Nārāyaṇa’s Generalisation of Mātrā-vṛttas-prastāra and the Generalised Virahānka-Fibonacci Representation of Numbers

Raja Sridharan*, R Sridharan** and M D Srinivas***

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Abstract

In his Vṛttajātisamuccaya (c. 600 AD), Virahānka discussed the prastāra or enumeration of mātrā-vṛttas or moric metres, metres of value \( n \) consisting of long and short syllables, laghu and guru, assigning value 1 to laghu and 2 to guru, the value of the metre being equal to the sum of the values of its constituent syllables. Virahānka noted that the saṃkhyākas or number of rows in the prastāra is given by a sequence of numbers, which were rediscovered much later in the 13th century by Fibonacci in a different context. Similar problems of enumeration were considered in the context of tāla-prastāra in music by Śāṅgītaratnākara (c. 1250 AD).

A general mathematical treatment of most of the combinatorial problems considered in the earlier literature was given by Nārāyaṇa Paṇḍita in his Gaṇitakumudī (c.1356 AD). One of the problems discussed by Nārāyaṇa is the prastāra of a general class of mātrā-vṛtta where, apart from the syllabic units laghu and guru (of values 1, 2), there could be other syllabic units (such as pluta, etc.) of values 3, 4,..., \( q \). This is also a general form of tāla-prastāra, but it does not subsume the specific tāla-prastāra considered by Śāṅgītaratnākara in his Saṅgītaratnākara, where the tāla-units have values 1, 2, 4 and 6.

The questions of naṣṭa and uḍḍiṣṭa, which consist of finding the row number associated with a generalised mātrā-vṛtta and its converse, in relation to this prastāra, have been dealt with by Nārāyaṇa Paṇḍita by means of a tabular form called unmeru. In this paper, we shall show that these naṣṭa and uḍḍiṣṭa processes are indeed based upon a certain representation of each number uniquely as a sum of the generalised Virahānka-Fibonacci numbers, i.e., saṃkhyākas associated with the generalised mātrā-vṛtta-prastāra of Nārāyaṇa.

Key words: Generalised mātrā-vṛttas, Prastāra or enumeration, Generalised Virahānka-Fibonacci numbers, Naṣṭa, Uḍḍiṣṭa, Unmeru, Virahānka-Fibonacci representation of numbers.

1. INTRODUCTION

Indian combinatorics goes as far back as Piṅgala (c. 300 BC), who introduced six combinatorial tools (pratayayas) to study Sanskrit prosody. Among these is prastāra, an enumeration rule for generating all possible metric patterns, naṣṭa and uḍḍiṣṭa the processes of finding any metrical pattern given its row number and its converse, in the prastāra. The technique of pratayayas introduced by Piṅgala provided a model for many enumeration problems discussed by later Indian mathematicians (Sridharan, 2005).

Calling the shorter syllable laghu (L) and the longer syllable guru (G), Piṅgala enumerated the \( 2^n \) metres of length \( n \) consisting of Ls and Gs. In view of the fact that it takes twice as much time to utter a long syllable (G) as it takes to utter the short syllable (L), Indian prosodists also...
considered the enumeration of mātrā-vrttas, metres of value n consisting of long and short syllables, assigning value 1 to L and 2 to G, the value of the metre being equal to the sum of the values of its constituent syllables. In his Vṛttajātisamuccaya (c. 600 AD), Virahānka showed that the saṅkhyaṅkas, or number of rows in the prastāra, is given by a sequence of numbers, which were rediscovered much later in the 13th century by Fibonacci in a different context (Sridharan, 2006).

In his treatise on music Sangitaratnakara, Śāṅgadeva (c. 1250 AD) discussed the enumeration of both tānas (musical phrases) and tālas (musical rhythms). The tāna-prastāra considered by Śāṅgadeva is essentially a procedure for enumerating the n! permutations of the symbols 1, 2, ..., n. The tāla-prastāra is a generalisation of the mātrā-vṛutta-prastāra, where Śāṅgadeva considers all possible rhythmic forms composed of the units druta, laghu, guru and pluta with values 1, 2, 4 and 6 respectively (Raja Sridharan et al., 2010).

Prior to the seminal work Gaṇitaśāstra of Nārāyaṇa Paṇḍita (c. 1356 AD), most of the work in combinatorics had its basis in practical problems. Nārāyaṇa Paṇḍita was the first to treat these problems from a general mathematical viewpoint, generalising and unifying many of the earlier results. Specifically, Nārāyaṇa considered prastāras of syllabic metres with more than two types of syllables generalising the work of Pīnagala, prastāras of permutations with repetitions generalising the work of Śāṅgadeva on svara-prastāra, and also prastāras of combinations which were briefly considered earlier by Varāhamihira (c. 550 AD) and Bhaṭṭotpala (c. 950 AD) (Raja Sridharan et al., 2012). In this paper we shall study Nārāyaṇa Paṇḍita’s generalisation of the mātrā-vṛutta-prastāra.

In Chapter XIII of Gaṇitaśāstra on Ankapāśa (net of numbers), Nārāyaṇa Paṇḍita gives an abstract mathematical formulation encompassing most of the combinatorial problems considered in earlier literature. After listing the various pratyayas, Nārāyaṇa defines a number of pankits (sequences) and merus (tabular figures) that are employed in various combinatorial problems. Nārāyaṇa then considers different kinds of prastāras which generalise those considered in prosody and music. He formulates the problem as one of enumerating the various possibilities which arise when there are p slots or places (sthānas) in which the q digits 1, 2, ..., q, are placed, subject to various conditions.

One of the problems discussed by Nārāyaṇa is the prastāra of a general class of mātrā-vṛttas where, apart from the syllabic units laghu and guru (of values 1, 2), there could be other syllabic units (such as pluta, etc.) of values 3, 4, ..., q. This is also a general form of tāla-prastāra, but it does not subsume the specific tāla-prastāra considered by Śāṅgadeva in Saṅgitaratnakara, where the tāla-units have values 1, 2, 4 and 6. Before discussing the prastāras considered by Nārāyaṇa, we first give a brief description of mātrā-vṛttas and their prastāra.

2. Mātrā-vṛttas and their prastāra

In the case of varṇa-vṛttas or syllabic metres, it is the number of syllables which are fixed and the prastāra or the enumeration consists of all possiblemetrical forms which are characterised by a sequence of laghus and gurus having the same number of syllables. In the case of mātrā-vṛttas or moric metres, it is the total value of the metrical form that is fixed, when we assign values of 1 mātrā for each laghu (L) and 2 mātrās for each guru (G).

