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HINDU TRIGONOMETRY

by

BIBHUTIBHUSAN DATTA AND AVADHESH NARAYAN SINGH

Revised by

KRIPA SHANKAR SHUKLA

Hussainganj Crossing, Behind Lakshman Bhawan
Lucknow

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1. TRIGONOMETRICAL FUNCTIONS. DEFINITIONS.

The Hindu name for the science of Trigonometry is *Jyotpatti-gaṇita* or "The science of calculation for the construction of the sine."[1] It is found as early as in the *Brāhma-sphuṭa-siddhānta* of Brahmagupta (628).[2] Sometimes that name is simplified into *Jyā-gaṇita* (or "The science of calculation of the sines").[3] In very recent years there has appeared the name *Trikoṇamiti*[4], which is a literal as well as phonetic rendering of the Greek name for the science.

The Hindus introduced and usually employed three trigonometrical functions, namely *jyā*, *koṭi-jyā* and *utkrama-jyā*. It should be noted that they are functions of an arc of a circle, but not of an angle. If AP be an arc of a circle with centre at O, then

[1]*Jyā* ("sine")+*utpatti* ("construction", "generating")+*gaṇita* ("the science of calculation").
[2]*BrSpSi*, xii. 66.
[3]Compare *SiTVi*, ii. 1.
[4]*Trikoṇa* ("triangle")+*miti* ("measure").

its *jyā*=*PM*, *koṭi-jyā*=*OM* and *utkrama-jyā*=*OA*—*OM*=*AM*. Hence their relation with modern trigonometrical functions will be

jyā AP=*R*sin θ, *koṭi-jyā AP*=*R*cos θ, *utkrama-jyā AP*=*R*—*R*cos θ=*R*versin θ, wheie *R* is the radius of the circle and θ the angle subtended at the centre by the arc *AP*. Thus the values of the Hindu trigonometrical functions vary with the radius chosen. The earliest Hindu treatise in which the above trigonometrical functions are now found recorded is the *Sūrya-siddhānta*.

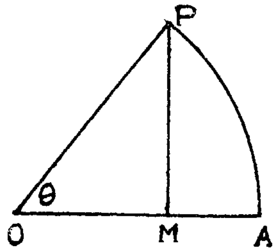

Fig. 1

Jyā

The Sanskrit word *jyā* means "a bow-string"; and hence "the chord of an arc", for the arc is called "a bow" (*dhanu*, *cāpa*). Its synonyms are *jīvā*, *siñjinī*,[1] *guṇa*, *maurvī*, etc. This trigonometrical function is also called *ardha-jyā*[2] ("half-chord") or *jyārdha*[3] ("chord-half"). Thus Bhāskara II (1150) explicitly observes, "It should be known that *ardha-jyā* is here called "*jyā*".[4] Parameśvara (1430) remarks:

"A part of a circle is of the form of a bow, so it is called the "bow" (*dhanu*). The straight line joining its two extremities is the "bow-string" (*jīvā*); it is really the "full-chord" (*samasta-jyā*). Half of it is here (called) the "half-chord" (*ardha-jyā*), and half that arc is called the "bow" of that half-chord. In fact the *R*sine (*jyā*) and *R*cosine (*koṭi-jyā*) of that bow are always half-chords."[5]

Kamalākara (1658) is more explicit. "Having seen the brevity", says he, "the half-chords are called *Jyā* by mathematicians in this (branch of) mathematics and are used accordingly."[6] The function *jyā* is sometimes distinguished as *krama-jyā*[7] or *kramārdha-jyā*,[8] from *krama*, "regular" or "direct" meaning "direct sine" or "direct half-chord".

It may be noted that the modern term *sine* is derived from the Hindu name. The Sanskrit term *jīvā* was adopted by the early Arab mathematicians but was pronounced as *jība*. It was subsequently corrupted in their tongue into *jaib*. The latter word was confused by the early Latin translators of the Arabic works such as Gherardo of Cremona (*c*. 1150 A.D.) with a pure Arabic word of alike phonetism but meaning differently "bosom" or "bay" and was rendered as *sinus*, which also signifies "bosom" or "bay".[9]

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[1] *ŚiDVṛ*, ii. 9; *MSi*, iii. 2.
[2] *Ā*, i .10; *BrSpSi*, ii. 2.
[3] *SūSi*, ii. 15; *Ā*, ii. 11, 12; *BrSpSi*, xxi. 17, 22.
[4] *SiŚi*, *Graha*, iii. 2.
[5] *Ā*, ii. 11 (*Com*.).
[6] *SiTVi*, ii. 52.
[7] *PSi*, iv. 28 (*Kramaśo jyā*); *BrSpSi*, ii. 15; vii. 12.
[8] *ŚiDVṛ*, ii. 1.
[9] Cf. *Nouv. Ann. Math.*, XIII (1854), p. 393; Smith, *History* II, p. 616.

The degeneration and variations of the term *kramajyā* are still more interesting. In the Arabic tongue it was corrupted into *karaja* or *kardaja*. According to *Fihrist*, the title of a work of Ya'kūb ibn Ṭārik (*c.* 770 A.D.) is "On the table of *kardaja*." This table was copied from the *Brāhma-sphuṭa-siddhānta* of Brahmagupta. In the same connexion, al-Khowārizmī (825) used the variant *karaja*. In the Latin translations of the term we find several variants such as *kardaga, karkaya, gardaga* or *cardaga*. These terms had in foreign lands also the restricted uses for the arc of 3° 45′, sometimes of 15°.[1]

Koṭi-Jyā

The Sanskrit word *koṭi* means, amongst others "the curved end of a bow" or "the end or extremity in general"; hence in Trigonometry it came to denote "the complement of an arc to 90°."[2] So the radical significance of the term *koṭi-jyā* is "the *jyā* of the complementary arc". But it began early to be used as an independent technical term.[3] The modern term cosine appears to be connected with *koṭijyā*, for in Hindu works, particularly in the commentaries *koṭijyā* is often abbreviated into *kojyā*. When *jyā* became *sinus, kojyā* naturally became *ko-sinus* or *co-sinus*.

Utkrama-Jyā

Utkrama means "reversed", "going out" or "exceeding". Hence the term *utkrama-jyā* literally means "reversed sine". This function is so called in contradistinction to *krama-jyā*, for it is, rather its tabular values are, derived from the tabular values of the latter by subtracting the elements from the radius in the reversed order. Or in other words it is the exceeding portion of the *krama-jyā* taken into consideration in the reversed order. Thus it is stated:

"The (tabular) versed sines are obtained by subtracting from the radius the (tabular) sines in the reversed order."[4]

"They (*jyārdha*), (being subtracted from the radius), in the reversed order beginning from the end, will certainly give the versed sines, that is, the arrows."[5]

Again, it is noteworthy that from a table of differences of sines, the successive sines are obtained by adding the differences in the direct order (from the top) whereas the corresponding versed sines will be found by adding the elements in the reversed order (from the end). This fact has been particularly noted by Sūryadeva Yajvā (born 1191 A.D.) and Śrīpati (1039 A.D.). The former observes:

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[1]Woepeke, F. "Sur le mot *kardaga* et sur une méthode indienne pour calcul les sinus", *Nouv. Ann. Math.*, XIII (1854), pp. 386-393; Braunmühl, A. *Geschichte der Trigonometrie*, 2 vols., Leipzig, 1900, 1903 (hereafter referred to as Braunmühl, *Geschichte*); Vol. I, pp. 44, 45, 78, 102, 110, 120; vide also Sarton's note on the point in *Isis*, xiv (1930), pp. 421f.

[2]In Hindu mathematics, the term *koṭi* also denotes "the side of a right-angled triangle."

[3]Compare *ŚiDVṛ*, ii. 30 (infra p. 10).

[4]*SūSi*, ii. 22.

[5]*BrSpSi*, xxi. 18; Compare also *MSi*, iii. 3.

"In order to get the direct sines (*krama-jyā*), these (tabular) differences of sines (*khaṇḍa-jyā*) should be added regularly from the beginning; and in order to determine the reversed sines (*utkrama-jyā*), they should be added in the reversed order from the end."[1]

Śrīpati says:

"The difference of sines are called *jyākhaṇḍa* (tabular "difference of sines"); (adding them) in the reversed way beginning from the end will be obtained the versed sines (*vyasta-jyā*) of the half-arcs equal to the 96th parts of the celestial circle."[2]

This function is also called *vyasta-jyā*[3] (from *vyasta*, "cast or thrown asunder", "reversed") or *viloma-jyā*[4] (from *viloma*, "reverse"). Occasionally it is termed *utkrama-jyārdha*.[5] Another name for it is "arrow" (*iṣu*, *bāṇa*).[6] Bhāskara II observes:

"What is really the arrow between the bow and the bowstring is known amongst the scholars here (i.e. in Trigonometry) as the versed sine."[7]

So also says Kamalākara (1658):

"What lies between the chord and the arc, like the arrow, is the versed sine."[8]

Tangent and Secant

The Hindus approached very near the tangent and secant functions and actually employed them in astronomical calculations, though they did not expressly recognise them as separate functions. The *Sūrya-siddhānta* gives the following rule for calculating the equinoctial midday shadow of the gnomon at a station:

"The sine of the latitude (of the station) multiplied by 12 and divided by the cosine of the latitude gives the equinoctial mid-day shadow."[9]

Here 12 is the usual height of a Hindu gnomon. So that

$$S = \frac{jy\bar{a}\ \phi \times h}{kojy\bar{a}\ \phi}$$

where ϕ denotes the latitude of the place, S=equinoctial mid-day shadow and h= gnomon. This is equivalent to

$$S = h \tan \theta.$$

[1] *Ā* i. 10 (*Com.*).
[2] *SiŚe*, xvi. 10,
[3] *BrSpSi*, ii. 5; *MSi*, iii. 3, 6.
[4] *ŚiDVṛ*, I, ii. 5.
[5] *SūSi*, ii. 22, 27.
[6] *BrSpSi*, xxi. 18.
[7] *SiŚi*, *Gola*, xiv. 5; Compare also *Graha*, ii. 20 (*Gloss*).
[8] *SiTVi*, ii. 58.
[9] *SūSi*, iii. 16.

