GENERALIZED COMPLEMENTS OF A GRAPH

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Let $G = (V, E)$ be a graph and $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of $V$ of order $k \geq 1$. For each set $V_r$ in $P$, remove the edges of $G$ inside $V_r$ and add the edges $\overline{G}$, (the complement of $G$) joining the vertices $V_r$. The graph $G^p_k (i)$ thus obtained is called the $k(i)$-complement of $G$ with respect to $P$. The graph $G$ is $k(i)$-self complementary $(k(i)-s.c.)$ if $G^p_k (i) \equiv G$ for some partition $P$ of $V$ of order $k$. Further, $G$ is $k(i)$-co-self complementary $(k(i)-c-o-s.c.)$ if $G^p_k (i) \equiv \overline{G}$.

We determine (1) all $k(i)$-s.c trees for $k = 2, 3$, and (2) $2(i)$-s.c. unicyclic graphs. Also, some necessary conditions for a tree/unicyclic graph to be $k(i)$-s.c. are obtained. We indicate how to obtain characterizations of all $k(i)$-co.s.c. trees, unicyclic graphs and forests from known results.

Key Words : Graphs; Complements; Trees; Unicyclic; Forests.

1. INTRODUCTION

Let $G = (V, E)$ be a graph and $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of $V$ of order $k \geq 1$. The $k$-complement $G^p_k$ of $G$ (with respect to $P$) is defined as follows: For all $V_i$ and $V_j$ in $P, i \neq j$, remove the edges between $V_i$ and $V_j$, and add the edges which are not in $G^p$. The graph $G$ is $k$-self complementary $(k-s.c)$ with respect to $P$ if $G^p_k \equiv G$. For $2 \leq k \leq p$, characterizations of all $k$-self complementary trees, forests and unicyclic graphs are obtained in (1).

In this paper, we study another type of complement defined as follows :-

For each set $V_r$ in the partition $P$, remove the edges of $G$ inside $V_r$, and add the edges of $\overline{G}$ (the complement of $G$) joining the vertices of $V_r$. The graph $G^p_k(i)$ thus obtained is called the $k(i)$-complement of $G$ with respect to the partition $P$ of $V$. The graph $G$ is $k(i)$-self complementary $(k(i)-s.c.)$ if $G^p_k(i) \equiv G$ for some partition $P$ of order $k$. 
We observe that the Petersen graph is 2(i)-s.c. with respect to the partition \( V_1 = \{1, 2, 3, 4, 5\}, \ V_2 = \{1', 2', 3', 4', 5'\} \). Also, it is 2-s.c. with respect to the partition 
\[
V_1 = \{1, 3, 4', 5'\}, \ V_2 = \{2, 4, 5, 1', 2', 3'\}. \ (\text{See Fig. 1}).
\]

![Fig. 1](image)

If \( G \) is a graph of order \( P \) and \( \overline{G} \) is the complement of \( G \), we note that

i) \( G^P_1 \equiv G \), and \( G^P_{1(i)} \equiv \overline{G} \), where \( P = \{V\} \), and

ii) \( G^P_p \equiv \overline{G} \), and \( G^P_{p(i)} \equiv \overline{G} \), where \( P \) is the partition of \( V \) into singleton sets.

A \( k(i) \)-complement is nontrivial if \( k < P \).

If \( G^P_k \equiv \overline{G} \), then \( G \) is \( k \)-co-self complementary (k-co-s.c.), and \( G \) is \( k(i) \)-co-self complementary (k(i)-co-s.c.) if \( G^P_{k(i)} \equiv \overline{G} \).

We determine (1) all \( k(i) \)-s.c. trees for \( k = 2, 3 \), and (2) all 2(i)-s.c., unicyclic graphs. Also, we obtain some necessary conditions for a tree/unicyclic graph to be \( k(i) \)-s.c. Further, we indicate how to obtain characterizations of all \( k(i) \)-co-s.c. trees, unicyclic graphs and forests from known results.

2. ELEMENTARY RESULTS

**Proposition 1 —** For any graph \( G \),

\[
i) \ G^P_k \equiv \overline{G}^P_k \quad \text{and} \quad ii) \ \overline{G}^P_{k(i)} \equiv \overline{G}^P_{k(i)}
\]

**PROOF:** i) Let \( u \) and \( v \) be two vertices in \( G \). Then \( u \) and \( v \) are adjacent in \( \overline{G}^P_k \)

\[\leftrightarrow u \text{ and } v \text{ are nonadjacent in } G^P_k.\]

\[\leftrightarrow u \text{ and } v \text{ are in the same set in } P, \text{ and are nonadjacent in } G, \text{ or they are in different sets in } P, \text{ and are adjacent in } G.\]

\[\leftrightarrow u \text{ and } v \text{ are in the same set in } P, \text{ and are adjacent in } \overline{G}, \text{ or they are in different sets in } P, \text{ and are nonadjacent in } \overline{G}.\]

\[\leftrightarrow u \text{ and } v \text{ are adjacent in } \overline{G}^P_k.\]
This proves (i).