The rule for the construction of the prastāra for mātrā-vṛttas is stated explicitly in the Vṛttajātisamuccaya of Virahānka (c. 600 AD) and is as follows:

- The first row consists of all Gs if the total value is even and an L followed by all Gs if the total value is odd.
The verses of fact, it can be shown that the 

\begin{align*}
\text{G} & = 1, 1, 2, 3, 5, 8, 13, 21, \ldots \\
\text{V} & = 1, 2, \text{G} \text{ and place an L below that. The elements to} \\
\text{F} & = 1, 1, 2, 3, 5, 8, 13, 21, \ldots \\
\text{S} & = 1, 2, \text{G} \text{ and place an L below that. The elements to} \\
\text{R} & = 1, 1, 2, 3, 5, 8, 13, 21, \ldots \\
\text{N} & = 1, 2, \text{G} \text{ and place an L below that. The elements to} \\
\text{M} & = 1, 1, 2, 3, 5, 8, 13, 21, \ldots \\
\end{align*}

The remaining mātrās to the left are filled by 

\begin{align*}
\text{G} & = 1, 1, 2, 3, 5, 8, 13, 21, \ldots \\
\text{V} & = 1, 2, \text{G} \text{ and place an L below that. The elements to} \\
\text{F} & = 1, 1, 2, 3, 5, 8, 13, 21, \ldots \\
\text{S} & = 1, 2, \text{G} \text{ and place an L below that. The elements to} \\
\text{R} & = 1, 1, 2, 3, 5, 8, 13, 21, \ldots \\
\text{N} & = 1, 2, \text{G} \text{ and place an L below that. The elements to} \\
\text{M} & = 1, 1, 2, 3, 5, 8, 13, 21, \ldots \\
\end{align*}

Go on till the last row is reached which is 

\begin{align*}
\text{G} & = 1, 1, 2, 3, 5, 8, 13, 21, \ldots \\
\text{V} & = 1, 2, \text{G} \text{ and place an L below that. The elements to} \\
\text{F} & = 1, 1, 2, 3, 5, 8, 13, 21, \ldots \\
\text{S} & = 1, 2, \text{G} \text{ and place an L below that. The elements to} \\
\text{R} & = 1, 1, 2, 3, 5, 8, 13, 21, \ldots \\
\text{N} & = 1, 2, \text{G} \text{ and place an L below that. The elements to} \\
\text{M} & = 1, 1, 2, 3, 5, 8, 13, 21, \ldots \\
\end{align*}

Table 1: Prastāra of vṛttas of 5-mātrās

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<tr>
<th></th>
<th>L</th>
<th>G</th>
<th>G</th>
</tr>
</thead>
<tbody>
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<tr>
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<td>L</td>
<td>G</td>
<td>L</td>
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<tr>
<td>7</td>
<td>G</td>
<td>L</td>
<td>L</td>
</tr>
<tr>
<td>8</td>
<td>L</td>
<td>L</td>
<td>L</td>
</tr>
</tbody>
</table>

Virahāṇa also gave the rule for the sanākhyā or the number of rows $S_n$ in the prastāra of vṛttas of value $n$, which is the following:

$$S_n = S_{n-1} + S_{n-2}$$

Taking $S_0 = 1$, and since $S_1 = 1$, and $S_2 = 2$, the above relation generates the Virahāṇa sequence (later rediscovered by Fibonacci): 1, 1, 2, 3, 5, 8, 13, 21, ...

The other pratyayas such as naṣṭa, uddiṣṭa and lagakriyā (the last one being a method for finding the number of metres in the prastāra with a given number of laghus or gurus) for māṭrā-vṛttas are worked out in later texts such as Prākṛtapaṅgala (of unknown authorship composed around 12th century), Vānībhūṣana of Dāmodara (c. 1550) and the commentary of Nārāyanabhaṭṭa (c.1555) on Vṛtaratnākara. In fact, it can be shown that the naṣṭa and uddiṣṭa processes for the māṭrā-vṛttas are indeed based on a certain representation of each number uniquely as a sum of non-consecutive Virahāṇa-Fibonacci numbers (Sridharan 2006). The lagakriyā process for these vṛttas also leads to an interesting relation between binomial coefficients and the Virahāṇa numbers.

3. Nārāyaṇa’s generalisation of māṭrā-vṛttas

Nārāyaṇa presents a generalisation of the māṭrā-vṛttta, which has, apart from laghu and guru with values 1, 2, other elements also (like pluta etc.) which have values 3, 4,.. .. $q$. Instead of using laghus and gurus, Nārāyaṇa considers sequences of variable length $p$ (the number of places (sthānas)) formed from the digits (aṅkas) 1, 2,.. . $q$ where $q$ is the highest or the last digit (antimāṅka), such that their total value or sum (yoga) $n$ is fixed. In other words, each row of the prastāra is an ordered partition of $n$ in terms of 1, 2,.. .. $q$.

Nārāyaṇa refers to this case as

निर्देश-अनिर्देश-निर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देशनिर्देश

To generate all the variations (all possible sequences) composed of digits 1, 2,.. . $q$, where both the total value or sum ($n$) of the sequence, as also the highest digit ($q$) that is allowed, are fixed, while the number of places ($p$) is variable.

If $q = 2$, we get the usual māṭrā-vṛttta-prastāra formed with laghus and gurus which are now denoted by 1, 2 respectively. For instance, the 5-māṭrā-prastāra displayed earlier will appear as shown in Table 2, in Nārāyaṇa’s notation.

In verses 79-80 of Chapter XIII of Gaṇitakaumudi on Ankapāśa (net of numbers), Nārāyaṇa Paṇḍita states the rule for the enumeration of generalised māṭrā-vṛttas as follows:

$\text{\textsuperscript{1}}$

$\text{\textsuperscript{1}}$ The verses of Gaṇitakaumudi have been cited from Kusuba T., Combinatorics and Magic-squares in India: A Study of Nārāyaṇa Paṇḍita’s Gaṇita-kaumudi, Chapters 13-14, Ph.D. Dissertation, Brown University (Unpublished), 1993.
Table 2: Prastāra with \( n = 5 \) and \( q = 2 \)

| 1 | 1 | 2 | 2 |
| 2 | 2 | 1 | 2 |
| 3 | 1 | 1 | 1 |
| 4 | 2 | 2 | 1 |
| 5 | 1 | 1 | 2 |
| 6 | 1 | 2 | 1 |
| 7 | 2 | 1 | 1 |
| 8 | 1 | 1 | 1 |

Write the final number at the beginning and fill out the sum of the numbers on the left. After one has put the smaller number below the first larger number, the rest (of the numbers are brought down) as above; the process [is repeated] till a row all of whose entries are 1s is obtained. This is the enumeration as declared by the ancients well-versed in Bharata.

Nārāyaṇa’s rule of prastāra is essentially the following:

- In the first row of the prastāra, write from the right to left as many last digits \( (q) \) as possible, such that their sum does not exceed \( n \). Follow this procedure the same way with the penultimate digit \( (q - 1) \), etc., till the sum of all the digits is \( n \).

- To go from any row of the prastāra to the next, scanning from the left, identify the first digit (say \( m \)) which is larger than 1. Below that, place the digit less than that by 1 (i.e., \( m - 1 \)). To the right of it, bring down the digits as they are from the row above.

- To the left, start from the highest digit \( q \) and so on (as in the first row) till the total sum of all the digits reaches the value \( n \).

- Repeat the process till the last row of the prastāra is reached, all of whose entries are 1.

This clearly, is a generalisation of the rule of prastāra for the usual mātrā-vṛttas given by Virahānka and others. Here, each prastāra is characterised by both the total value \( n \) and the last digit \( q \). To illustrate this rule, Nārāyaṇa gives two examples which are displayed in Tables 3, 4. The first one is the prastāra of total value \( n = 7 \) and highest digit \( q = 3 \). This is nothing but 7-mātrā-prastāra with laghu, guru and pluta being the syllabic elements. The second one is the prastāra of total value \( n = 7 \) and highest digit \( q = 7 \).