Again to find the mid-day shadow (s) of the gnomon (h) and the hypotenuse (d), having known the meridian zenith distance (z) of the sun, we have the rules:[1]

$$s = h \tan z, \qquad d = h \sec z.$$

Similar rules occur in other astronomical works also.[2] In the *Gaṇita-sāra-saṃgraha* of Mahāvīra (850) by the term "shadow" of a gnomon is sometimes meant the ratio of the actual shadow to the height of the gnomon.[3] This ratio, as has been just stated, is equal to the tangent of the zenith distance of the sun.

Quadrants

A circle is ordinarily divided into four equal parts, called *vṛtta-pāda*, by two perpendicular lines, usually the east-to-west line and the north-to-south line. The quadrants are again classified into odd (*ayugma*, *viṣama*) and even (*yugma*, *sama*). Earlier Hindu writers do not explain this fact fully and particularly. Thus Bhāskara I (629) simply observes: "Three signs form a quadrant".[4] Lalla writes:

"Three anomalistic signs form a quadrant. The quadrants are successively distinguished as odd and even."[5]

But the description of Bhāskara II (1150) is very full. He says:

"Three signs together form a quadrant. In a circle there will be four such; and they should be successively called odd and even."[6]

He then explains it further thus:

"On a plane surface describe a circle of any specified radius with a pair of compasses. Mark on its circumference 360 degrees. Draw the east-to-west and north-to-south lines through its centre. These lines will divide the circle into quadrants, which should be taken into consideration in the leftwise manner (*savya-krama*, that is 'anti-clockwise'[7]) proceeding from the east-point (*prācī*); they should be called odd and even (quadrants) successively."[8]

Variation in value

As regards the variation in the value of a trigonometrical function as its argument changes, Bhāskara II observes as follows:

[1] *SūSi*, iii. 21.
[2] *PSi*, iv. 22.
[3] *GSS*, ix. 8½.
[4] *MBh*, iv. 1; *LBh*, ii. 1.
[5] *ŚiDVṛ*, ii. 10.
[6] *SiŚi*, *Graha*, ii. 19.
[7] The Sanskrit term *savya-krama* ordinarily signifies the "clockwise direction"; but it may also denote the "anti-clockwise direction."
[8] *SiŚi*, *Graha*, ii. 19 (*Gloss*).

"In the first quadrant, mark a point on the circumference of the circle at any optional distance from the east point. The perpendicular distance of that point from the east-to-west line is called the *R*sine (*doḥ-jyā*); and its distance from the north-to-south line is the *R*cosine (*koṭi-jyā*). The corresponding arcs are called *bhuja* and *koṭi*. (Starting from the east point) as the point gradually moves forward in the same way (i.e. anti-clockwise), the *R*sine increases and the *R*cosine decreases. When the point arrives at the end of the quadrant, the *R*cosine vanishes and the *R*sine is equal to the radius. Then in the second quadrant, the *R*cosine increases; at the end of that quadrant the *R*cosine is maximum (irrespective of sign) and the *R*sine vanishes."[1]

One fact perhaps deserves a particular notice here. It is that in Hindu trigonometry the *jyā* of an arc of 90° in a circle is equal to the radius of that circle. On account of that, the radius is called in Hindu mathematics by the terms *tri-jyā*, *tri-bha-jyā*, *tribhavana-jyā*, etc., every one of which literally means the "sine of three signs." The radius is also called *viṣkambhārdha*, *vyāsārdha*, or *ardha-vyāyāma* meaning the "semi-diameter". All these terms are very old.[2]

Functions of a complement or supplement

Sūrya-siddhānta says:

"In odd quadrants, the arc passed over gives the *R*sine, while the arc to be passed over gives the *R*cosine; and in the even quadrants, the arc to be passed over gives the *R*sine and that passed over gives the *R*cosine."[3]

Bhāskara I writes:

"In the odd quadrants the arc described and that to be described should respectively be known as the *bhuja* and *koṭi*; but in the even quadrants they are respectively the *koṭi* and *bhuja*; this is the fact."[4]

Lalla remarks:

"When (the anomaly[5] is) greater than 90°, it is subtracted from the semi-circle (i.e. 180°); when greater than the semi-circle, 180° is subtracted from it; when greater than 270°, it is subtracted from the complete circle (i.e. 360°); the remainder is called the (corresponding) *bhuja* by the expert in the subject.'[6]

[1] *Ibid*, ii. 20 (*Gloss*). The Sanskrit terms *jyā* and *kojyā* have been translated as *R*sine and *R*cosine because they are equal to $R \times$ sine and $R \times$ cosine respectively.

[2] Compare *ĀpŚlSū*, vii. 11 (*ardha-vyāyāma*); *Jambūdvīpasamāsa* of *Umāsvāti*, iv (*vyasārdha*); *Tattvārthādhigama-sūtra-bhāṣya*, iv. 14 (*viṣkambhārdha*).

[3] *SūSi*, ii. 30.

[4] *LBh*, ii. 1-2; Compare also *MBh*, iv. 8-9.

[5] It is in connection with the treatment of the anomaly that the remark of Lalla, as of several other Hindu mathematicians, occurs.

[6] *ŚiDVṛ*, ii. 10-11.

In the words of Brahmagupta:

"The *R*sine and *R*cosine (are obtained) in the odd quadrants from the arc passed over and to be passed over (respectively); and in the even quadrants in the reverse way."[1] Or,

"In the odd quadrants (the *R*sine is determined) from the arc described and in the even quadrants from the arc to be described."[2]

"(For the determination of) the *R*sine (proceed with the anomaly as it is) when the anomaly is less than three signs (i.e. 90°); when greater than three signs subtract it from six signs; when greater than six signs, subtract six signs (from it); when greater than nine signs, subtract it from the complete circle."[3]

Mañjula (932) says:

"In the odd quadrants, the *bhuja* and *koṭi* are (to be calculated) from the arc described and that to be described (respectively); but in the even quadrants in the contrary way."[4]

His commentator and younger contemporary Praśastidhara (962) dilates upon this point thus:

"In the odd quadrant, where the anomaly is less than three signs (i.e. 90°), the *R*sine should be calculated from it and the *R*cosine should be calculated after subtracting that from 90°. In the even quadrant, where the anomaly exceeds 90° but is less than 180°; in that case the *R*sine should be taken after subtracting it from 180° and the cosine after subtracting 90° from it. In the odd quadrant, where the anomaly is greater than 180°, but less than 270°, the *R*sine should be calculated after subtracting 180° from it and the *R*cosine after subtracting it from 270°. In the even quadrant when the anomaly exceeds 270°, but is less than 360°, the *R*sine is determined after subtracting it from 360°, and the *R*cosine after subtracting 270° from it."[5]

Śrīpati (*c.* 1039) remarks:

"In the odd and even quadrants, the arc passed over and to be passed over (respectively) is the *bhuja* and the *koṭi* is otherwise. Or, as the learned have said, the *R*sine of 90° minus the anomaly is the *R*cosine (of the anomaly)."[6]

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[1] *BrSpSi*, ii. 12.
[2] *KK*, *I*, *i*. 16.
[3] *KK*. *I*, *i*. 16.
[4] *LMā*, ii. 2.
[5] Commentary on the same.
[6] *SiŚe*, iii. 13.

And Bhāskara II:

"In the odd quadrants, the arc passed over and in the even quadrants the arc to be passed is the *bhuja*. Ninety degrees minus the *bhuja* is said to be the *koṭi*."[1]

The above results can be represented graphically thus:

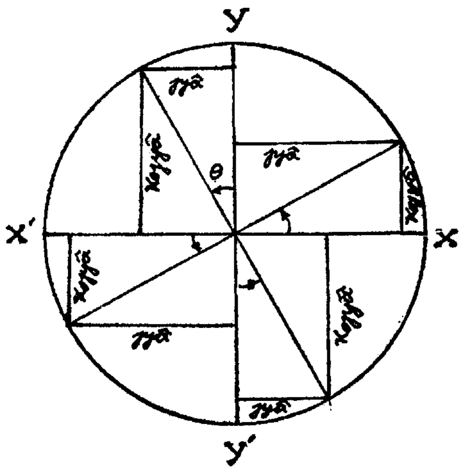

Fig. 2

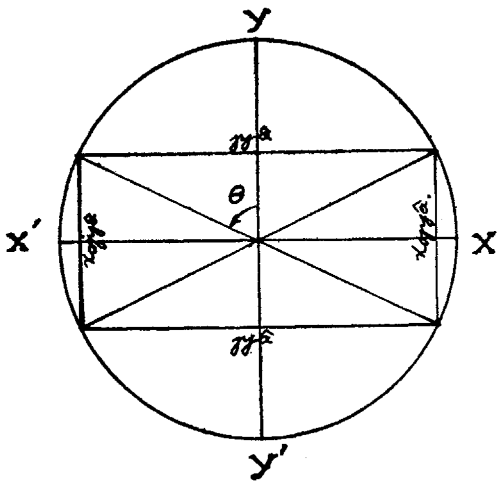

Fig. 3

Relation between Functions

Varāhamihira says:

"The *R*sine of 90° minus latitude is the *R*cosine of the latitude."[2]

Lalla:

"The square of the base-sine (*bhuja-jyā*) is subtracted from the square of the radius; the square root of the remainder is the *R*cosine; or it is the *R*sine of 90° minus the *bhuja* arc."[3]

$$\sqrt{R^2 - (jy\bar{a}\,\alpha)^2} = kojy\bar{a}\,\alpha$$

or

$$kojy\hat{a}\,\alpha = jy\hat{a}\;(90^\circ - \alpha)$$

where *kojyā* is the usual Hindu symbol for *koṭi-jyā*.

[1] *SiŚi, Graha*, ii. 19.
[2] *PSi*, iv. 28.
[3] *ŚiDVṛ*, ii. 30.