\( ii \) \quad u \text{ and } v \text{ are adjacent in } G^P_{k(i)} \)

\[ \iff \text{ they are nonadjacent in } G^P_{k(i)}. \]

\[ \iff u \text{ and } v \text{ are in the same set in } P \text{ and are adjacent in } G, \text{ or they are in different sets in } P \text{ and are nonadjacent in } G. \]

\[ \iff u \text{ and } v \text{ are in the same set in } P \text{ and are nonadjacent in } \overline{G}, \text{ or they are in different sets in } P \text{ and are adjacent in } \overline{G}. \]

\[ \iff u \text{ and } v \text{ are adjacent in } G^P_{k(i)}. \]

This proves (ii).

As a consequence of Proposition 1, we have

**Corollary 1.1** — For any graph \( G \),

\[ i) \quad G^P_k \equiv G \quad \iff \quad \overline{G^P_k} \equiv \overline{G}; \quad \text{and} \]

\[ ii) \quad G^P_{k(i)} \equiv G \quad \iff \quad \overline{G^P_{k(i)}} \equiv \overline{G}. \]

In other words, \( G \) is \( k \)-s.c. if, and only if, \( \overline{G} \) is so, and \( G \) is \( k(i) \)-s.c. if, and only if, \( \overline{G} \) is so.

The \( k \)-complement and \( k(i) \) complement of \( G \) are related as follows :-

**Proposition 2** — \( i) \overline{G^P_k} \equiv G^P_{k(i)} \) and \( ii) \overline{G^P_{k(i)}} \equiv G^P_k \).

**PROOF :** \( i) \) Let \( u \) and \( v \) be two vertices in \( G \). Then

\[ u \text{ and } v \text{ are adjacent in } \overline{G^P_k}. \]

\[ \iff \text{ they are in the same set of the partition } P, \text{ and are nonadjacent in } G, \text{ or they are in different sets in } P, \text{ and adjacent in } G. \]

\[ \iff \text{ they are adjacent in } \overline{G^P_{k(i)}}. \]

From Propositions 1 and 2, we have the following:

**Corollary 2.1** — For any graph \( G \),

\[ i) \quad G^P_k \equiv G_k \equiv G^P_{k(i)} \]

and \[ ii) \quad G^P_{k(i)} \equiv G_{k(i)} \equiv G^P_k \]

By Proposition 2, we have

**Corollary 2.2** — \( i) \quad G^P_k \equiv G \quad \iff \quad G^P_{k(i)} \equiv \overline{G}. \)

\[ ii) \quad G^P_{k(i)} \equiv G \quad \iff \quad G_k \equiv \overline{G}. \]

In other words, \( G \) is \( k \)-s.c. if, and only if, \( G \) is \( k(i) \)-o.s.c., and \( G \) is \( k(i) \)-s.c. if, and only if, it is \( k \)-co-s.c.
3. \(k(i)-\text{Co-Self Complementary Graphs}\)

As usual, let \(P_r\) and \(C_r\) respectively denote a path and a cycle on \(r\) vertices.

Every complete bipartite graph \(K_{m,n}\) is \(2(i)\)-co-s.c. with respect to the partition \(P = \{V_1, V_2\}\), where \(V_1\) consists of two vertices corresponding to the end vertices of an edge and \(V_2\), the other vertices. In particular, every star \(K_{1,n}\), \(n \geq 2\), is \(2(i)\)-co-s.c. A double star is a tree with exactly two vertices of degree greater than one. Every double star is \(2(i)\)-co-s.c., \((V_1\) is composed of the two vertices of degree greater than one), as are the cube \(Q_3\) \((V_1\) is composed of the vertices on two diagonally opposite edges), the paths \(P_3\) (a star), \(P_4\) (a double star), \(P_5\) \((V_1\) is the middle vertices \(P_6(V_1)\) is composed of the two middle vertices and the two end vertices), \(P_7\) \((V_1\) is composed of every other vertex), the cycle \(C_4\) \((V_1\) is composed of two adjacent vertices), \(C_6\) \((V_1\) is composed of two opposite vertices, and \(C_8\) \((V_1\) is composed of every other vertex).