Table 3: Prastāra with total value \( n = 7 \) and the highest digit \( q = 3 \)

<table>
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<th>1</th>
<th>133</th>
<th>12</th>
<th>2122</th>
<th>23</th>
<th>2131</th>
<th>34</th>
<th>3211</th>
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<td>35</td>
<td>12211</td>
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<td>111111</td>
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Table 4: Prastāra with total value \( n = 7 \) and the highest digit \( q = 7 \)

<table>
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4 . The Sāmāśi-pañkti and the Saṅkhyā or the Number of Rows in the Prastāra

In order to discuss the various pratyayas of the generalised mātrā-vṛttas-prastāra, Nārāyaṇa...
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makes use of various sequences (pañktis) and tabular forms (merus) which he introduces right at the beginning of the chapter on combinatorics (ānapāṣa). Of these, the most important is the sāmāsikī-pankti (additive sequence), which is a generalisation of the Virahāṅka-Fibonacci sequence, and gives the sankhyā or the total number of rows in the prastāra. In verses 13-14 of the Chapter on Āṅkāpāṣa, Nārāyaṇa defines the sāmāsikī-pankti as follows:

Below the sāmāsikī-pankti, write 0 followed by 1. Then, the number above the last number is added to the sum in reverse order of those numbers which occupy the places equal in number to the last digit. In this way, all the places in front are to be filled. In the absence of places equal in number to the last digit, the sum of those present is written.

The pātāla-pankti when the last digit is \( q \) is thus defined by the relations

\[
P_0^q = P_1^q = 1
\]

\[
P_r^q = S_{n-1}^q + P_{r-2}^q + \ldots + P_0^q \quad \text{when } 1 < r \leq q.
\]

\[
P_n^q = S_{n-1}^q + P_{n-1}^q + P_{n-2}^q + \ldots + P_{n-q}^q \quad \text{when } n > q.
\]

In verses 78-79, Nārāyaṇa notes the significance of the sāmāsikī-pankti and the pātāla-pankti as follows:

The end of the sāmāsikā sequence gives the number of possible variations (metrical forms of generalised maṭrā-vṛttas, or the number of rows in the prastāra). The numbers of the sequence taken in the reverse order, from the penultimate element are the number of variations ending in 1, etc. The numbers of the pātāla sequence, which correspond to the same last digit and the sum, when taken in reverse order give the (number of occurrences) of 1, etc. The sum of all these is the occurrence of all the digits.

Thus, the sankhyā or the number of rows in the prastāra is given by the last element of the sāmāsikā sequence and the penultimate and other elements in the reverse order give the number of rows in the prastāra which end in digits 1, 2, etc. Similarly the numbers in the pātāla sequence, taken in the reverse order give the number of times the different digits 1, 2, etc. occur in the prastāra. The sum of all the elements of the pātāla sequence,
gives the total number of all the digits occurring in the prastāra. We display below two examples of the sāmāsikī-pankti and pātāla-pankti given by Nārāyaṇa.

Table 5, displays the sāmāsikī-pankti and pātāla-pankti for the prastāra with total value 7 and the last digit 3. This is the generalised mātrā-vṛtta, where we include apart from the syllabic units laghu and guru (of values 1, 2), a third syllabic unit pluta, which has value 3. Recalling the prastāra that we displayed earlier in Table 3, we can easily verify that 44 is the sankhyā or the number of rows in the prastāra; and that the last 24 rows end in 1, the previous 13 rows end in 2 and the first 7 rows end in 3. We can also verify that the digit 1 appears 118 times in the prastāra, 2 appears 56 times, and 3 appears 26 times. The total number of digits appearing in the prastāra is 26+56+118=200.

Table 5: The Sāmāsikī-pankti and pātāla-pankti for n = 7 and q = 3

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>7</th>
<th>13</th>
<th>24</th>
<th>44</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>12</td>
<td>26</td>
<td>56</td>
<td>118</td>
<td></td>
</tr>
</tbody>
</table>

Table 6: The Sāmāsikī-pankti and pātāla-pankti for n = 7 and q = 7

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>12</td>
<td>28</td>
<td>64</td>
<td>144</td>
<td></td>
</tr>
</tbody>
</table>

Unlike in the case of varṇa-vṛtta-prastāra where the number of syllables is fixed to start with, in a mātrā-vṛtta-prastāra where only the total mātrā of the metric form is held fixed, the number of syllables in different rows of the prastāra can be different. This is clearly seen for example in the example of 5-mātrā-prastāra that was displayed in Table 1. In the same way, in the generalised mātrā-vṛtta-prastāra as defined by Nārāyaṇa, the lengths of the sequences of digits that constitute different rows of a prastāra (characterised by a given sum n and last digit q) can be different as can be readily seen from the examples of prastāra displayed in Tables 3, 4. That is why, Nārāyaṇa refers to this case as aniyata-sthāna, where the number of places (the number of elements in each row of the prastāra) is not fixed.

Thus, for a generalised mātrā-vṛtta-prastāra, we need to answer the question as to how many rows in the prastāra (where the total sum is n and the last digit is q), have length p. Nārāyaṇa answers this question by means of the elements of the so called needle sequence (sūcī-pankti) or the arrow-head sequence (nārācikā-pankti), which are also put together in a fish-tabular figure (matsya-meru). The elements $U_{p,q}(r)$ of the needle sequence, as defined by Nārāyaṇa, are essentially multinomial coefficients, the coefficients of various powers $x$ in the expansion of $(1 + x + x^2 + \ldots + x^{q-1})^n$. Nārāyaṇa also gives an interesting expression for the elements of the sāmāsikī-pankti or the generalised Virahānika-Fibonacci sequence of sankhyānkas $S_n^q$ as sums of suitable elements of the sūcī-pankti $U_{p,q}(r)$. We have summarised Nārāyaṇa’s discussion of sūcī-pankti and its application to the study of the generalised mātrā-vṛtta-prastāra in the Appendix.

5. The Unmeru and the Naṣṭa and Udīśa Processes

In verses 84-86 of the Chapter on Ankapāśa, Nārāyaṇa introduces the following unmeru, which helps in carrying out the naṣṭa and udīśa processes in a generalised mātrā-vṛtta-prastāra.

The number of cells in each row, starting with one and increasing by one at each step, are equal to the total sum plus one. In the bottom row, the elements of the
additive sequence (sāmasikī-pankti) are written in order. In the others, the arithmetical sequence (caya-pankti 1, 2, . . . , q . . . ) is written in the reverse (from right to left). In the upper horizontal rows they appear naturally. In those cells where numbers larger than the last digit is to be written, an omission is made. This figure is called the lofty tabular figure (unmeru).

The bottom row of unmeru has a number of entries which is one more than total sum (n) and subsequent rows have one number less at each step. The bottom row is filled with the sāmasikī-pankti or the generalised Virahāṅka sequence of sankhyāṅkas. In the subsequent rows the numbers 1, 2, . . . , q, are written from the right end. In Table 7 and Table 8, we display two examples of unmeru given by Nārāyaṇa.