Brahmagupta says:

"The radius diminished by the versed *R*sine of an arc or of its complement will give the *R*sine of the other. The square-root of the difference of the square of the radius and that of the *R*sine of an arc or of its complement will be the *R*sine of the other."[1]

$$R - utjyā\,\alpha = jyā\,(90° - \alpha),$$
$$R - utjyā\,(90° - \alpha) = jyā\,\alpha,$$
$$\sqrt{R^2 - (jyā\,\alpha)^2} = jyā\,(90° - \alpha),$$
$$\sqrt{R^2 - \{jyā(90° - \alpha)\}^2} = jyā\,\alpha$$

where *utjyā* is the usual abbreviation for *utkrama-jyā.*

"The direct *R*sine of the excess of an arc over 90° added to the radius will give versed *R*sine of that arc."[2]

$$R + jyā\,(\alpha - 90°) = utjyā\,\alpha,$$

where $\alpha > 90°$.

Śrīpati writes:

"The square of the radius is diminished by the square of the *R*sine; the square-root of the remainder will be the *R*cosine. Again the square-root of the square of the radius minus the square of the *R*cosine will be the *R*sine. The radius minus the versed *R*sine of the complement of an arc is equal to the *R*sine of the arc, and minus the versed *R*sine of the arc becomes the *R*sine of the other (i.e. complement)."[3]

The treatment of Bhāskara II is exhaustive. He says:

"Subtract from the radius the direct *R*sine of an arc and of its complement; the results will be the versed *R* sines of the complement and the arc (respectively). Subtract from the radius the versed *R*sine of an arc and of its complement; the remainders will be the direct *R*sines of the complement and the arc (respectively)."[4]

$$R - jyā\,\alpha = utjyā\,(90° - \alpha),$$
$$R - jyā\,(90° - \alpha) = utjyā\,\alpha,$$
$$R - utjyā\,\alpha = jyā\,(90° - \alpha),$$
$$R - utjyā\,(90° - \alpha) = jyā\,\alpha.$$

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[1] *BrSpSi*, xiv. 7.
[2] *Ibid*, vii. 12.
[3] *SiŚe*, iii. 14.
[4] *SiŚi*, *Graha*, ii. 20; also *Gola*, v. 2; xiv. 5.

"The square of the *R*sine of an arc and of its complement are (severally) subtracted from the square of the radius, the square-roots of the results are (respectively) the *R*sines of the complement and of the arc."[1]

$$\sqrt{R^2-(jy\bar{a}\ \alpha)^2}=jy\bar{a}\ (90^\circ-\alpha);\quad \sqrt{R^2-\{jy\bar{a}\ (90^\circ-\alpha)\}^2}=jy\bar{a}\ \alpha.$$

"The square of the radius is diminished by the square of the *R*sine of an arc; the square-root of the result is the *R*cosine of the arc."[2]

$$\sqrt{R^2-(jy\bar{a}\ \alpha)^2}=kojy\bar{a}\ \alpha.$$

Kamalākara writes:

"The square-root of the square of the radius diminished by the square of the *R*sine of an arc, is the *R*cosine of the arc; similarly, the square-root of the square of the radius diminished by the *R*cosine of an arc, is the *R*sine of the arc. Again, the *R*sines of an arc and its complement when subtracted from the radius will give the versed *R*sines of the complement and the arc (respectively)."[3]

$$\sqrt{R^2-(jy\bar{a}\ \alpha)^2}=kojy\bar{a}\ \alpha,\quad \sqrt{R^2-(kojy\bar{a}\ \alpha)^2}=jy\bar{a}\ \alpha,$$

$$R-jy\bar{a}\ \alpha=utjy\bar{a}\ (90^\circ-\alpha),\quad R-jy\bar{a}\ (90^\circ-\alpha)=utjy\bar{a}\ \alpha.$$

Change of Sign of a Function

The Hindus were fully aware of the changes of sign of a trigonometrical function according as its argument lies in different quadrants. Though nowhere do we find any systematic treatment of this principle in any Hindu work there are ample concrete instances of its application in almost all their important astronomical treatises. Thus it is stated in the *Sūrya-siddhānta:*

"The *śīghra koṭiphala* is positive, when the *kendra* (mean anomaly) lies in a position beginning with the Capricorn; and it is to be subtracted from the radius in a position beginning with the Cancer."[4]

Now according to the *Sūrya-siddhānta* and other Hindu astronomical works, the *śīghra koṭiphala* (the result derived from the complement of the distance from the conjunction) is given by $D\cos\theta$, where θ is the *śīghra kendra*[5] (the distance of the mean planet from its apex of swiftest motion; hence mean *śīghra* anomaly) and D, a certain known constant. The Cancer is the fourth sign of the Zodiac and Capricorn is the tenth sign. Again the motion of the mean planet is anti-clockwise. Hence it is clear from

[1] *SiŚi, Graha,* ii. 21.
[2] *SiŚi, Gola,* v. 2; xiv. 4.
[3] *SiTVi,* ii. 56-7.
[4] *SūSi,* ii. 40.
[5] "Subtract the longitude of a planet from that of its apex of slowest motion (*mandocca*); so also subtract it from that of its apex of swiftest motion (conjunction); the result (in either case) is its *Kendra.*" *SūSi,* ii. 29.

the above rule that the author was aware that the cosine of an angle lying between 0° and 90° or between 270° and 360° is positive and that it is negative when the angle lies between 90° and 270°.

Again it has been said:

"In case of the *manda* and *śīghra* corrections of all planets, the *phala* (equation) will be positive, if the *kendra* lies in the six signs beginning with the Aries and it will be negative in the six signs beginning with the Libra."[1]

Now the *phala* is defined as arc ($D'\sin\theta$), where D' does not change sign. Hence clearly the author knows that the sign is positive in the first two quadrants and negative in the other two quadrants.

Similar rules are found in other treatises of astronomy.[2] The statement of Mañjula (932) is more explicit and fuller. He says:

"The (mean) planet when diminished by its apogee or aphelion is the *kendra* (mean anomaly). Its *R*sine is positive or negative in the upper or lower halves (of the quadrants); and its *R*cosine is positive, negative, negative, and positive (respectively) according to the (successive) quadrants."[3]

Thus the Hindus knew very early what in modern trigonometrical notations will be expressed as—

$$\sin(\pi \mp \theta) = \pm\sin\theta, \qquad \cos(\pi \mp \theta) = -\cos\theta$$

$$\sin(2\pi - \theta) = -\sin\theta, \qquad \cos(2\pi - \theta) = +\cos\theta$$

$$\sin\left(\frac{\pi}{2} \mp \theta\right) = +\cos\theta, \qquad \cos\left(\frac{\pi}{2} \mp \theta\right) = \pm\sin\theta$$

$$\sin\left(\frac{3\pi}{2} \mp \theta\right) = -\cos\theta, \qquad \cos\left(\frac{3\pi}{2} \mp \theta\right) = \mp\sin\theta$$

Again it has been stated before that according to a rule of Brahmagupta

$$R + jy\bar{a}\,(\alpha - 90°) = utjy\bar{a}\ \alpha.$$

But by definition,

$$utjy\bar{a}\ \alpha = R - kojy\bar{a}\ \alpha = R - jy\bar{a}\,(90° - \alpha).$$

[1] *SūSi*, ii. 45.
[2] For instance *Ā*, iii. 22; *MBh*, iv. 5, 9; *LBh*, ii. 6; *ŚiDVṛ*, ii. 32; *BrSpSi*, ii. 14ff.
[3] *LMā*, ii. 1.

These clearly show that the author knows that the value of the sine function changes sign along with its argument.

Or symbolically

$$\sin(\pm\theta) = \pm\sin\theta.$$

2. Trigonometrical Formulae

(1) $\sin^2\theta + \cos^2\theta = 1$

It has been stated before that according to Hindu astronomers, if α be an arc of a circle of radius R

$$\sqrt{R^2 - (jy\bar{a}\ \alpha)^2} = kojy\bar{a}\ \alpha,\ \sqrt{R^2 - (kojy\bar{a}\ \alpha)^2} = jy\bar{a}\ \alpha.$$

These are of course equivalent to the modern formulae

$$\sqrt{1 - \sin^2\theta} = \cos\theta,\ \sqrt{1 - \cos^2\theta} = \sin\theta$$

or $\sin^2\theta + \cos^2\theta = 1$,

where θ is the angle subtended at the centre of the circle by the arc α.

(2) $4\sin^2\dfrac{\theta}{2} = \sin^2\theta + \text{versin}^2\,\theta$

This formula has been stated first by Varāhamihira (505).

He says:

"To find the *R*sine of any other desired arc, double the arc and subtract from the quarter of a circle; diminish the radius by the *R*sine of the remainder. The square of half the result is added to the square of half the *R*sine of the double arc. The square-root of the sum is the desired *R*sine."[1]

Let the arc XP = arc $PQ = \alpha$; then arc $QY = 90° - 2\alpha$.

Now

$$XQ^2 = QT^2 + TX^2$$
$$\text{or } 4\,XD^2 = QT^2 + TX^2$$
$$\text{or } XD^2 = (QT/2)^2 + (TX/2)^2.$$

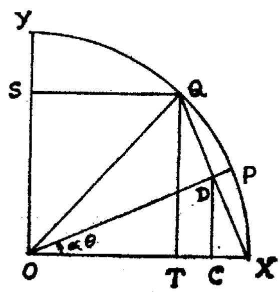

Fig. 4

[1] *PSi*, iv. 2f.

Hence

$$(jy\bar{a}\ \alpha)^2 = \left(\frac{jy\bar{a}\ 2\alpha}{2}\right)^2 + \left(\frac{R - jy\bar{a}\ (90^\circ - 2\alpha)}{2}\right)^2;$$

which is equivalent to

$$4 \sin^2 \theta = \sin^2 2\theta + \text{versin}^2\ 2\theta.$$

Āryabhaṭa I (499) seems to have been aware of this formula before Varāhamihira. It reappears also in later works.