We now state a number of results on \(k(i)\)-co-s.c. graphs without proofs. These are immediate consequences of Corollary 2.2 and the results in [1] as indicated.

**Proposition 3** — (Cf. Proposition 1 of [1]). If a \((P, q)\)-graph \(G\) is \(k(i)\)-co-s.c., then

\[i) \quad G \text{ has a vertex of degree at least } \frac{p(k-1)}{2k}, \text{ and} \]

\[ii) \quad \frac{(k-1)(2p-k)}{2k} \leq q \leq \frac{2p(p-k)+k(k-1)}{4}. \]

**Proposition 4** — (Cf. Corollary 1.1. of [1]) The following statements are true :

\[i) \quad \text{A } (4(i)\text{-co-s.c. tree has order at most four.} \]

\[ii) \quad \text{There are no } (k(i)\text{-co-s.c. trees for } k \geq 5. \]

\[iii) \quad \text{A forest with at least two components is not } (k(i)\text{-co-s.c. for } k \geq 3. \]

\[iv) \quad \text{A connected } 4(i)\text{-co-s.c. unicyclic graph has order at most six.} \]

\[v) \quad \text{A connected } 5(i)\text{-co-s.c. unicyclic graph has order five.} \]

\[vi) \quad \text{There do not exist connected } (k(i)\text{-co-s.c. unicyclic graphs for } k \geq 6.} \]

3.1. **Characterizations of \(k(i)\)-co-s.c. Trees**

As mentioned above, using Corollary 2.2, characterizations of \(k(i)\)-co-s.c. trees can be easily deduced from the characterizations of \(k\)-s.c. trees obtained in [1]. We simply state the results without proofs.

**Proposition 5** — (Cf. Theorem 3 of [1]) A tree of order \(P\) is \(2(i)\)-co-s.c. if, and only if, one of the following holds:-

\[i) \quad \text{\(P = 7\) and the tree is either } P_7 \text{ or it consists of a path } v_1v_2v_3v_4v_5v_6 \text{ together with a pendent edge } v_4v_7. \]

\[ii) \quad \text{The vertex set of } T \text{ can be partitioned into two sets } V_1 \text{ and } V_2 \text{ such that one of } (a), \]

\((b)\text{ and } (c) \text{ is true.} \]

\[(a) \quad p \geq 5, \, |V_1| = 1 \text{ and the subgraph } \langle V_2 \rangle \text{ has exactly } (P-1)/2 \text{ components.} \]
(b) \( V_1 \) consists of exactly two nonadjacent vertices and exactly one component of \( \langle V_2 \rangle \) is 
\( K_2 \), and all others, of which there is at least one, are \( K_1 \)'s.

(c) \( V_2 = \{u, v\} \) where \( uv \) is an edge and \( V_1 \) is independent.

**Corollary 5.1** — (Cf. Corollary 3.1 of [1]) Any \( 2(i)-co-s.c. \) tree has diameter at most six.

Let \( T_1 \) and \( T_2 \) be two trees obtained from a path \( P_5 : v_1v_2v_3v_4v_5 \) as follows:

\[ T_1 \text{ is } P_5 \text{ plus two pendent edges } v_3v_6 \text{ and } v_3v_7; \text{ and} \]

\[ T_2 \text{ is } P_5 \text{ plus two pendent edges } v_2v_6 \text{ and } v_4v_7. \]

**Corollary 5.2** — (Cf. Corollary 3.2 of [1]) All trees on \( P \) vertices, \( 3 \leq p \leq 7 \), except \( T_1 \) and \( T_2 \) are \( 2(i)-co-s.c. \).

In a similar way, one can deduce characterizations of \( k(i)-co-s.c. \) forests and unicyclic graphs from the known results. The reader is referred to [1] for details.

### 4. \( k(i)\)-Self Complementary Trees

We first obtain some necessary conditions for a tree to be \( k(i)-s.c. \).

**Proposition 6** — Suppose \( P = \{V_1, V_2, \ldots, V_k\} \) is a partition of the vertex set \( V(T) \) of a tree \( T \), and \( G_r = \langle V_r \rangle \) be the subgraph induced by \( V_r \), \( 1 \leq r \leq k \). If \( T_{k(i)}^P \equiv T \), then -

1. \( \beta_0(G_r) \leq 2 \) for \( 1 \leq r \leq k \), where \( \beta_0(G_r) \) is the independence number of \( G_r \);
2. each set \( V_r \) in \( P \) contains at most four vertices; and
3. each \( G_r \) is exactly one of the following graphs:

   (a) \( rK_1, 1 \leq r \leq 2 \), (b) \( K_2 \), (c) \( K_1 \cup K_2 \), (d) \( K_{1,2} \), (e) \( P_4 \).