Table 7: Unmeru for n = 7 and q = 3

Table 8: Unmeru for n = 7 and q = 7

In verses 87-89, Nārāyaṇa presents a rather cryptic description how the naṣṭa process is carried out with the help of unmeru:

\[ 44 - 36 = 8, 8 - 7 = 1, \text{ and } 1 - 1 = 0 \]

Thus the second 1 and 7 are patita and the rest are apatita-Sāṅkhyāṅkas. Note the entry 1 in the first row above the last apatita 24. In the row above the topmost entry of the column of that apatita, move left till you reach the column of the

\[ \text{subtract the row number for which the variation or the corresponding sequence of digits is to be determined)} \]

\[ \text{from the last number of the sāmasikī-pankti (the sequence of sankhyāṅkas or the generalised Virahāṅka sequence).} \]

\[ \text{Then, subtract from the result the next number and so on till no subtraction is possible. The number in the cell which is the intersection of the row and column associated (with the subtracted number) is the first digit of the variation. The rest of the elements of the row are similarly obtained, by repeating the process with unmeru.} \]

The method is essentially as follows: First, subtract the given row number from the last entry of the bottom row of unmeru which consists of the sequence of sankhyāṅkas or the generalised Virahāṅka sequence. Then, from what remains, see if the penultimate number can be subtracted and so on. In this way, amongst the numbers of the bottom row, identify those which are patita (numbers which have been subtracted) and apatita (numbers which have not been subtracted). Then, starting from right, the digit just above the first apatita number is the last element of the desired sequence. Now, go one cell above the top entry of that column and move left till the intersection with the column above the next apatita number is reached. That digit is the penultimate element of the desired sequence of digits. And so on.

Nārāyaṇa’s Example 1: To Find the 36th row of the 7-mātrā-prastāra with highest digit 3

\[ 44 - 36 = 8, 8 - 7 = 1, \text{ and } 1 - 1 = 0 \]

Subtract the naṣṭa (row number for which the variation or the corresponding sequence of digits is to be determined) from the last number of the sāmasikī-pankti (the sequence of sankhyāṅkas or the generalised Virahāṅka sequence). Then, subtract from the result the next number and so on till no subtraction is possible. The number in the cell which is the intersection of the row and column associated (with the subtracted number) is the first digit of the variation. The rest of the elements of the row are similarly obtained, by repeating the process with unmeru.
next apatita 13. Note the corresponding entry 1. And so on. Thus the sequence of digits in 36th row of the prastāra is 21211.

Table 9: Example of naṣṭa process when \( n = 7 \) and \( q = 3 \)

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>13</td>
<td>24</td>
</tr>
<tr>
<td>44</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Nārāyaṇa’s Example 2: To find the row where 2212 appears in the 7-mātrā-prastāra with highest digit 7

64 - 36 = 28, 28 - 16 = 12, 12 - 8 = 4 and 4 - 4 = 0

Thus 16, 8 and 4 are patita and the rest are apatita-saṅkhyaṅkas. We find that the desired sequence in the 36th row of the prastāra is 1141

Table 10: Example of naṣṭa process when \( n = 7 \) and \( q = 7 \)

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
</tr>
<tr>
<td>64</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Nārāyaṇa’s description of the uddiṣṭa process, presented in verse 90, is even more cryptic than his description of naṣṭa process:

\[ \text{Saṅghyāṅkā process:} \]

Those numbers in the (bottom row of) unmeru which were subtracted in the naṣṭa process, if these are subtracted from the number (saṅghyāṅka) of the enumeration, one gets the row-number of a given variation.

The uddiṣṭa process is indeed the reverse of the naṣṭa process as will be clear from the following example:

Nārāyaṇa’s Example: To find the row where 2212 appears in the 7-mātrā-prastāra with highest digit 3:

From the right, we first identify the apatita 13 above which 2 appears. Then we move left in the row above the topmost entry of the column of that apatita till we get 1 and note the apatita number 7. And so on.

Row number = 44 - sum of the patitas = 44 - (24+4+1) = 15

Table 11: Example of uddiṣṭa process when \( n = 7 \) and \( q = 3 \)

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
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</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>13</td>
<td>24</td>
</tr>
<tr>
<td>44</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6 Alternate method for the naṣṭa and uddiṣṭa processes

We here describe an alternative method for the naṣṭa and uddiṣṭa processes, by making use of a correspondence between the various digits 1, 2, . . . , \( q \), and associated sequences of patita and apatita saṅkhyaṅkas. This method is similar to the one discussed in works such as the commentary of Nārāyaṇabhaṭṭa on Vṛttaratnākara for the usual mātrā-ṛttaśas (with only laghus and gurus). Such a method was also used by, Śāṅgadeva for the uddiṣṭa process in the case of tāla-prastāra in Saṅgītaratnākara (Raja Sridharan et al., 2010).
NĀRĀYĀNA’S GENERALISATION OF MĀṬRĀ-VṚTTA-PRĀṬĀRA

Table 12: Signature of different digits as sequences of a and p

<table>
<thead>
<tr>
<th>$S_{r-q}$</th>
<th>$S_{r-(q-1)}$</th>
<th>$S_{r-4}$</th>
<th>$S_{r-3}$</th>
<th>$S_{r-2}$</th>
<th>$S_{r-1}$</th>
<th>$S_{r}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(a)</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>a</td>
<td>p</td>
<td></td>
<td></td>
<td>(a)</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>a</td>
<td>p</td>
<td>p</td>
<td>(a)</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>a</td>
<td>p</td>
<td>p</td>
</tr>
<tr>
<td>$q$</td>
<td>a</td>
<td>p</td>
<td>p</td>
<td>p</td>
<td>p</td>
<td>(a)</td>
</tr>
</tbody>
</table>

Write down the sequence of saṅkhyaṅkas or the generalised Virahāṅka sequence for the prastāra. For the given row-number, identify patita and apatita-saṅkhyaṅkas and place the mark $a$ or $p$ below each saṅkhyaṅka. From the right, scan the sequence of these marks and identify the corresponding element of the sequence of digits, successively, via their signatures as given in Table 12.

We illustrate the method with the following examples which are displayed in Tables 13, 14.

Table 13: To Find the 36th row of the 7-māṭrā-prastāra with highest digit 3

\[
\begin{array}{cccccccc}
1 & 1 & 2 & 4 & 7 & 13 & 24 & 44 \\
1 & 1 & 2 & 4 & p & a & a & a \\
2 & 1 & 2 & 1 & 1 & & & \\
\end{array}
\]

Table 14: To Find the 36th row of the 7-māṭrā-prastāra with highest digit 7

\[
\begin{array}{cccccccc}
1 & 1 & 2 & 4 & 8 & 16 & 32 & 64 \\
a & a & a & p & p & p & p & a \\
1 & 1 & 4 & 1 & & & & \\
\end{array}
\]

We can also similarly devise an alternative method for uddiṣṭa process by using the above correspondence between the various digits and associated sequences of patita and apatita saṅkhyaṅkas. The procedure is as follows.

Write the sequence of saṅkhyaṅkas (the generalised Virahāṅka sequence) such that one number is written above 1, two numbers above 2, and so on and $q$ numbers are written above $q$. Omit the number above 1, and also the first number above each of the digits 2, 3, . . . , $q$, and sum the rest. Subtract the sum from the last entry of the sequence of saṅkhyaṅkas (which is nothing but the number of rows in the given prastāra), and the result is the row-number associated with the given sequence of digits. We illustrate the above process in Tables 15-18.

Table 15: To Find the row where 2212 appears in the 7-māṭrā-prastāra with highest digit 3

Row-number = 44-(1+4+24) = 15

\[
\begin{array}{cccccccc}
1 & 1 & 2 & 4 & 7 & 13 & 24 & 44 \\
2 & 2 & 1 & 2 & & & & \\
\end{array}
\]

Table 16: To Find the row where 322 appears in the 7-māṭrā-prastāra with highest digit 3

Row-number = 44-(1+2+7+24) = 10

\[
\begin{array}{cccccccc}
1 & 1 & 2 & 4 & 7 & 13 & 24 & 44 \\
3 & 2 & 2 & & & & & \\
\end{array}
\]

Table 17: To Find the row where 21211 appears in the 7-māṭrā-prastāra with highest digit 3

Row-number = 44-(1+7) = 36

\[
\begin{array}{cccccccc}
1 & 1 & 2 & 4 & 7 & 13 & 24 & 44 \\
2 & 1 & 2 & 1 & 1 & & & \\
\end{array}
\]

Table 18: To Find the row where 1141 appears in the 7-māṭrā-prastāra with highest digit 7

Row-number = 64-(4+8+16) = 36

\[
\begin{array}{cccccccc}
1 & 1 & 2 & 4 & 8 & 16 & 32 & 64 \\
1 & 1 & 4 & 1 & & & & \\
\end{array}
\]
One important feature that we notice in the above examples of nasṭa and uddiṣṭa processes, is that they are based on certain representations of numbers as sums of the generalised Virahāṅka numbers \( \{S_n\} \).