Brahmagupta says:

"The sum of the squares of the *R*sine and versed *R*sine of the same arc is divided by four; subtract this quotient from the square of the radius. Take the square-root of the two results. The former will be the *R*sine of half that arc, and the other the *R*sine of the arc equal to the quarter circle less that half."[1]

The formula has been described almost similarly by Śrīpati (1039)[2]. Bhāskara II (1150) writes very briefly thus:

"Half the square-root of the sum of the square of the *R*sine and of the versed *R*sine of an arc, will be the *R*sine of half that arc."[3]

Parameśvara (1430) says:

"The square-root of the sum of the square of the *R*sine and of the versed *R*sine of an arc is the 'whole chord' (*samasta-jyā*) of that arc. Half that is the half-chord (i.e. the *R*sine) of half that arc."[4]

(3) $2 \sin^2 \frac{\theta}{2} = 1 - \cos \theta$

This is given first by Varāhamihira. He says:

"Twice any desired arc is subtracted from three signs (i.e. 90°), the *R*sine of the remainder is subtracted from the *R*sine of three signs. The result multiplied by sixty is the square of the *R*sine of that arc."[5]

[1] *BrSpSi*, xxi. 20f.
[2] *SiŚe*, xvi. 14-5.
[3] *SiŚi*, *Gola*, v. 4; xiv. 10.
[4] Quoted in his commentary of *Ā*, ii. 11.
[5] *PSi*, iv. 5.

In the fig. 4, page 50, since the triangles *XCD* and *XDO* are similar, we have:

$$XD : XC :: XO : XD$$
$$\therefore XD^2 = XO.XC = \tfrac{1}{2}\, XO.XT$$

Hence $(jy\bar{a}\ \alpha)^2 = \frac{1}{2} R\ \{R - jy\bar{a}\ (90^\circ - 2\alpha)\}$.

The factor $\frac{1}{2}$ *R* on the right-hand side has been stated by Varāhamihira as 60 since he has taken the value of the radius to be equal to 120. In modern notations, the above formula becomes—

$$\sin^2 \theta = \tfrac{1}{2}\,(1 - \cos 2\theta).$$

This also follows easily from the preceding formula.

Brahmagupta says:

"The square-root of the fourth part of the versed *R*sine of an arc multiplied by the diameter is the *R*sine of half that arc."[1]

Bhāskara II writes:

"Or, the square-root of half the product of the radius and the versed *R*sine of an arc, will be the *R*sine of half that arc."[2]

He has further given the following proof of it.[3]

Since $kojy\bar{a}\ \alpha = R - utjy\bar{a}\ \alpha$

so that squaring $(kojy\bar{a}\ \alpha)^2 = R^2 + (utjy\bar{a}\ \alpha)^2 - 2R.\ utjy\bar{a}\ \alpha$

Therefore $R^2 - (kojy\bar{a}\ \alpha)^2 = 2R.\ utjy\bar{a}\ \alpha - (utjy\bar{a}\ \alpha)^2$

Or, $(jy\bar{a}\ \alpha)^2 = 2R.\ utjy\bar{a}\ \alpha - (utjy\bar{a}\ \alpha)^2$

$(jy\bar{a}\ \alpha)^2 + (utjy\bar{a}\ \alpha)^2 = 2R.\ utjy\bar{a}\ \alpha$

But by the formula (2), the righthand side is equal to

$$4\left(jy\bar{a}\ \frac{\alpha}{2}\right)^2.$$

Hence, $jy\bar{a}\ \frac{\alpha}{2} = \sqrt{\tfrac{1}{2} R.\, utjy\bar{a}}$.

This rule of Bhāskara II together with his proof has been reproduced by Kamalākara.[4]

[1] *BrSpSi*, xxi. 23.
[2] *SiŚi, Gola*, v. 5; xiv. 10.
[3] *SiŚi, Gola* (Gloss).
[4] *SiTVi*, ii. 78 and its commentary.

(4) $\sin \frac{1}{2} (90° \pm \theta) = \sqrt{\frac{1}{2} (1 \pm \sin \theta)}$

This formula first appears in the works of Āryabhaṭa II (950). He says:

"The *R*sine of any arc multiplied by the radius is subtracted from or added to the square of the maximum value of the *R*sine; the square-root of half the results are extracted. These will be the *R*sine of 45° decreased or increased by half that arc."[1]

Let the arc *XP* be denoted by α. Bisect the complementary arc *YP* at *Q*. Then

$$YP^2 = YN^2 + NP^2$$
$$= (OY - PM)^2 + PN^2$$
$$= OY^2 + PM^2 + OM^2 - 2\, OY.PM.$$

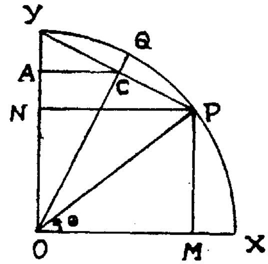

Fig. 5

Therefore,

$$4\, PC^2 = 2\, (OP^2 - OP.PM).$$

Hence, $jy\bar{a}\ \frac{1}{2} (90° - \alpha) = \sqrt{\frac{1}{2} (R^2 - R.\, jy\bar{a}\, \alpha)}$.

Similarly it can be proved that

$$jy\bar{a}\ \frac{1}{2} (90° + \alpha) = \sqrt{\frac{1}{2} (R^2 + R.\, jy\bar{a}\, \alpha)}.$$

These are of course equivalent to

$$\sin \frac{1}{2} (90° \pm \theta) = \sqrt{\frac{1}{2} (1 \pm \sin \theta)}.$$

Bhāskara II (1150) writes:

"The square of the radius is diminished or increased by the product of the radius and the *R*sine of an arc; the square-root of half the results will be the *R*sine of the half of 90° minus or plus that arc."[2]

Kamalākara defines:

"The product of the radius and the *R*sine of an arc is added to or subtracted from the square of the radius. The square-root of the half of the results are taken. They will respectively be the *R*sine of the half of three signs plus or minus the arc."[3]

---

[1] *MSi*, iii. 2.
[2] *SiŚi*, *Gola*, xiv. 12.
[3] *SiTVi*, ii. 93

He adduces the following proof of it:[1]

$R \pm jy\bar{a}\ \alpha = utjy\bar{a}\ (90^\circ \pm \alpha)$

Squaring and adding $\{jy\bar{a}\ (90^\circ \pm a)\}^2$ to both the sides, we get
$R^2 + (jy\bar{a}\ \alpha)^2 + \{jy\bar{a}\ (90^\circ \pm \alpha)\}^2 \pm 2R\ jy\bar{a}\ \alpha = \{jy\bar{a}\ (90^\circ \pm \alpha)\}^2 + \{(utjy\bar{a}\ (90^\circ \pm \alpha)\}^2$
or, $2\ (R^2 \pm R\ jy\bar{a}\ \alpha) = 4\ \{jy\bar{a}\ \frac{1}{2}\ (90^\circ \pm \alpha)\}^2$, by formulae (1) and (2).

(5) $2\ \cos^2 \frac{\theta}{2} = 1 + \cos\theta$

Bhāskara II remarks that if the arc α in the formula

$jy\bar{a}\ \frac{1}{2}\ (90^\circ \pm \alpha) = \sqrt{\frac{1}{2}(R^2 \pm R\ jy\bar{a}\ (90^\circ - \alpha)}$

be substituted by its complement $90^\circ - \alpha$, it will still be true.[2]

So that, $jy\bar{a}\ \frac{1}{2}\ (90^\circ \pm \overline{90^\circ - \alpha}) = \sqrt{\frac{1}{2}\{R^2 \pm R\ jya\,(90^\circ - \alpha)\}}$

which leads to, $2\cos^2 \frac{\theta}{2} = 1 + \cos\theta$, $2\sin^2 \frac{\theta}{2} = 1 - \cos\theta$.

Kamalākara says:

"Half the *R*cosine of an arc is added to the *R*sine of one sign (i.e. 30°) and the sum is multiplied by the radius; the square-root of the product should be known by the intelligent as the *R*cosine of half that arc."[3]

$kojy\bar{a}\frac{\alpha}{2} = \sqrt{R\ (jy\bar{a}\ 30^\circ + \frac{1}{2}\ kojy\bar{a}\ \alpha)}$

or $\cos^2 \frac{\theta}{2} = \sin 30^\circ + \frac{1}{2}\cos\theta = \frac{1}{2}(1 + \cos\theta)$

(6) $\sin^2 (45^\circ - \theta) = \frac{1}{2}\ (\cos\theta - \sin\theta)^2$

Bhāskara II says:

"The square of the difference of the *R*sine and *R*cosine of an arc is halved; the square-root of the result is equal to the *R*sine of half the difference between that arc and its complement."[4]

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[1] *SiTVi*, (Gloss).
[2] *SiŚi, Gola*, xiv. 12.
[3] *SiTVi*, ii. 91.
[4] *SiŚi, Gola*, xiv. 14.

Denote the arc XP by α ; cut off the arc YQ equal to the arc XP. Bisect the chord PQ by the point D.

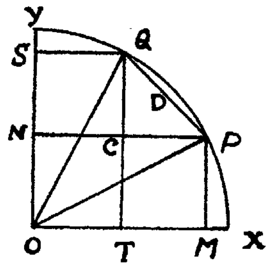

Fig. 6

Then, $CP = PN—CN = PN—QS$
$= kojy\bar{a}\ \alpha—jy\bar{a}\ \alpha$
$= QT—PM = CQ.$

Therefore, $PQ^2 = 2\ CP^2$
$PD^2 = \frac{1}{2}\ CP^2$

or, $jy\bar{a}\ \frac{1}{2}\ (\overline{90°—\alpha}—\alpha) = \sqrt{\frac{1}{2}(kojy\bar{a}\ \alpha\ —jy\bar{a}\ \alpha)^2}$

which is equivalent to

$\sin\ (45°—\theta) = \sqrt{\frac{1}{2}\ (\cos\theta—\sin\theta)^2}.$

Kamalākara writes :

"The *R*sine of half the difference between an arc and its complement should be known by the intelligent in this (science) as equal to the square-root of half the square of the difference of the *R*sine of the arc and of its complement."[1]

His proof of the formula is substantially the same as that stated above.