**Proof:**

1. If \( \beta_0(G_r) \geq 3 \) for some \( G_r \), then \( T_{k(i)}^P \) contains a cycle, which is not true.
2. This follows from (i), since otherwise \( \beta_0(G_r) \geq 3 \) for some \( G_r \).
3. By (i) and (ii), each \( G_r \) has at most four vertices with \( \beta_0(G_r) \leq 2 \). If a \( G_r \) has at most three vertices, clearly, \( G_r \) must be one of the graphs mentioned in (a)-(d). If a \( G_r \) has four vertices, then it must have exactly three edges, since otherwise \( \overline{G_r} \) has a cycle.

   Also \( G_r \neq K_{1,3} \) by (i). This implies \( G_r \equiv P_4 \).

**Corollary 6.1** — If a tree \( T \) of order \( P \) is \( k(i)-s.c. \), then \( k \leq p \leq 4k \).

The following result gives another necessary condition for a tree to be \( k(i)-s.c. \).

**Proposition 7** — Let \( T \) be a \( k(i)-s.c. \) tree for some \( k \geq 2 \), with respect to a partition \( P = \{V_1, V_2, \ldots, V_k\} \) of the vertex set \( V \).

Let \( v \) be a vertex of maximum degree in \( T \), and \( f \) be an isomorphism of \( T \) onto \( T_{k(i)}^P \). Then \( \deg v \leq k \) if \( f(v) = v \); and
\[ \deg v \leq \frac{k + 4}{2} \text{ if } f(v) \neq v. \]

**Proof**: Let \( v \in V_1 \). Then by Proposition 6, \( v \) has at most two neighbours in \( V_1 \). We now consider various cases.

**Case 1**: \( f(v) = v \).

If \( v \) has two neighbours in \( V_1 \), then by Proposition 6, the subgraph \( \langle V_i \rangle \) is either \( P_4 \) or \( K_{1, 2} \), and the degree of \( v \) in \( T_{k(i)}^P \) is less than that in \( T \), a contradiction. Hence, \( v \) has at most one neighbour in \( V_1 \). Also, \( v \) can have at most one neighbour in each of the other sets in \( P \), for otherwise there will be a cycle in \( T_{k(i)}^P \). Thus \( \deg v \leq k \).

**Case 2**: \( f(v) = u \neq v \).

**Subcase 2.1**: \( u \in V_1 \). In this case there exists at most one set other than \( V_1 \) containing neighbours of both \( u \) and \( v \). Each other set has at most one neighbour of either \( v \) or \( u \). This implies that \( \deg v \leq 2 + 1 + \frac{k - 2}{2} = \frac{k + 4}{2} \).

**Subcase 2.2**: \( u \in V_2 \).

**Subcase 2.2.1 (a)**: \( v \) is adjacent to \( u \) in \( T \).

In this situation there is no set containing neighbours of both \( u \) and \( v \).

Hence, \( \deg v \leq 2 + 1 + \frac{k - 2}{2} = \frac{k + 2}{2} \).  

**Subcase 2.2.1 (b)**: \( v \) is adjacent to a vertex different from \( u \) in \( V_2 \). Now, if \( u \) has a neighbour in \( V_1 \), then as in Subcase 2.2.1(a), \( \deg v \leq \frac{k + 4}{2} \). Otherwise, there exists at most one set \( V_j, j \neq 1, 2 \) in \( P \) having neighbours of both \( u \) and \( v \). Also, if \( v \) has two neighbours in \( V_1 \), then \( u \) is adjacent to a vertex in a set \( V_r \) which has no neighbours of \( v \), since \( \deg u \) in \( T_{k(i)}^P \) must be equal to \( \deg v \) in \( T \). Hence \( \deg v \leq 2 + 1 + \frac{k - 4}{2} = \frac{k + 4}{2} \).

**Subcase 2.2.2**: \( v \) has no neighbours in \( V_2 \).

In this case there may exist at most two sets different from \( V_1 \) and \( V_2 \) having neighbours of both \( u \) and \( v \). Hence,

\[ \deg v \leq 2 + 1 + 1 + \frac{k - 4}{2} = \frac{k + 4}{2}. \]

Thus in all cases, \( \deg v \leq \frac{k + 4}{2} \).