For the case when \( q = 3 \), we have the representations:

- \( 8 = 1 + 7 \)
- \( 29 = 1 + 4 + 24 \)
- \( 34 = 1 + 2 + 7 + 24 \)

For the case when \( q = 7 \), we have the representation:

- \( 28 = 4 + 8 + 16 \)

These are particular instances of a general result that every number can be decomposed uniquely as a sum of the generalised Virahāṅka-Fibonacci numbers. We shall discuss this, and other mathematical aspects of the generalised mātrā-vṛtta-prastāra, in the next section.

7. The Generalised Virahāṅka-Fibonacci Representation of Numbers

While discussing the mathematics underlying the generalised mātrā-vṛtta-prastāra of Nārāyaṇa, we shall, for simplicity consider the case \( q = 3 \), i.e., there are only three digits, 1, 2, 3, in each row of the prastāra.

7.1 An inductive construction of the prastāra and the saṅkhyā rule

We consider the symbols 1, 2, 3, having the same value as the numbers that these digits represent. We write down all rows consisting of these symbols, such that the total value of each row is \( n \), the value of a row being defined to be the sum of the values of the constituent symbols in the row. A prastāra of metres of value \( n \) is a table enumerating all such ordered partitions (allowing repetitions) of \( n \) in terms of the numbers 1, 2, 3.

The prastāra of rows of total value 0 consists of one row with no symbols. The prastāra of rows of total value 1 consists of a single row with the entry 1. The prastāra of rows of value 2 is given in Table 19. This is obtained by appending the symbol 2 to the prastāra of rows of value 0 and 1 to the prastāra of rows of total value 1.

**Table 19: Prastāra of value 2**

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

The prastāra of rows of total value 3 is given in Table 20. This is obtained by appending the symbol 3 to the prastāra of rows of value 0 and then appending the symbol 2 to the prastāra of rows of value 1 and finally appending 1 to the prastāra of rows of total value 2.

**Table 20: Prastāra of value 3**

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

More generally, we can construct the prastāra of rows of total value \( n > 3 \) inductively as follows: Write down the prastāra of rows of value \( n - 3 \) and append 3, then write down the prastāra of rows of value \( n - 2 \) and append 2, and finally write down the prastāra of rows of value \( n - 1 \) with 1 appended at the end. It follows by induction on \( n \) that this is indeed the prastāra obtained by following Nārāyaṇa’s rule for generalised mātrā-vṛttas total value \( n \), and highest digit \( q = 3 \).

It follows that if \( S_n^q \) is the number of rows in the prastāra of metres of value \( n \), then \( S_0^3 = 1 \), \( S_1^3 = 1 \), \( S_2^3 = 2 \) and

\[
S_n^3 = S_{n-1}^3 + S_{n-2}^3 + S_{n-3}^3, \quad n \geq 3.
\]

We shall now see how this also follows from an argument that uses generating functions.
Let us assume that $a$, $b$, $c$ are non-commuting variables and consider the formal expression

$$1 + (a + b + c) + (a + b + c)^2 + ...$$

This formal sum will consist of monomials $a_1 a_2 \ldots a_n$, where each $a_i$ is $a$, $b$ or $c$. Further, because of non-commutativity we cannot permute the various constituent symbols of the monomials. Now suppose that we assign values 1, 2, 3 to the symbols $a$, $b$, $c$, and define the total value of a monomial to be the sum of the values of its constituent symbols. If we now specialize $a = x$, $b = x^2$, $c = x^3$ in the formal sum, we obtain the following power series

$$1 + (x + x^2 + x^3) + (x + x^2 + x^3)^2 + ...$$

$$= [1 - (x + x^2 + x^3)]^{-1}$$

Now, the co-efficient of the power $x^n$ in the above power series is really the *sankhyānaka* $S^n_3$, as it is equal to the number of monomials of total value $n$ in $a$, $b$, $c$, when we assign the values 1, 2, 3 to the symbols $a$, $b$, $c$. Hence we have the equality

$$1 + (x + x^2 + x^3) + (x + x^2 + x^3)^2 + ...$$

$$= S^3_0 + S^3_1 x + S^3_2 x^2 + ...$$

Therefore we obtain

$$[1 - (x + x^2 + x^3)]^{-1} [S^3_0 + S^3_1 x + S^3_2 x^2 + ...] = 1$$

By equating the coefficients of $x^n$, we obtain the relation

$$S^n_3 = S^n_{n-1} + S^n_{n-2} + S^n_{n-3}, \quad n \geq 3$$

**Remark:** In the case of the usual mātrā-vṛttas, with *laghus* and *gurus* only, with values 1, 2 respectively, the *sankhyānakas* are the usual Virahānaka-Fibonacci numbers \{$S^n_3$\}, which satisfy

$$S^n_0 = 1, S^n_1 = 1, \text{ and } S^n_2 = S^n_{n-1} + S^n_{n-2}, \quad \text{for } n \geq 2$$

These relations for $S^n_3$ can be obtained following a similar argument as above as a consequence of the identity

$$[1 - (x + x^2)]^{-1} [S^3_0 + S^3_1 x + S^3_2 x^2 + ...] = 1$$

### 7.2 Representation of Numbers as Sums of Generalised Virahānaka- Fibonacci Numbers

We first recall the following result for Virahānaka-Fibonacci numbers \{$S^n_3$\}, which was proved in an earlier investigation (Sridharan, 2006) on the mathematical theory of the usual mātrā-vṛttas-prastāra:

**Proposition 1:** Any natural number $n$ is either equal to a Virahānaka-Fibonacci number $S^2_1$ or can be uniquely expressed as a sum of non-consecutive Virahānaka-Fibonacci numbers \{$S^2_1$\}.

Now let us consider the case of the generalised Virahānaka-Fibonacci numbers $S^n_3$. We have $S^3_1 = 1$, $S^3_2 = 2$, $S^3_3 = 4$ and $S^3_4 = S^3_{n+1} + S^3_{n+2} + S^3_{n+3}$, $n \geq 3$, and we have the following:

**Proposition 2:** Any natural number $n$ is either a generalised Virahānaka-Fibonacci number $S^3_1$ or can be uniquely expressed uniquely as a sum of generalised Virahānaka-Fibonacci numbers $S^3_1$, no three of which are consecutive.

To prove this we first need the following Lemma:

**Lemma 1:** For all $i > 0$, we have

$$2S^3_i \geq S^3_{i+1}$$

with equality holding only when $i = 1, 2$.

**Proof:** We have to show that $2S^3_i > S^3_i + S^3_{i+1} + S^3_{i+2}$. It is thus enough to show that $S^3_i > S^3_{i+1} + S^3_{i+2}$. But this follows from the fact that $S^3_i = S^3_{i+1} + S^3_{i+2} + S^3_{i+3} > S^3_{i+1} + S^3_{i+2}$.