(7) $\cos 2\theta = 1—2\sin^2\theta$

Bhāskara II gives:

"The square of the *R*sine of an arc is divided by half the radius; the difference between this quotient and the radius is equal to the *R*sine of the difference between that arc and its complement."[2]

In the fig. 4 on page 50

$QX^2 = QT^2 + TX^2 = QT^2 + (OX—OT)^2$
$= QT^2 + OT^2 + OX^2—2\ OX.OT.$

or $4\ XD^2 = 2\ OX^2—2\ OX.QS.$

Hence, $QS = OX—\dfrac{XD^2}{OX/2}$

So that $jy\bar{a}(90°—2\alpha) = R—\dfrac{(jy\bar{a}\alpha)^2}{R/2}$

[1]*SiTVi*, ii. 95.
[2]*SiŚi, Gola*, xiv. 15.

which is the same as

$$\cos 2\theta = 1 - 2\sin^2\theta.$$

This formula is practically the same as (3). In the words of Kamalākara:

"Twice the square of the *R*sine of an arc is divided by the radius, the quotient is subtracted from the radius; the remainder will be the *R*sine of the difference of the arc and its complement."[1]

(8) $\sin^2\theta + \text{versin}^2\theta = 2\,\text{versin}\,\theta$

(9) $2\sin\theta\cos\theta + [\text{versin}\,\theta - \text{versin}\,(90^\circ - \theta)]^2 = 1$

(10) $(1+\sin\theta).\ \text{versin}\,(90^\circ - \theta) = \cos^2\theta$

(11) $2\sin\theta \pm [\text{versin}\,\theta \sim \text{versin}\,(90^\circ - \theta)]$
$= \sqrt{2 - [\text{versin}\,\theta \sim \text{versin}\,(90^\circ - \theta)]^2}$,
according as $\sin\theta \lesseqgtr \cos\theta$

(12) $(\cos\theta + \sin\theta)^2 + [\text{versin}\,\theta \sim \text{versin}\,(90^\circ - \theta)]^2 = 2.$

Formulae (8) to (12) and similar others occur in the *Vaṭeśvara-siddhānta* of Vaṭeśvara (904).

3. Addition and Subtraction Theorems

Bhāskara II (1150) says:

"The *R*sines of any two arcs of a circle are reciprocally multiplied by their *R*cosines; the products are then divided by the radius; the sum of the quotients is equal to the *R*sine of the sum of the two arcs; and their difference is the *R*sine of the difference of the arcs."[2]

If α and β be any two arcs, then the rule says:

$$jy\bar{a}\,(\alpha \pm \beta) = \frac{jy\bar{a}\,\alpha .\ kojy\bar{a}\,\beta}{R} \pm \frac{kojy\bar{a}\,\alpha .\ jy\bar{a}\,\beta}{R}$$

which is equivalent to

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi.$$

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[1] *SiTVi*, ii. 96.
[2] *SiŚi*, *Gola*, xiv. 21f.

In the words of Kamalākara (1658):

"The quotients of the *R*sines of any two arcs of a circle divided by its radius are reciprocally multiplied by their *R*cosines; the sum and difference of them (the products) are equal to the *R*sine of the sum and difference respectively of the two arcs."[1]

The rule for finding the *R*cosine of the sum and difference of two arcs of a circle is enunciated by Kamalākara thus:

"The product of the *R*cosines and of the *R*sines of two arcs of a circle are divided by its radius; the difference and sum of them (the quotients) are equal to the *R*cosine of the sum and difference (respectively) of the two arcs."[2]

$$kojy\bar{a}\ (\alpha \pm \beta) = \frac{kojy\bar{a}\ \alpha .\ kojy\bar{a}\ \beta}{R} \mp \frac{jy\bar{a}\ \alpha .\ jy\bar{a}\ \beta}{R}$$

which is equivalent to

$$\cos(\theta \pm \phi) = \cos\theta \cos\phi \pm \sin\theta \sin\phi .$$

Though we do not find this *R*cosine theorem in the printed editions of the works of Bhāskara II, we are quite sure that it was known to him. For it has been attributed to him by his most relentless critic Kamalākara[3] as well as by his commentator Muniśvara.

The above theorems can be proved by methods algebraical as well as geometrical. Several such proofs were given by previous writers, observes Kamalākara[4] (1658). Unfortunately we have not been able to trace them as yet. The following two geometrical proofs are found in the *Siddhānta-tattva-viveka*[5] of Kamalākara.

First Proof. Let the arc $YP=\beta$, and arc $YQ=\alpha$; α being greater then β. Join OP, OQ.

Draw PN, PM perpendicular to OY, OX respectively. Also draw QS perpendicular to OY, and produce it to meet the circle again at Q'. Draw QT, $Q'T'$ perpendicular to OP. Then $PN = jy\bar{a}\ \beta$, $ON = kojy\bar{a}\ \beta$, $QS = jy\bar{a}\ \alpha$, $OS = kojy\bar{a}\ \alpha$; $PG = kojy\bar{a}\ \beta - kojy\bar{a}\ \alpha$, $QG = jy\bar{a}\ \alpha + jy\bar{a}\ \beta$, $QT = jy\bar{a}\ (\alpha+\beta)$, $PT = R - kojy\bar{a}\ (\alpha+\beta)$.

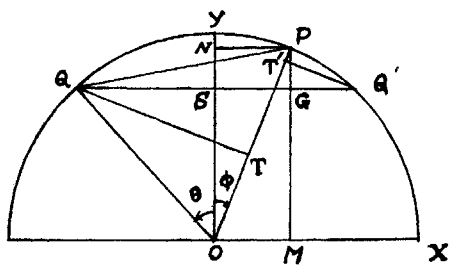

Fig. 7

[1] *SiTVi*, ii. 68.
[2] *Ibid*, ii. 69.
[3] Kamalākara remarks:
"*Evamānayanaṁ cakre pūrvaṁ svīyaśiromaṇau,*
Bhāvanābhyāmatispaṣṭaṁ saṁyagāryo' pi Bhāskaraḥ"
—*SiTVi*, ii. 70.
Or "This theorem, which is evident from the two *Bhāvanās*, was stated before also by the highly respected Bhāskara in his (*Siddhānta-*) *Śiromaṇi*."
[4] "*Tasya cānayanasyāryaiḥ siddhāntajñaiḥ puroditā,*
Vāsanā bahubhiḥ svasvabuddhivaicitryataḥ sphuṭāḥ"—*SiTVi*, ii. 71.
Or "Many correct proofs of this theorem were given before by the learned authors of the *Siddhāntas* according to the manifoldness of their intelligence."
[5] ii. 68-9 (Gloss).

Now $PG^2+QG^2=QP^2=QT^2+PT^2$.

Therefore, substituting the values

$$(kojy\bar{a}\ \beta - kojy\bar{a}\ \alpha)^2+(jy\bar{a}\ \alpha+jy\bar{a}\ \beta)^2 = \{jy\bar{a}\ (\alpha+\beta))\}^2+\{R - kojy\bar{a}\ (\alpha+\beta)\}^2$$

Simplifying we get

$$kojy\bar{a}\ (\alpha+\beta)=\frac{1}{R}(kojy\bar{a}\ \alpha.\ kojy\bar{a}\ \beta - jy\bar{a}\ \alpha.\ jy\bar{a}\ \beta)$$

which is equivalent to

$$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi.$$

Again

$$R^2-\{kojy\bar{a}\ (\alpha+\beta)\}^2=\frac{1}{R^2}\{R^4-(kojy\bar{a}\ \alpha.\ kojy\bar{a}\ \beta - jy\bar{a}\ \alpha.\ jy\bar{a}\ \beta)^2\}$$

$$=\frac{1}{R^2}[\{(jy\bar{a}\ \alpha)^2+(kojy\bar{a}\ \alpha)^2\}\times$$

$$\{(jy\bar{a}\ \beta)^2+(kojy\bar{a}\ \beta)^2\}-(kojy\bar{a}\ \alpha.kojy\bar{a}\ \beta - jy\bar{a}\ \alpha.jy\bar{a}\ \beta)^2]$$

or, $jy\bar{a}\ (\alpha+\beta)=\frac{1}{R}(jy\bar{a}\ \alpha.\ kojy\bar{a}\ \beta+kojy\bar{a}\ \alpha.\ jy\bar{a}\ \beta)$,

which is

$$\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi.$$

Since

$$PG^2+Q'G^2=Q'P^2=Q'T'^2+PT'^2,$$

we have

$$(kojy\bar{a}\ \beta - kojy\bar{a}\ \alpha)^2+(jy\bar{a}\ \alpha - jy\bar{a}\ \beta)^2=\{jy\bar{a}\ (\alpha-\beta)\}^2+\{R-kojy\bar{a}\ (\alpha-\beta)\}^2.$$

Therefore,

$$kojy\bar{a}\ (\alpha-\beta)=\frac{1}{R}(kojy\bar{a}\ \alpha.\ kojy\bar{a}\ \beta+jy\bar{a}\ \alpha.jy\bar{a}\ \beta),$$

which is

$$\cos(\theta-\phi)=\cos\theta\cos\phi+\sin\theta\sin\phi,$$

whence, proceeding as before, we get

$$jy\bar{a}\ (\alpha-\beta)=\frac{1}{R}\ (jy\bar{a}\ \alpha.\ kojy\bar{a}\ \beta-kojy\bar{a}\ \alpha.jy\bar{a}\ \beta)$$

or $\sin(\theta-\phi)=\sin\theta\cos\phi-\cos\theta\sin\phi$.

Alternative Proof. Let the arc YP=arc $PP_1=\beta$ and the arc YQ=arc $QQ_1=\alpha$. Then it is obvious from the figure that

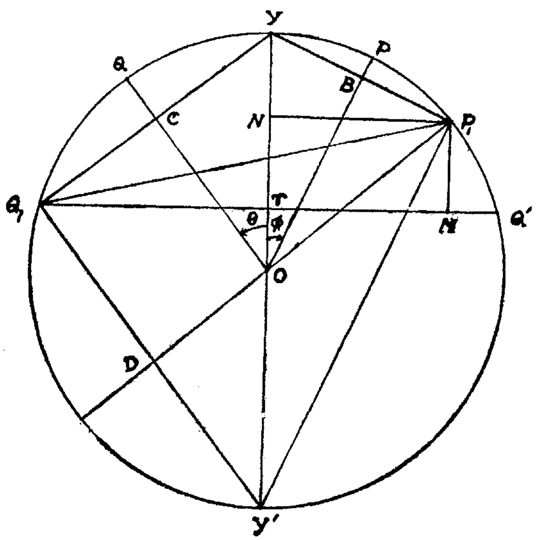

Fig. 8

$$YP_1=2\ jy\bar{a}\ \beta,\ Y'P_1=2\ kojy\bar{a}\ \beta,$$
$$YQ_1=2\ jy\bar{a}\ \alpha,\ Y'Q_1=2\ kojy\bar{a}.\alpha.$$

Also

$$Q_1P_1^2=Q_1D^2+DP_1^2=\{jy\bar{a}\ (2\alpha+2\beta)\}^2+\{(utjy\bar{a}\ (2\alpha+2\beta)\}^2.$$

Therefore, $Q_1P_1=2\ jy\bar{a}\ (\alpha+\beta)$.
Similarly $Q'_1P_1=2\ jy\bar{a}\ (\alpha-\beta)$.