**Corollary 7.1**: If a tree \( T \) is \( k \)-co-s.c. for some \( k \geq 3 \), then \( \Delta(T) \leq k \).

We now characterize \( 2(i) \)-s.c. trees. In fact, one can find them.

**Proposition 8**: There are exactly five trees which are \( 2(i) \)-s.c.
PROOF: Let $T$ be a $2(i)$-s.c. tree of order $P$ with respect to the partition $P = \{V_1, V_2\}$ of the vertex set $V$ of $T$. Let $G_r$ be the subgraph of $T$ induced by $V_r$, $r = 1, 2$. By Corollary 6.1, $p \leq 8$. We consider various cases.

Case 1 — $P = 8$.

In this case it follows by Proposition 6 that every partition $P = \{V_1, V_2\}$ of $V$ is such that $|V_1| = |V_2| = 4$. Again, by Proposition 6, $G_1 = G_2 = P_4$. The only tree that is $2(i)$-s.c. with respect to such a partition is the one in which a vertex of degree 2 in $G_1$ is adjacent to a vertex of degree 1 in $G_2$ as shown in Fig. 2.

![Fig. 2](image)

Thus, there is exactly one tree on eight vertices which is $2(i)$-s.c.

Case 2 — $P = 7$.

In this case, it follows by Proposition 6 that $|V_1| = 4$ and $|V_2| = 3$, $G_1 = P_4$ and $G_2$ is either $K_1 \cup K_2$ or $K_{1,2}$. Both choices of $G_2$ yield a $T_{2(i)}^p$ in which the number of edges is different from that in $T$. This proves that no tree on seven vertices is $2(i)$-s.c.

Case 3 — $P = 6$.

As above we observe that either $|V_1| = 4$ and $|V_2| = 2$, or $|V_1| = |V_2| = 3$. In the first case, $G_1 = P_4$ and $G_2$ is either $K_2$ or $2K_1$. Both the choices of $G_2$ yield a $T_{2(i)}^p$ with different number of edges than $T$. If $|V_1| = |V_2| = 3$, then $G_1 = K_{1,2}$ and $G_2 = K_1 \cup K_2$ or vice-versa in order that both $T$ and $T_{2(i)}^p$ have the same number of edges. There are exactly two trees on six vertices which are $2(i)$-s.c. with respect to such a partition. (See Fig. 3)

![Fig. 3](image)
Case 4 — $P = 5$.

In this case either $|V_1| = 4$ and $|V_2| = 1$ or $|V_1| = 3$ and $|V_2| = 2$. In the former case $G_1 = P_4$, and $G_2 = K_1$, and no tree is $2(i)$-s.c. with respect to such a partition. In the latter case, $G_1 = K_{1,2}$ or $K_1 \cup K_2$ and $G_2 = 2K_1$ or $K_2$. The possible choices such that both $T$ and $T_{2(i)}^P$ have the same number of edges are $G_1 = K_{1,2}$, $G_2 = 2K_1$, and $G_1 = K_1 \cup K_2$ and $G_2 = K_2$. One can easily verify that no tree obtained by joining a vertex of $G_1$ to a vertex of $G_2$ is $2(i)$-s.c. Thus, no tree on five vertices is $2(i)$-s.c.

Case 5 — $2 \leq p \leq 4$.

Out of the four trees in this case, only $P_4$ and $K_2$ are $2(i)$-s.c. This proves the Proposition.

From Corollary 2.2, we have

Corollary 8.1 — There are exactly five trees which are $2$-co-s.c., and these are the trees which are $2(i)$-s.c.

We now characterize $3(i)$-s.c. trees.

**Proposition 9** — There are exactly seven $3(i)$-s.c. trees.

**Proof**: Let $T$ be a $3(i)$-s.c. tree of order $P$ with respect to the partition $\{V_1, V_2, V_3\}$ of $V(T)$. Then by Proposition 6 $3 \leq p \leq 12$. Suppose $G = \langle V_r \rangle$, $1 \leq r \leq 3$. We consider various cases. In the proof we only consider partitions such that

(i) Both $T$ and $T_{k(i)}^P$ have the same number of edges.

Case 1 — $P = 12$. By Proposition 6, $G_r = P_4$, $r = 1, 2, 3$, and one can verify that no tree with this partition is $3(i)$-s.c.

Case 2 — $P = 11$. Again by Proposition 6, $G_1 = G_2 = P_4$, and $G_3$ is either $K_{1,2}$ or $K_2 \cup K_1$. There is no $3(i)$-s.c. tree with this partition.