**Remark:** We note that the *sankhyānaka* of the *varna-vṛttas* are $S_i = 2^i$ which satisfy $2S_i = S_{i+1}$. The *sankhyānaka* of ordinary mātrā-vṛttas, the Virahānaka-Fibonacci numbers $S^2_i$, also satisfy $2S^2_i > S^2_{i+1}$.

Now, let $n$ be a natural number. We define the canonical decomposition of $n$ as follows: We choose the largest $i_1$ such that $S^3_{i_1} \leq n$. Then, we choose the largest $i_2$ such that $S^3_{i_2} \leq n - S^3_{i_1}$. 
Continuing in this manner, we obtain a decomposition

\[ n = S_{i_1}^3 + S_{i_2}^3 + \ldots + S_{i_k}^3, \]

which we refer to as the canonical decomposition of \( n \).

Since \( 2S_i > S_{i+1}^3 \), every \( S_i \) occurs only once in the canonical decomposition. For, otherwise, if \( S_i \) were to occur more than once in the canonical decomposition of \( n \), we would have at a certain stage of the canonical decompositions, \( i \) as the largest number such that

\[ S_i^3 \leq m - S_i^3, \]

and that \( i \) also satisfies \( S_i^3 < m \). In other words, \( 2S_i \leq m \), which implies that \( S_{i+1}^3 \leq m \), contradicting the fact that \( i \) is the largest number such that \( S_i^3 \leq m \).

A similar argument can also be made to show that three consecutive generalised Virahāṅka-Fibonacci numbers \( \{S_i^3\} \) do not appear in any canonical decomposition.

**Lemma 2:** Let

\[ m = S_a^3 + S_c^3 + \ldots + S_p^3 + S_t^3, \]

where, \( a < c < \ldots < l < p < t \) are natural numbers with no three of \( a, c, \ldots, l, p, \) and \( t \) being consecutive. Then

\[ m < S_{i+1}^3. \]

**Proof:** Under the hypothesis of the lemma, we have to prove that

\[ S_a^3 + S_c^3 + \ldots + S_l^3 + S_p^3 + S_t^3 < S_{i+1}^3. \]

Note that since \( p < t \), we have \( p \leq t - 1 \). It is enough to prove that

\[ S_a^3 + S_c^3 + \ldots + S_l^3 + S_p^3 < S_{i+1}^3 - S_t^3 = S_{i+1}^3 - S_{i-2}. \]

If \( p \leq t - 2 \), we have, by induction

\[ S_a^3 + S_c^3 + \ldots + S_l^3 + S_p^3 < S_{i+1}^3 \leq S_{i-1}^3 \leq S_{i-1}^3 + S_{i-2}, \]

thus proving the result.

If \( p > t - 2 \); since \( p \leq t - 1 \), we have \( p = t - 1 \); and we thus have to prove that

\[ S_a^3 + S_c^3 + \ldots + S_{i-1}^3 + S_{i+1}^3 < S_{i+1}^3. \]

Now since no three consecutive \( \{S_i^3\} \) occur in the decomposition, we have \( l < t - 2 \), and by induction

\[ S_a^3 + S_c^3 + \ldots + S_l^3 < S_{i-2}^3 + S_{i-1}^3 + S_{i+1}^3 = S_{i+1}^3, \]

proving the lemma.

**Remark:** The above lemma also holds for the sankhyāṇkas of the varna-vṛttas \( S_i = 2^i \), and also for the sankhyāṇkas of ordinary mātrā-vṛttas, the Virahāṅka-Fibonacci numbers \( \{S_i^3\} \), where in the latter case we only need to impose the condition that the decomposition does not contain consecutive Virahāṅka-Fibonacci numbers.

Now we state a proposition, which follows immediately from the above lemma.

**Proposition 3:** Let \( m \) be a natural number and

\[ m = S_{i_1}^3 + S_{i_2}^3 + \ldots S_{i_k}^3 \] with \( i_1 < i_2 < \ldots < i_k \)

\[ = S_{j_1}^3 + S_{j_2}^3 + \ldots S_{j_r}^3 \] with \( j_1 < j_2 < \ldots < j_r \)

be two decompositions of \( m \) such that both the decompositions do not contain three consecutive sankhyāṇkas \( \{S_i^3\} \). Then the two decompositions are the same. In particular, any decomposition of \( m \) having no three consecutive sankhyāṇkas \( \{S_i^3\} \) is the canonical decomposition.

If we associate the \((n+1)\)-tuple \( (a_0, a_1, a_2, \ldots, a_n) \), where each \( a_i \) is either 0 or 1, with the number \( a_0S_0^3 + a_1S_1^3 + a_2S_2^3 + \ldots a_nS_n^3 \), then we can formulate the result regarding the existence and uniqueness of the generalised Virahāṅka-Fibonacci decomposition as follows:
Proposition 4: There exists a bijection between numbers $0, 1, 2, \ldots, S_n^0 - 1$, and the $(n + 1)$-tuples $(a_0, a_1, a_2, \ldots, a_n)$, where each $a_i$ is either 0 or 1, satisfying the following conditions:

(i) $a_0 = 0, a_n = 0$.

(ii) No three consecutive $a_i$'s are 1.

The bijection is obtained by associating each number $k$ (such that $0 \leq k \leq S_n^0 - 1$) with its canonical decomposition.

Remark: We can also prove that the above assignment is a bijection by showing that the cardinality of the set of all $(n + 1)$-tuples satisfying (i) and (ii) is $S_n^0$, which can be done via a simple argument using induction.

In the above discussion, for the sake of simplicity, we have considered the case of the generalised mātra-vṛtta-prastāra of 3 digits, 1, 2, 3, or, to use Nārāyaṇa’s terminology, the case where the maximum digit $q = 3$. All the results which were demonstrated above for $q = 3$, can be easily generalised to the case where $q$ is any positive integer greater than or equal to 2. We shall here just state (without proof) the basic result on the representation of any natural number uniquely as a sum of the generalised Virahāṅka-Fibonacci numbers $\{S_n^k\}$:

Proposition 5: Let $q \geq 2$ be an integer. Then, every natural number is either a generalised Virahāṅka-Fibonacci number $S_n^k$ for some $k > 0$, or can be uniquely expressed as a sum of generalised Virahāṅka-Fibonacci numbers $\{S_1^1, S_2^1, S_3^1, \ldots\}$, under the condition that the sum does not contain any set of $q$ consecutive generalised Virahāṅka-Fibonacci numbers.

7.3 The Mathematical Basis of the naṣṭa and Uḍḍiṣṭa Processes

We again restrict ourselves, for the sake of simplicity, to the case of the generalised mātra-vṛtta-prastāra of the three digits, 1, 2, 3. We propose to enumerate the rows of the prastāra of value $n$, using symbols 1, 2, 3, starting from the last row (consisting of $n$ ones), numbering the last row as 0, the penultimate row as 1, and so on. If the rank of the row under our enumeration is $t$, then the numbering of the row in the usual enumeration (starting from the top row and taking its row-number as 1) is $S_n^0 - t$.

The process of naṣṭa involves writing down the row explicitly (without the aid of the prastāra) given the row-number associated with it.