By the geometrical rules for finding the height and the segments of the base of a triangle whose sides are known, it can be easily proved that

$$Y'N=\frac{2}{R}\ (kojy\bar{a}\ \beta)^2,\ YN=\frac{2}{R}\ (jy\bar{a}\ \beta)^2,\ P_1N=\frac{2}{R}\ jy\bar{a}\ \beta.\ kojy\bar{a}\ \beta,$$

$$Y'T=\frac{2}{R}\ (kojy\bar{a}\ \alpha)^2,\ \ YT=\frac{2}{R}(jy\bar{a}\ \alpha)^2,\ QT=\frac{2}{R}\ 2\ jy\bar{a}\ \alpha.\ kojy\bar{a}\ \alpha.$$

Now $Q_1P_1^2 = Q_1M^2+P_1M^2$

$$= (Q_1T+P_1N)^2+(YT-YN)^2$$

$$= \frac{4}{R^2}\,(jy\bar{a}\ \alpha.\ kojy\bar{a}\ \alpha+jy\bar{a}\ \beta.\ kojy\bar{a}\ \beta)^2+\frac{4}{R^2}\,\{(jy\bar{a}\ \alpha)^2-(jy\bar{a}\ \beta)^2\}^2$$

$$= \frac{4}{R^2}\,(jy\bar{a}\ \alpha.\ kojy\bar{a}\ \beta+kojy\bar{a}\ \alpha.\ jy\bar{a}\ \beta)^2$$

$$\therefore \quad jy\bar{a}\ (\alpha+\beta)= \frac{1}{R}\,(jy\bar{a}\ \alpha.\ kojy\bar{a}\ \beta+kojy\bar{a}\ \alpha.jy\bar{a}\ \beta),$$

which is equivalent to

$\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi.$

Again

$$Q'P_1^2 = Q'M^2+MP_1^2$$

$$= (Q_1T-P_1N)^2+(YT-YN)^2$$

$$=\frac{4}{R^2}\,(jy\bar{a}\ \alpha.\ kojy\bar{a}\ \alpha-jy\bar{a}\ \beta.\ kojy\bar{a}\ \beta)^2+\frac{4}{R^2}\,\{(jy\bar{a}\ \alpha)^2-(jy\bar{a}\ \beta)^2\}^2$$

whence

$$jy\bar{a}\ (\alpha-\beta)=\frac{1}{R}\,(jy\bar{a}\ \alpha.\ kojy\bar{a}\ \beta-kojy\bar{a}\ \alpha.\ jy\bar{a}\ \beta),$$

which is equivalent to

$\sin(\theta-\phi)=\sin\theta\cos\phi-\cos\theta\sin\phi.$

The above theorems are called *Bhāvanā* ("demonstration" or "proof" meaning "any thing demonstrated or proved", hence "theorem").

They are again divided into *Samāsa-bhāvanā* or *Yoga-bhāvanā* ("Addition Theorem") and *Antara-bhāvanā* or *Viyoga-bhāvanā* ("Subtraction Theorem").[1]

In the proofs given above the arcs α and β have been tacitly assumed to be each less than 90°. But the theorems are quite general and hold true even when the arcs are greater than 90°.

Thus Kamalākara observes:

"Even when the two arcs go beyond 90° to any even or odd quadrant (the theorems) will remain the same, not otherwise. That is the opinion of those who are aware of the true facts."[2]

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[1] *SiŚi, Gola,* xiv. 21 (*Gloss*); *SiTVi,* ii. 65.
[2] *SiTVi,* ii. 66f.

Functions of Multiple Angles

As corollaries to the general case of the theorems for expanding sin $(\theta \pm \phi)$ and cos $(\theta \pm \phi)$, Bhāskara II (1150) indicates how to derive the functions of multiple angles. He observes:

"This being proved, it becomes an argument for determining the values of other functions. For example, take the case of the combination of functions of equal arcs: by combining the functions of any arc with those of itself, we get the functions of twice that arc; by combining the functions of twice the arc with those of twice the arc, we get functions of four times that arc; and so on. Next take the case of combination of functions of unequal arcs: on combining the functions of twice an arc with those of thrice that arc, by the addition theorem we get the functions of five times that arc; but by the subtraction theorem, we get the functions of one time that arc; and so on."[1]

The theorems meant here are clearly these:

$$\begin{aligned}
\sin 2\theta &= 2 \sin \theta \cos \theta, \\
\cos 2\theta &= \cos^2\theta - \sin^2\theta, \\
\sin 4\theta &= 2 \sin 2\theta \cos 2\theta, \\
&= 4 \sin \theta \cos \theta (\cos^2 \theta - \sin^2 \theta), \\
\cos 4\theta &= \cos^2 2\theta - \sin^2 2\theta, \\
&= \cos^4 \theta - 6 \sin^2\theta\cos^2\theta + \sin^4\theta, \\
\sin 3\theta &= 3 \sin\theta \cos^2\theta - \sin^3\theta, \\
\cos 3\theta &= \cos^3\theta - 3 \cos\theta \sin^2\theta, \\
\sin 5\theta &= \sin 2\theta \cos 3\theta + \cos 2\theta \sin 3\theta, \\
\cos 5\theta &= \cos 2\theta \cos 3\theta - \sin 2\theta \sin 3\theta, \\
\sin\theta &= \sin 3\theta \cos 2\theta - \cos 3\theta \sin 2\theta, \\
\cos\theta &= \cos 3\theta \cos 2\theta + \sin 3\theta \sin 2\theta.
\end{aligned}$$

All these theorems have been expressly stated by Kamalākara (1658).

He says:

"Hereafter I shall describe how to find the *R*sine of twice, thrice, four times or five times an arc, having known the *R*sine of the sum of two arcs. The product of the *R*sine and *R*cosine of an arc is multiplied by 2 and divided by the radius; the result is the *R*sine of twice that arc."[2]

"The difference of the squares of the *R*sine and *R*cosine of an arc is divided by the radius; the quotient is certainly the *R*cosine of twice that arc."[3]

He has given the following proof of the above two formulae.[4]

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[1] *SiŚi, Gola*, xiv. 21-2 (*Gloss*).
[2] *SiTVi*, ii. 73.
[3] *Ibid*, ii. 90.
[4] See his own gloss on the preceding rules.

Let the arc XP=arc PQ, then $X'Q=2OC=2$ *kojyā* α. Now from the right angled triangles OPM, $X'QT$, we have

$PO : PM : : X'Q : QT$,
and $OP : OM : : X'Q : X'T$.
$\therefore\ OP.QT=PM.X'Q=2PM.OC$,
and $OP.X'T=OM.X'Q$,
or $OP(X'O+OT)=OM.\ 2OC=2OM^2$, because $OC=OM$
or $OP(OP+OT)=2OM^2$,
or $OP.OT=2OM^2-OP^2=OM^2-PM^2$.

Therefore,

$$QT=\frac{2\,PM.OC}{R}$$

$$OT=\frac{OM^2-PM^2}{R}.$$

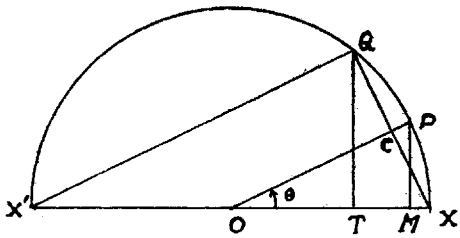

Fig. 9

Hence,

$$jy\bar{a}\ 2\alpha=\frac{2jy\bar{a}\ \alpha.\ kojy\bar{a}\ \alpha}{R}.$$

$$kojy\bar{a}\ 2\alpha=\frac{(kojy\bar{a}\ \alpha)^2-(jy\bar{a}\ \alpha)^2}{R}.$$

That is,

$$\sin 2\theta = 2\sin\theta\cos\theta,$$
$$\cos 2\theta = \cos^2\theta-\sin^2\theta.$$

It has been further observed that these results can be easily deduced from the Addition Theorem by putting $\phi=\theta$.

"The sine of an arc is divided by the sine of one sign (i.e. 30°); the square of the quotient is subtracted from 3 and the remainder is multiplied by the R sine of the arc; the result is the R sine of thrice that arc."[1]

$$jy\bar{a}\ 3\alpha=jy\bar{a}\ \alpha\left\{3-\left(\frac{jya\ \alpha}{jy\bar{a}\ 30^\circ}\right)^2\right\}.$$

That is, $\sin 3\theta=\sin\theta\left(3-\dfrac{\sin^2\theta}{\sin^2 30^\circ}\right)$.

[1] *SiTVi*, ii. 74; also the Gloss.