Case 3 — $P = 10$. As above we find that the only possible partitions are as follows:

$G_1 = P_4$, $G_2 = K_{1,2}$, $G_3 = K_2 \cup K_1$, or

$G_1 = P_4$, $G_2 = P_4$, $G_3 = K_2$ or $\overline{K}_2$

With these partitions, we find that the tree $T$ is not $3(i)$-s.c.

Case 4 — $P = 9$. In this case the only possible partitions are

$G_1 = G_2 = P_4$, $G_3 = K_1$,

$G_1 = P_4$, $G_2 = K_{1,2}$, $G_3 = \overline{K}_2$
\[ G_1 = G_2 = G_3 = K_{1,2} \quad \text{or} \quad \overline{K}_{1,2} \]

The first one yields a 3\((i)\)-s.c. tree as follows:

![Diagram of 3\((i)\)-s.c. tree](image)

Fig. 4

The other partitions do not give 3\((i)\)-s.c. trees.

Case 5 — \(P = 8\). Among all possible partitions, we find that the only partition where 
\(G_1 = P_4, G_2 = K_2\) and \(G_3 = \overline{K}_2\) yield a 3\((i)\)-s.c. tree as follows:

![Diagram of 3\((i)\)-s.c. tree](image)

Fig. 5

Case 6 — \(P = 7\). In this case, the only partition which yields a 3\((i)\)-s.c. tree is \(G_1 = P_3, G_2 = K_2 \cup K_1, G_3 = K_1\).

![Diagram of 3\((i)\)-s.c. tree](image)

Fig. 6

There are exactly three trees which are 3\((i)\)-s.c. with respect to such a partition, and they are as follows:

![Additional diagrams of 3\((i)\)-s.c. trees](image)

Fig. 6

Case 7 — \(P = 6\). In this case, the only possible partitions are
\[ G_1 = P_4, \ G_2 = G_3 = K_1 \]
\[ G_1 = K_{1,2}, \ G_2 = \overline{K_2}, \ G_3 = K_1. \]

There are no 3(i)-s.c. trees with such partitions.

Case 8 — \( P = 5 \). In this case there is exactly one tree namely, \( P_5 \) which is 3(i)-s.c. with respect to the partition \( G_1 = K_2, \ G_2 = K_1, \ G_3 = \overline{K_2}. \)

Case 9 — \( 3 \leq p \leq 4 \). There is exactly one tree, namely, \( P_3 \) which is 3(i)-s.c. trivially.

By Corollary 2.2, we have

Corollary 9.1 — There are exactly seven trees which are 3-co-s.c.

In general, characterization of \( k(i) \)-s.c. trees for \( k \geq 4 \) appears to be difficult. We now state some open problems.

Problem 1 : Characterize \( k(i) \)-s.c. trees for \( k \geq 4 \).

Problem 2 : Characterize trees which have nontrivial \( k(i) \)-complement which are also trees.

Problem 3 : Characterize graphs which are \( k(i) \)-complements of \( i \) trees, and \( ii \) cycles.

5. \( k(i) \)-SELF COMPLEMENTARY UNICYCLIC GRAPHS

We consider only connected unicyclic graphs here.

First we give some necessary conditions for a unicyclic graph to be \( k(i) \)-s.c., and then determine all 2(i)-s.c. unicyclic graphs.

Proposition 10 — Let \( G \) be a unicyclic graph which is \( k(i) \)-s.c. with respect to the partition \( P = \{V_1, V_2, \ldots, V_k\} \) of \( V(G) \). Then

i) \( |V_r| \leq 5 \) for \( 1 \leq r \leq k \), and

ii) for at most one \( V_r \), \( |V_r| = 5 \), and in this case the subgraph \( \langle V_r \rangle \) is a self complementary graph on five vertices.

Proof: If a set has six vertices, or two sets in \( P \) have at least five vertices each, then \( G_{k(i)}^P \) has at least two cycles, a contradiction. Now, suppose \( |V_1| = 5 \). Then the subgraph \( \langle V_1 \rangle \) should have exactly five edges, for otherwise \( G_{k(i)}^P \) will have more than one cycle. Also, the subgraph \( \langle V_1 \rangle \) must be a self complementary graph on five vertices, for otherwise, the lengths of the cycles in \( G \) and \( G_{k(i)}^P \) will not be the same.

Corollary 10.1 — If \( G \) is a \( k(i) \)-s.c. unicyclic graph of order \( P \) then, \( k \leq p \leq 4k + 1 \).

