation graphs associated with the extensions of each of these sets.

TABLE I

*Summary of computational results*

| | # of bad permutations of order 12 | # of nonhamiltonian cycle permutation graphs |
|---|---|---|
| catenate (8) | 8 | 1 |
| catenate (108) | 424 | 21 |
| splice (92) | 11,352 | 63 |
| splice (108) | 14,000 | 64 |
| Totals | ≥ 14,000 | 64 |

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