In a prastāra of value $n$:

i The last symbol of the $m$-th row is 1 if and only if $m < S_{n-1}^0$, and the penultimate symbol of the $m$-th row is the last symbol of the $m$-th row in the prastāra of value $n - 1$ (where the last row consists of $n - 1$ ones).

ii The last symbol of the $m$-th row is 2 if and only if $S_{n-1}^0 \leq m < S_{n-1}^0 + S_{n-2}^0$, and the penultimate symbol of the $m$-th row is the last symbol of the $(m - S_{n-1}^0)$-th row in the prastāra of value $n - 2$ (where the last row consists of $n - 2$ ones).

iii The last symbol of the $m$-th row is 3 if and only if $S_{n-1}^0 + S_{n-2}^0 \leq m < S_{n-1}^0 + S_{n-2}^0 + S_{n-3}^0$, and the penultimate symbol of the $m$-th row is the last symbol of the $(m - S_{n-1}^0 - S_{n-2}^0)$-th row in the prastāra of value $n - 3$ (where the last row consists of $n - 3$ ones).

Proceeding as above by induction, we can obtain each symbol of the $m$-th row of the prastāra of value $n$.

Given any integer, $0 < m < S_{n-1}^0$, we have the canonical decomposition of $m$:

$$m = a_0 S_0^0 + a_1 S_1^0 + a_2 S_2^0 + \ldots + a_n S_n^0,$$

where, $a_0 = 0, a_n = 0$, and the rest of the $a_i$ could be 0 or 1, as discussed above. To this canonical decomposition, we associate the string $(a_0, a_1, a_2, \ldots, a_n)$ of zeroes and ones, with $a_0 = 0, a_n = 0$. Summarising the discussion above, we have:
Proposition 6: In a prastāra of value \( n \):

i. If the string associated with \( m \) ends with \((0, 0)\), the last symbol of the \( m \)-th row is 1, and the penultimate symbol is given by the truncated binary sequence with the last zero removed in the prastāra of value \( n - 1 \).

ii. If the string associated with \( m \) ends with \((0, 1, 0)\), then the last symbol of the \( m \)-th row is 2 and the penultimate symbol is given by the truncated binary sequence with the end portion \((1, 0)\) removed in the prastāra of value \( n - 2 \).

iii. If the string associated with \( m \) ends with \((0, 1, 1, 0)\), the last symbol of the \( m \)-th row is 3 and the penultimate symbol is given by the truncated binary sequence with the end portion \((1, 1, 0)\) removed in the prastāra of value \( n - 3 \).

The above proposition gives a way of writing down the \( m \)-th row of a prastāra of value \( n \), given the canonical decomposition of \( m \). Let the canonical decomposition of \( m \) be given by

\[
m = a_0 S^3_0 + a_1 S^3_1 + a_2 S^3_2 + \ldots + a_n S^3_n,
\]

with \( a_0 = 0 \), \( a_n = 0 \), and the rest of the \( a_i \) = 0 or 1.

We call the \( S^3_i \) where the \( a_i \) = 0 as an apatita-sankhyānka, and the \( S^3_i \) where the \( a_i \) = 1 as a patita-sankhyānka. The sequence of patitas and apatitas corresponding to \( m \) determine the sequence of entries from 1, 2, 3, comprising the \( m \)-th row. The above proposition actually gives the signature of 1, 2, 3, in terms of the apatitas (a) and the patitas (p) as shown in Table 21 (this is a particular case of Table 12).

Table 21: Signature of the digits 1; 2; 3 in terms of \( a \) and \( p \)

<table>
<thead>
<tr>
<th>( S^3_{r-3} )</th>
<th>( S^3_{r-2} )</th>
<th>( S^3_{r-1} )</th>
<th>( S^3_r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( a )</td>
<td>(a)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( a )</td>
<td>( p )</td>
<td>(a)</td>
</tr>
<tr>
<td>3</td>
<td>( a )</td>
<td>( p )</td>
<td>( p )</td>
</tr>
</tbody>
</table>

We now briefly consider the converse process uttīṣṭa, where the objective is to determine the row-number (in the enumeration given above) given the sequence of symbols that occur in that particular row. Suppose that the row-number that we wish to determine is \( m \) and we now describe a process to determine that.

Suppose that the last symbol in the given row is 1, then \( m \leq S^3_{n-1} \) and therefore \( S^3_{n-1} \) does not occur in the decomposition of \( m \). If we delete the last symbol 1, we obtain a row in the prastāra of value \( n - 1 \), where the row number \( t \) can be determined by following the same process by induction, and we have \( m = t \).

Suppose that the last symbol of the \( m \)-th row is 2, then \( S^3_{n-1} \) occurs in the decomposition of \( m \). If we delete the last symbol 2, we obtain a row of value \( n - 2 \), whose row-number \( t \) in the prastāra of value \( n - 2 \), can be determined by following the same process by induction. We then set \( m = S^3_{n-1} + t \).

If the last symbol of the \( m \)-th row is 3, then \( S^3_{n-1} \) and \( S^3_{n-2} \) both occur in the canonical decomposition of \( m \). If we delete the last symbol 3, we obtain a row of value \( n - 3 \), whose row number \( t \) in the prastāra of value \( n - 3 \), can be determined following the same process by induction. We then set \( m = S^3_{n-1} + S^3_{n-2} + t \).

The above process is exactly the same as the one discussed in the previous section, where one, two and three sankhyānkas were written above each of the symbols 1, 2, 3, and only the second sankhyānka above 2, and the second and the third sankhyānkas above 3, were summed together. The total so obtained is the row-number in the alternate enumeration (starting with row-number 0 from the bottom of the prastāra) that we have used in this section. The total subtracted from \( S^3_n \) gives the row-number in the usual enumeration which starts with row number 1 from the top of the prastāra.
The above discussion can be generalised to the case where the largest entry in each row of the prastāra is any natural number q. Clearly, the essential mathematical result that is at the heart of the naṣṭa and uḍḍiṣṭa processes is the canonical decomposition of any positive integer uniquely as a sum of the generalised Virahāṅka-Fibonacci numbers \( \{S^n_q\} \).

**Appendix**

**The sūci-pāntki, matsya-meru and the number of rows of prastāra with p-sthānas**

In verse 21 of the Chapter on Ankapāśa, Nārāyaṇa defines the sūci-pāntki (needle sequence) or the nārācikā-pāntki (arrow-head sequence) as follows:

Vaiśeṣikī-panktis measured by the last digit, which are equal in number to the number of places are kept separately. Their product is the sūci-pāntki (needle sequence) or the nārācikā-pāntki (arrow-head sequence)

If \( p \) is the number of places and \( q \) is the final digit, then the arrow-head sequence is defined by multiplying the vaiśeṣikī-pankti, which is of the form 1, 1, 1, . . . , 1 (1s repeated \( q \) times), multiplied by itself \( p \) times, by the door-junction (kapāṭa-sandhi) method, which is nothing but the algebraic method of multiplying keeping in mind the different place-values (see Table 22).

Hence, the \((r+1)\)-th element of the sequence is a sum of multinomial co-efficients:

\[
U_{p,q}(r) = \text{Coefficient of } x^r \text{ in } (1 + x + x^2 + \ldots + x^q)^p
\]

In particular, for \( q = 2 \), \( U_{p,2}(r) = C(p,r) \), i.e., in this case the elements of the arrow-head sequence are nothing but the binomial coefficients.

Nārāyaṇa gives an example of how the needle-sequence is obtained for the case when \( p = q = 3 \), which is shown in Table 22. Clearly the elements of this sequence are nothing but the coefficients of different powers of \( x \) in the expansion of \((1 + x + x^2)^q\).