Proposition 11 — Let \( G \) be a unicyclic graph \( P = \{V_1, V_2, \ldots, V_k\} \ k \geq 3 \) be a partition of the vertex set \( V(G) \) of \( G \), and \( G \equiv G_{k(i)}^P \). Then \( \Delta(G) \leq k + 2 \), where \( \Delta(G) \) is the maximum degree of
G. Further, if for any vertex \( v \) of maximum degree in \( G \) and any isomorphism \( f \) of \( G \) onto \( G^P_{k(i)}, f(v) \neq v \), then \( \Delta(G) \leq \frac{k+6}{2} \).

**Proof:** First we make two observations:

1. A vertex in a self complementary graph on five vertices has degree at most three. This together with Proposition 10 implies that a vertex in any \( V_j, 1 \leq j \leq k \), has at most three neighbours in \( V_j \).

2. A vertex \( v \) in any \( V_j, 1 \leq j \leq k \), can be adjacent to at most two vertices in any \( V_s, s \neq j \). For otherwise, \( G^P_{k(i)} \) will have more than two cycles.

Let \( v \) be a vertex of maximum degree in \( G \), and \( v \in V_1 \). We consider various cases.

**Case 1** — \( f(v) = v \).

Since \( v \) has the same degree both in \( G \) and \( G^P_{k(i)} \), we have

\[ (A) : \text{the number of neighbours of } v \text{ in } V_1 \text{ both in } G \text{ and } G^P_{k(i)} \text{ must be equal.} \]

Since \( |V_1| \leq 5 \), clearly \( (A) \) implies that in \( G \), \( v \) cannot have three neighbours in \( V_1 \). We now consider various subcases.

**Subcase 1.1** — In \( G \), \( v \) has no neighbours in \( V_1 \).

It follows from \( (A) \) that \( |V_1| = 1 \), and there can be at most two other sets in \( P \), say \( V_2, V_3 \) in each of which \( v \) may have two neighbours. Also, \( v \) may have at most one neighbour in each of the other sets. Hence, \( \deg v \leq 0 + 2 + 2 + k - 3 = k + 1 \).

**Subcase 1.2** — \( v \) has exactly one neighbour in \( V_1 \).

As in Subcase 1.1, \( v \) may have two neighbours in each of the sets \( V_2, V_3 \), and at most one neighbour in each of the remaining sets. So \( \deg v \leq 1 + 2 + 2 + k - 3 = k + 2 \).

**Subcase 1.3** — \( v \) has exactly two neighbours in \( V_1 \).

In this case, Proposition 10 and the condition \( (A) \) imply that the subgraph \( \langle V_j \rangle \) of \( G \) should be a self complementary graph on five vertices. So, \( v \) can have at most one neighbour in each of the other sets, and \( \deg v \leq 2 + k - 1 = k + 1 \).

**Case 2** — \( f(v) = u \neq v \).

**Subcase 2.1** \( (B) \) — \( v \) has three neighbours in \( V_1 \).

There may exist at most one set \( V_j, 2 \leq j \leq k \), say \( V_2 \) in which \( v \) has two neighbours. (For otherwise \( G^P_{k(i)} \) will have more than one cycle).

**Subcase 2.1.1** — \( v \) has two neighbours in \( V_2 \).

In this case it follows by \( (B) \) that

1. \( v \) belongs to a triangle in \( G \), and since \( f(v) = u \), \( u \) should also belong to a triangle in \( G^P_{k(i)} \), and

2. \( u \) belongs to either \( V_1 \) or \( V_2 \). (See Fig. 7)
Fig. 7

\( u \) belongs to either \( V_1 \) or \( V_2 \). (See Fig. 7)

**Subcase 2.1.1 (a) \( u \in V_1 \)**

If \( u \) has a neighbour in \( V_2 \), then both \( u \) and \( v \) cannot have neighbours in any \( V_j, 3 \leq j \leq k \). Also, \( u \) has exactly one neighbour in two sets in which \( v \) has no neighbours, since \( \deg v \) in \( G \) is equal to \( \deg u \) in \( G_{k(i)}^P \). The remaining sets in \( P \) contain at most one neighbour of either \( u \) or \( v \).

Hence, \( \deg v \leq 3 + 2 + \frac{k - 4}{2} = \frac{k + 6}{2} \).

On the other hand, if \( u \) has no neighbours in \( V_2 \), then there exists at most one set \( V_j, 3 \leq j \leq k \), containing neighbours of both \( u \) and \( v \). As above, we find that \( u \) has exactly one neighbour in three remaining sets in \( P \) in which \( v \) has no neighbour. The other sets in \( P \) contain a neighbour of at most one of \( u \) or \( v \). Hence, \( \deg v \leq 3 + 2 + 1 + \frac{k - 6}{2} = \frac{k + 6}{2} \).