By the successive application of the Addition Theorems, Kamalākara obtains the formulae:[1]

$$\begin{aligned}
jy\bar{a}\ 3\alpha &= \{3R^2\ jy\bar{a}\ \alpha - 4(jy\bar{a}\ \alpha)^3\}/R^2, \\
kojy\bar{a}\ 3\alpha &= \{4\ (kojy\bar{a}\ \alpha)^3 - 3R^2\ kojy\bar{a}\ \alpha\}/R^2, \\
jy\bar{a}\ 4\alpha &= 4\ \{(kojy\bar{a}\ \alpha)^3\ jy\bar{a}\ \alpha - (jy\bar{a}\ \alpha)^3\ kojy\bar{a}\ \alpha\}/R^3, \\
kojy\bar{a}\ 4\alpha &= \{(kojy\bar{a}\ \alpha)^4 - 6\ (kojy\bar{a}\alpha)^2\ (jy\bar{a}\alpha)^2 + (jy\bar{a}\ \alpha)^4\}/R^3, \\
jy\bar{a}\ 5\alpha &= \{(jy\bar{a}\ \alpha)^5 - 10\ (jy\bar{a}\ \alpha)^3\ (kojy\bar{a}\ \alpha)^2 + 5\ jy\bar{a}\ \alpha\ (kojy\bar{a}\ \alpha)^4\}/R^4, \\
kojy\bar{a}\ 5\alpha &= \{(kojy\bar{a}\ \alpha)^5 - 10\ (kojy\bar{a}\ \alpha)^3\ (jy\bar{a}\ \alpha)^2 + 5\ kojy\bar{a}\ \alpha\ (jy\bar{a}\ \alpha)^4\}/R^4;
\end{aligned}$$

which are of course equivalent to

$$\begin{aligned}
\sin 3\theta &= 3 \sin \theta - 4 \sin^3 \theta, \\
\cos 3\theta &= 4 \cos^3 \theta - 3 \cos \theta, \\
\sin 4\theta &= 4\ (\cos^3 \theta \sin \theta - \sin^3 \theta \cos \theta), \\
\cos 4\theta &= \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta, \\
\sin 5\theta &= \sin^5 \theta - 10 \sin^3\theta \cos^2 \theta + 5 \sin \theta \cos^4 \theta, \\
\cos 5\theta &= \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta.
\end{aligned}$$

Functions of Submultiple Angles

It has been stated before that the following two formulae for the sine of half an angle were known to almost all the Hindu astronomers:

$$\sin \frac{\theta}{2} = \tfrac{1}{2}\sqrt{\sin^2\theta + \text{versin}^2\theta},$$

$$\sin \frac{\theta}{2} = \sqrt{\tfrac{1}{2}\ (1 - \cos\theta)}.$$

Besides these[2] Kamalākara has given formulae for the functions of the third, fourth and fifth parts of an arc.

"Find the cube of one-third the *R*sine of an arc; divide it by the square of the radius; the quotient is added to its one-third and the sum again to one-third the *R*sine of the arc; the result is nearly the *R*sine of one-third that arc. From the cube of this again further accurate values can be obtained."[3]

$$jy\bar{a}\ \frac{\alpha}{3} = \tfrac{1}{3}\,jy\bar{a}\ \alpha + \frac{4}{3\ R^2}\left(\frac{jy\bar{a}\ \alpha}{3}\right)^3.$$

The *rationale* of this formula has been stated to be this: As has been proved before

$$jy\bar{a}\ 3\beta = 3\ jy\bar{a}\,\beta - \frac{4}{R^2}\ (jy\bar{a}\ \beta)^3.$$

[1] *SiTVi*, ii. 75-7 and also the Gloss on them.
[2] *SiTVi*, ii. 78 f.
[3] *Ibid*, ii. 81.

Put $3\beta=\alpha$; then this formula will become

$$jy\bar{a}\,\frac{\alpha}{3} = \tfrac{1}{3}\,jy\bar{a}\alpha + \frac{4}{3R^2}\left(jy\bar{a}\frac{\alpha}{3}\right)^3. \qquad (i)$$

Now $jy\bar{a}\,\frac{\alpha}{3}$ can be taken, says Kamalàkara, as a *rough approximation* (*sthūla*) to be equal to $\left(\frac{jy\bar{a}\,\alpha}{3}\right)^3$. So that approximately

$$jy\bar{a}\,\frac{\alpha}{3} = \tfrac{1}{3}\,jy\bar{a}\,\alpha + \frac{4}{3R^2}\left(\frac{jy\bar{a}\,\alpha}{3}\right)^3, \qquad (ii)$$

as stated in the rule. Very nearer approximation (*sūkṣmāsanna*) to the value of $jy\bar{a}\,\frac{\alpha}{3}$ can be found by substituting the cube of this value in the last term of (*i*) and by repeating similar operations.

The form (*ii*) is equivalent to

$$\sin\,\frac{\theta}{3} = \frac{1}{3}\,\sin\theta + \frac{4}{81}\,\sin^3\theta.$$

"From the known value of the *R*sine of an arc, first calculate the value of the *R*sine of half that arc; the *R*sine of the arc is divided by that and multiplied by the square of the radius; the result is subtracted from twice the square of the radius. Half the square-root of the remainder is the value of the *R*sine of one-fourth that arc."[1]

$$jy\bar{a}\,\frac{\alpha}{4} = \frac{1}{2}\sqrt{2R^2 - R^2\,\frac{jy\bar{a}\,\alpha}{jy\bar{a}\,(\alpha/2)}}\,.$$

The *rationale* of this formula is given thus: It is known that

$$jy\bar{a}\,4\beta = \frac{4}{R^3}\left\{(kojy\bar{a}\,\beta)^3\,jy\bar{a}\,\beta - (jy\bar{a}\,\beta)^3\,kojy\bar{a}\,\beta\right\},$$

$$= \frac{4}{R^3}\left\{R^2\,jy\bar{a}\,\beta\,kojy\bar{a}\,\beta - 2\,(jy\bar{a}\beta)^3\,kojy\bar{a}\,\beta\right\},$$

Putting α for 4β, we get

$$R^3\,jy\bar{a}\,\alpha = 4\,jy\bar{a}\,\frac{\alpha}{4}\,kojy\bar{a}\,\frac{\alpha}{4}\,\left\{R^2 - 2\,(jy\bar{a}\,\frac{\alpha}{4})^2\right\},$$

$$= 2R\,jy\bar{a}\,\frac{\phi}{2}\,\left\{R^2 - 2\,(jy\bar{a}\,\frac{\alpha}{4})^2\right\};$$

[1] *SiTVi*, ii. 82-83.

whence

$$jy\bar{a}\,\frac{\alpha}{4}=\frac{1}{2}\sqrt{2R^2-R^2\,(jy\bar{a}\,\alpha)/jy\bar{a}\,\frac{\alpha}{2}}$$

or

$$\sin\frac{\theta}{4}=\frac{1}{2}\sqrt{2-\frac{\sin\theta}{\sin(\theta/2)}}.$$

"The intelligent should first find the one-fifth of the *R*sine of the given arc; divide four times the cube of that by the square of the radius; the quotient should be called the "first". Multiply the "first" by the square of the fifth part of the *R*sine and divide the product by the square of the radius; lessen this quotient by its fifth part and mark the remainder as the "second". One-fifth of the *R*sine of the arc added with the "first" and diminished by the "second", will be clearly the value of the *R*sine of the fifth part of the arc. Finding the value of the "first" again from this, further approximate value to the *R* sine of one-fifth the arc can be found. Still closer approximations can be obtained by repeating the process stated above."[1]

$$jy\bar{a}\,\frac{\alpha}{5}=\frac{1}{5}\,jy\bar{a}\,\alpha+\frac{4}{R^2}\left(\frac{jya\,\alpha}{5}\right)^3-\frac{16}{5R^4}\left(\frac{jy\breve{a}\,\alpha}{5}\right)^5.$$

The *rationale* is stated to be this: It has been established before that
$R^4\,jy\bar{a}\,5\beta=(jy\bar{a}\,\beta)^5-10\,(jy\bar{a}\,\beta)^3\,(kojy\bar{a}\,\beta)^2+5jy\bar{a}\,\beta\,(kojy\bar{a}\,\beta)^4$.

Substituting the value $R^2-(jy\bar{a}\,\beta)^2$ for $(kojy\bar{a}\,\beta)^2$ in this, we get
$R^4\,jy\bar{a}\,5\,\beta=16\,(jy\bar{a}\,\beta)^5-20R^2\,(jy\bar{a}\,\beta)^3+5R^4\,jy\bar{a}\,\beta$

Putting α for 5β,

$$jy\bar{a}\,\frac{\alpha}{5}=\frac{1}{5}\,jy\bar{a}\,\alpha+\frac{4}{R^2}(jy\bar{a}\,\frac{\alpha^3}{5})-\frac{61}{5R^4}(jy\bar{a}\,\frac{\alpha}{5})^5. \qquad (i)$$

In the last two terms on the right hand side, one may take as a rough approximation

$$jy\bar{a}\frac{\alpha}{5}=\frac{1}{5}\,jy\bar{a}\,\alpha;$$

so that

$$jy\bar{a}\frac{\alpha}{5}=\frac{1}{5}\,jy\bar{a}\alpha+\frac{4}{R^2}\left(\frac{jy\bar{a}\,\alpha}{5}\right)^3-\frac{16}{5R^4}\left(\frac{jy\bar{a}\,\alpha}{5}\right)^5\quad. \qquad (ii)$$

[1] *SiTVi*, ii. 84-87.

Again substituting this value of *jyā* $\frac{\alpha}{5}$ in the last two terms of (*i*) and repeating similar operations, closer approximations to the value of *jyā* $\frac{\alpha}{5}$ can be obtained.

The formula (*ii*) is equivalent to

$$\sin\frac{\theta}{5}=\frac{1}{5}\sin\theta+4\left(\frac{\sin\theta}{5}\right)^3-\frac{16}{5}\left(\frac{\sin\theta}{5}\right)^5.$$

Kamalākara then observes that "in this way, the *R*sines of other desired submultiples of an arc should be obtained."[1]

Sin $\frac{(\theta-\phi)}{2}$.