If we look at the above definition of the \( U_{p,q}(r) \), that it is the coefficient of \( x^{p+r} \) in \((x + x^2 + \ldots + x^q)^p\), we can easily see that this is also the number of sequences of length \( p \), formed out of the digits 1, 2, . . ., \( q \), such that the sum of the sequence is \( p + r \). In other words, the number of rows of length \( p \) in the generalised mātrā-vṛttaprasāra characterised by the total sum \( n \) and last digit \( q \), is \( U_{p,q}(n-p) \), where clearly we need to have \( \frac{n}{q} \leq p \leq n \), (where \( \frac{n}{q} \) denotes the integral part of \( \frac{n}{q} \)), in order that the sum of a sequence of length \( p \) composed of the digits 1, 2, . . ., \( q \), may have the sum \( n \). We shall now see that Nārāyaṇa also states essentially the same result while identifying the number of rows of different lengths in the prastāra with the elements of the so called matsya-meru (fish tabular figure).

**Table 22:** Needle sequence for \( p = q = 3 \) by door-junction method

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<td>1</td>
<td>3</td>
<td>6</td>
<td>7</td>
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<td>3</td>
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In verses 40-43 of the chapter on Ankapāśa, Nārāyaṇa defines the tabular figure called the matsya-meru associated with a given sum \( n \), last digit \( q \), and also the number of places \( p \), as follows:
Rows of cells are made beginning with one cell and increasing each time by the last digit minus one. The number of rows is measured by the number of places plus one. Leaving the first cell above, horizontal rows equal in number to the sum of numbers are to be made below the second. One should place one in the first cell and, in each row below, write the sum in the reverse order of entries in the row above equal in number to the last digit. If there are not as many numbers as the last digit, one should write the sum of the available digits.

When we consider the prastāra which is defined by the sum \( n \) and the last digit \( q \), the number of rows in matsya-meru is equal to \( n + 1 \). The first row has a single cell which has entry 1. The successive rows have their first cell moved to the right and each of them have \( (q - 1) \) cells more than the previous row. This later rule is followed till the number of rows is equal to the total number of places \( p + 1 \). What is really meant by this condition is not clear, though from the example given below we can see that the number of columns is not increased after this stage. The cells of the successive rows are filled as follows. Each cell is filled by the sum of the entries in \( q \) cells of the row above which are just to the left of the given cell.

After defining the matsya-meru, Nārāyaṇa displays the form of the meru shown in Table 23, when the sum \( n = 3 \), the last digit \( q = 3 \) and the number of places \( p = 3 \).

While working out this example, Nārāyaṇa also notes that

In this matsya-meru, we will have [the rows given by] the arrow-head and partial arrow-head sequences.

In fact, excluding the first row, the succeeding rows of the matsya-meru are filled with elements of the arrow-head sequences \( U_{1,q}(r) \), \( U_{2,q}(r) \), and so on. The process of generating the meru is actually based on the recurrence relation

\[
U_{p,q}(r) = U_{p-1,q}(r - 1) + U_{p-1,q}(r - 2) + \ldots + U_{p-1,q}(r - q),
\]

which can be derived easily from the definition of \( U_{p,q}(r) \).

Later in the same Chapter on Ankapāśa, while discussing the generalised mātrā-vṛttaprastāra, Nārāyaṇa seems to characterise the matsya-meru in terms of just the sum \( n \) and last digit \( q \) without any reference to some fixed number of places \( p \). All the defining features of the matsya-meru as described above are retained, except that both the number of rows and columns of the meru are now fixed at \( n + 1 \). This is clear from the example of matsya-meru, displayed in Table 24, which is merely characterised by the sum \( n = 3 \) and the last digit \( q = 3 \). This form of the meru is presented by Nārāyaṇa while discussing an example following verse 79 in the Chapter on Ankapāśa.

In verse 80 of the Chapter on Ankapāśa, Nārāyaṇa indicates how the matsya-meru can be used in order to obtain the number of rows in the prastāra which have a given length:
Table 24: Matsyameru for $n = 3; q = 3$

\[
\begin{array}{ccccccc}
 & 1 & 1 & 1 & 1 & 2 & 3 & 2 & 1 \\
1 & 1 & 2 & 3 & 2 & 1 & 1 & 3 & 6 & 7 & 6 & 1 & 4 & 10 & 16 \\
 & 1 & 5 & 15 & 6 & 1 & 1 & 6 & 1
\end{array}
\]

The column (vertical row) at the end of the matsya-meru, from that are known the (number of) variations (or rows of the prastāra) for each number of places (sthāna)... In other words, the last column of the matsya-meru gives the number of rows in the prastāra with different lengths, or number of places. For the case where $n = q = 3$, the last column of the matsya-meru (shown in Table 24) is given by 6, 16, 15, 6, 1. These indeed are the number of rows in the prastāra which are of length 3, 4, 5, 6 and 7 respectively. The sum of this last column of the meru is equal to 44 which is indeed $S^n_3$, the sankhyā or the total number of rows in the prastāra characterised by $n = q = 3$. In fact, the various column sums of the above matsyameru are 1, 1, 2, 4, 7, 13, 24 and 44, which are nothing but the generalised Virahānka sequence \{$S^n_3$\}.

The last property mentioned above is indeed a particular case of a general relation between the elements of the nārācikā-pankti $U_{r}(n)$ (which are also entries of matsya-meru) and the sāmāsikī-pankti or the generalised Virahānka sequence \{$S^n_r$\}, which is noted by Nārāyaṇa in verses 43-44 of the Chapter on Ankapāśā, while discussing the sums of the rows and columns of matsya-meru.

The elements of the arrowhead sequence in each row are measured by the number of places. The sum of each of these (rows) separately becomes a geometric sequence (guṇottara-pankti). The columns are measured by the sum. The columns summed separately become equal to the elements of the sāmāsikī-pankti.

Here, Nārāyaṇa first mentions that the sum of the sūcī-pankti (for a given value of $q$) form a geometrical sequence. This is essentially the relation

\[
\sum_{r=0}^{n-1-q} U_{r,q}^p(r) = (1+1+...+1)^p = q^p
\]

Nārāyaṇa then gives the following important relation between the coefficients $U_{r,q}(n)$ and the sāmāsikī-pankti or the generalised Virahānka-Fibonacci sequence:

\[
\sum_{r=0}^{n-q} U_{n-r,q}(r) = S^n_q
\]

where $t$ is such that $(t-1)q < n \leq tq$.

The above relation, gives an important relation between the multinomial coefficients $U_{n-r,q}(r)$ and the elements of the generalised Virahānka-Fibonacci sequence \{$S^n_q$\}. It has also an important interpretation in the prastāra of the generalised mātrā-vṛttas which is the following.

Since $U_{n-r,q}(r)$ is the total number of generalised mātrā-vṛttas (sequences of 1, 2, .., $q$), which are of length $n-r$ and whose total value or sum is $n$, the sum of all such numbers for various possible lengths, should clearly be equal to $S^n_q$, which is the sankhyā or the total number of all the generalised mātrā-vṛttas of value $n$.

**Remark:** In mathematical terms, Nārāyaṇa’s matsya-meru is a part of the triangle of multinomial coefficients. If we tilt this triangle in a certain way and take the sum of the rows, we get the generalised Virahānka-Fibonacci numbers. In a similar manner, in case we tilt the Piṅgala-
Pascal triangle and take the sum of the rows, we get the standard Virahāṅka-Fibonacci numbers (Raja Sridharan et al., 2012a)

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