**Subcase 2.1.1. (b) \( u \in V_2 \)**

In this case both \( u \) and \( v \) cannot have neighbours in any \( V_j, 3 \leq j \leq k \). (For otherwise, \( G_{k(i)}^P \) will have more than one cycle). Since \( \deg v \) in \( G \) is equal to \( \deg u \) in \( G_{k(i)}^P \), it follows by (B) that if \( u \) has \( r \) neighbours (one each) in some \( r \) sets in the collection \( P' = \{ V_3, V_4, \ldots, V_k \} \), then \( v \) has at most \((r - 2)\) neighbours (one each) in some other \((r - 2)\) sets in \( P' \). This implies that in \( G \), \( \deg v \leq 3 + 2 + \frac{k - 4}{2} = \frac{k + 6}{2} \).

**Subcase 2.1.2 — \( v \) has at most one neighbour in each set \( V_j, 2 \leq j \leq k \).** In this case, both \( v \) and \( u \) can have neighbours in at most one set \( V_j, 2 \leq j \leq k \). Hence, as in Subcase 2.1.1., we have
\[ \text{deg } v \leq 3 + 1 + \frac{k - 2}{2} = \frac{k + 6}{2} \]

\textbf{Subcase 2.2 — } \text{\textit{V \textnormal{} has at most two neighbours in } } V_1. \\

In this case, there exist at most two other sets, say \( V_2, V_3 \) in each of which \( v \) may have two neighbours. In this case also, by similar arguments as above, one can show that \( \text{deg } v \leq \frac{k + 6}{2}. \)

We now find all 2\((i)\)-s.c. unicyclic graphs

\textbf{Proposition 12 — } There are exactly eighteen 2\((i)\)-s.c. unicyclic graphs.

\textbf{Proof:} Let \( G \) be a 2\((i)\)-s.c. unicyclic graph of order \( P \) with respect to the partition \( P = \{ V_1, V_2 \} \) of \( V(G) \). By Corollary 10.1, \( p \leq 9 \). We consider various cases, and in each case, we consider partitions satisfying the condition

\[ (D): \text{the number of edges in } G \text{ and } G_{k(i)}^P \text{ are equal.} \]

\textbf{Case 1 — } \( P = 9. \)

The only possible partition of \( V(G) \) is such that \( |V_1| = 5 \) and \( |V_2| = 4 \). There is no 2\((i)\)-s.c. unicyclic graph with respect to such a partition.

\textbf{Case 2 — } \( P = 8. \)

The only possible partitions are such that \( |V_1| = |V_2| = 4 \).

\textbf{Subcase 2.1 — } Each subgraph \( \langle V_1 \rangle \) and \( \langle V_2 \rangle \) has exactly three edges. There are exactly four unicyclic graphs \( G_1 - G_4 \) as in Fig. 8 which are 2\((i)\)-s.c. with respect to such partitions.

\textbf{Subcase 2.2 — } The subgraph \( \langle V_1 \rangle \) has four edges and the subgraph \( \langle V_2 \rangle \) has two edges.

In this case, there are exactly five 2\((i)\)-s.c. graphs \( G_5 - G_9 \) as shown in Fig. 8.

\textbf{Case 3 — } \( P = 7. \) There are no partitions satisfying the condition \( (D). \)

\textbf{Case 4 — } \( P = 6. \)

There exist two types of partitions satisfying the condition \( (D), \) namely \( |V_1| = 5, |V_2| = 1, \) and \( |V_1| = |V_2| = 3. \)

\textbf{Subcase 4.1 — } \( |V_1| = 5, |V_2| = 1. \)

There are exactly two 2\((i)\)-s.c. unicyclic graphs \( G_{10} \) and \( G_{11} \) with this partition as shown in Fig. 8.

\textbf{Subcase 4.2 — } \( |V_1| = |V_2| = 3. \)

In this case there are exactly five graphs \( G_{12} - G_{16} \) as in Fig. 8.

\textbf{Case 5 — } \( P = 5. \) There are two possibilities for the sets as follows:

\( |V_1| = 4, |V_2| = 1 \) and \( |V_1| = 3, |V_2| = 2. \)
Out of these only the latter gives a 2(i)-s.c. graph $G_{17}$ as in Fig. 8.

Case 6 — $p \leq 4$. In this case $G_{18}$ is the only graph which is 2(i)-s.c.

Corollary 12.1 — There are exactly eighteen 2-co-s.c. unicyclic graphs namely, those in Fig. 8.

REFERENCE