Bhāskara II says:

"Find the difference of the *R*sines of two arcs and also of their *R*cosines; then find the square-root of the sum of the squares of the two results; half this root will be the *R*sine of half the difference of the two arcs."[2]

That is,

$$jy\bar{a}\ \frac{1}{2}(\alpha-\beta)=\frac{1}{2}\{(jy\bar{a}\ \alpha-jy\bar{a}\ \beta)^2+(kojy\bar{a}\ \alpha-kojy\bar{a}\ \beta)^2\}^{\frac{1}{2}}.$$

or, in modern notations,

$$\sin\tfrac{1}{2}(\theta-\phi)=\tfrac{1}{2}\{(\sin\theta-\sin\phi)^2+(\cos\theta-\cos\phi)^2\}^{\frac{1}{2}}.$$

Kamalākara writes:

"Half the square-root of the sum of the squares of the differences of *R*sines and *R*cosines of two arcs is certainly equal to the *R*sine of half the difference of the two arcs."[3]

The latter has given the following proof of it.[4] Let the arc *XP* be denoted by β and the arc *XQ* by α; then

[1] *SiTVi*, ii. 87 (c-d).
[2] *SiŚi, Gola*, xiv. 13.
[3] *SiTVi*, ii. 94.
[4] *Ibid*, (Gloss).

$$QC = QT - PM = jy\bar{a}\ \alpha - jy\bar{a}\ \beta,$$
$$PC = OM - OT = kojy\bar{a}\ \beta - kojy\bar{a}\ \alpha.$$

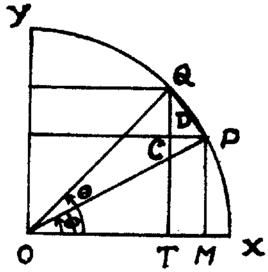

Fig.10

Now,

$$PQ^2 = QC^2 + PC^2.$$

Hence,

$$jy\bar{a}\ \tfrac{1}{2}\ (\alpha - \beta) = \tfrac{1}{2}\left\{(jy\bar{a}\ \alpha - jy\bar{a}\ \beta)^2 + (kojy\bar{a}\ \alpha - kojy\bar{a}\ \beta))^2\right\}^{\frac{1}{2}};$$

which is equivalent to

$$\sin \tfrac{1}{2}\ (\theta - \phi) = \tfrac{1}{2}\left\{(\sin\ \theta - \sin\phi)^2 + (\cos\ \theta - \cos\phi)^2\right\}^{\frac{1}{2}}.$$

Theorem of Sines

Brahmagupta[1] has made use of the important relation

$$\frac{a}{jy\bar{a}\ A} = \frac{b}{jy\bar{a}\ B} = \frac{c}{jy\bar{a}\ C}$$

which is of course equivalent to

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

between the sides (a, b, c) and angles (A, B, C) of a plane triangle.

4. Functions of Particular Angles

Sine of 30°, 45° and 60°. Preliminary to the calculation of tables of trigonometrical functions almost all the Hindu writers have stated the values of the *R*sines of 30°, 45° and 60°.

$jy\bar{a}\ 30° = \sqrt{R^2/4}$, $jy\bar{a}\ 45° = \sqrt{R^2/2}$, $jy\bar{a}\ 60° = \sqrt{3R^2/4}$ or, in modern notations, $\sin\ 30° = \frac{1}{2}$, $\sin\ 45° = 1/\sqrt{2}$, $\sin\ 60° = \frac{1}{2}\sqrt{3}$.

Śrīpati indicates the proof thus:

"The experts in spherics say that the circum-radius of a regular hexagon is equal to a side. So it will be perceived that the chord of the sixth part of the circumference

[1]*KK*, Part I, viii. 2. Our attention to this was first drawn by Professor P. C. Sen Gupta, who was then preparing a new edition of *Khaṇḍakhādyaka* with English translation and critical notes. This rule occurs in the works of other Hindu astronomers also.

of a circle is equal to its semi-diameter. The hypotenuse arising from the base and perpendicular (of a right-angled triangle) each equal to the semi-diameter is the chord of the fourth part of the circumference. Half those (chords) will be the *R*sines of half those arcs."[1]

The same proof is also given by Bhāskara II:[2]

"The side of a regular hexagon inscribed in a circle is equal to its radius; this is well known and has also been stated in (my) Arithmetic. Hence follows that the *R*sine of 30° is half the radius."

"Suppose a right-angled triangle whose base and perpendicular are each equal to the radius; the square-root of the sum of the squares of these will be equal to the side of a square inscribed in that circle and it is again the chord of 90°. Take the half of that. Hence the sum of the squares (of the sides) is divided by four; and the result is half the square of the radius. The square-root of that, it thus follows, is the *R*sine of 45°."

"The *R*sine of 60° is equal to the *R*cosine of 30°, the *R*sine of which is equal to half the semi-diameter."

Sin 18° and sin 36°. Bhāskara II says:

"The square-root of five times the square of the radius is diminished by the radius and the remainder is divided by four; the result is the exact value of the *R*sine of 18°."[3]

$$jy\bar{a}\ 18^\circ = \tfrac{1}{4}\left(\sqrt{5R^2} - R\right);$$

or $$\sin 18^\circ = \tfrac{1}{4}\left(\sqrt{5} - 1\right).$$

"The square-root of five times the square of the square of the radius is subtracted from five times the square of the radius and the remainder is divided by eight; the square-root of the quotient is the *R* sine of 36°."

"Or the radius multiplied by 5878 and divided by 10000, is the *R*sine of 36°. The *R*cosine of that is the *R*sine of 54°."[4]

$$jy\bar{a}\ 36^\circ = \sqrt{\tfrac{1}{8}\left(5R^2 - \sqrt{5R^4}\right)} = \frac{5878R}{10000}.$$

That is, $$\sin 36^\circ = \sqrt{\tfrac{1}{8}\left(5 - \sqrt{5}\right)} = \frac{5878}{10000}$$

[1] *SiŚe*, xvi. 11-2.
[2] *SiŚi, Gola*, v. 3-4 (Gloss).
[3] *Ibid, Gola*, xiv. 9.
[4] *Ibid, Gola*, xiv. 7-8.

Since $\sqrt{5} = 2{\cdot}237411$ approximately

$\therefore \quad 5 - \sqrt{5} = 2{\cdot}762589 \ldots$

$\therefore \quad \sqrt{\tfrac{1}{8}(5-\sqrt{5})} = \sqrt{.345323 \ldots} = {\cdot}5878$ approximately.

Kamalākara proved the results thus:

Let x denote *jyā* 18°; then

$$\tfrac{1}{2} R (R-x) = \tfrac{1}{2} R.\ utjy\bar{a}\ 72^\circ = (jy\bar{a}\ 36^\circ)^2;$$

$$\frac{2x^2}{R} = R - kojy\bar{a}\ 36^\circ = utjy\bar{a}\ 36^\circ;$$

$$\begin{aligned}\tfrac{1}{2} R (R-x) + \left(\frac{2x^2}{R}\right)^2 &= (jy\bar{a}\ 36^\circ)^2 + (utjy\bar{a}\ 36^\circ)^2 \\ &= 4\,(jy\bar{a}\ 18^\circ)^2 \\ &= 4x^2,\end{aligned}$$

or $8\,x^2 R^2 = 8\,x^4 - R^3 x + R^4$,

or, multiplying by 8 and arranging,

$$16R^2 x^2 + 8\,R^3 x + R^4 = 9\,R^4 - 48\,R^2 x^2 + 64\,x^4,$$

whence taking the positive square roots of the two sides, we get

$$4\,Rx + R^2 = 3\,R^2 - 8\,x^2,$$
or $(4\,x + R)^2 = 5R^2$.

Therefore $x = \tfrac{1}{4}(\sqrt{5R^2} - R)$;

the other sign is neglected since x must be less than R.

Again $(jy\bar{a}\ 36^\circ)^2 = \tfrac{1}{2}R(R-x)$,

$$= \frac{R}{8}(5R - \sqrt{5\,R^2}),$$

$$\therefore jy\bar{a}\ 36^\circ = \sqrt{\tfrac{1}{8}(5R^2 - \sqrt{5R^4})}$$

Sin π/N

In his treatise on arithmetic, Bhāskara II has given a rule which yields the *R*sine of certain particular angles to a very fair degree of approximation.

"Multiply the diameter of a circle by 103923, 84853, 70534, 60000, 52055, 45922 and 41031 severally and divide the products by 120000; the quotients will be the sides of regular polygons inscribed in the circle from the triangle to the enneagon respectively."[1]

If S_n be a side of a regular polygon of n sides inscribed in a circle of diameter D, then according to Bhāskara II,

$$S_3 = D\,\frac{103923}{120000} = D \times {\cdot}866025$$

$$S_4 = D\,\frac{84853}{120000} = D \times {\cdot}707108\dot{3}$$

$$S_5 = D\,\frac{70534}{120000} = D \times {\cdot}58778\dot{3}$$

$$S_6 = D\,\frac{60000}{120000} = D \times {\cdot}5$$

$$S_7 = D\,\frac{52055}{120000} = D \times {\cdot}433791\dot{6}$$

$$S_8 = D\,\frac{45922}{120000} = D \times {\cdot}38268\dot{3}$$

$$S_9 = D\,\frac{41031}{120000} = D \times {\cdot}341925$$

where are given the formulae of Bhāskara II first in their original forms and then in decimals. Now, we know that

$$s_n = D \sin\frac{\pi}{n}\,.$$

Hence it is found that

$$\left.\begin{aligned} \sin 60^\circ &= {\cdot}866025 \\ \sin 45^\circ &= {\cdot}707108\dot{3} \\ \sin 36^\circ &= {\cdot}58778\dot{3} \end{aligned}\right\} \qquad \begin{aligned} \sin \pi/7 &= {\cdot}433791\dot{6} \\ \sin \pi/8 &= {\cdot}38268\dot{3} \\ \sin \pi/9 &= {\cdot}341925 \end{aligned}$$

According to modern computation

$$\left.\begin{aligned} \sin 60^\circ &= {\cdot}8660254\ldots. \\ \sin 45^\circ &= {\cdot}7071067\ldots. \\ \sin 36^\circ &= {\cdot}5877853 \end{aligned}\right\} \qquad \begin{aligned} \sin \pi/7 &= {\cdot}4338819 \\ \sin \pi/8 &= {\cdot}3826834 \\ \sin \pi/9 &= {\cdot}3420201 \end{aligned}$$

[1] *L*, vss. 206-7, p. 207.

Comparing the two tables we find that except in case of sin $\pi/7$ and sin $\pi/9$ Bhāskara'a approximations are correct up to five places of decimals; in these two latter cases the results are near enough.

Approximate Formula of Bhāskara I

Bhāskara I (629) has given the following rule for the calculation of the *R*sine and *R*cosine of an arc without the help of a table.

"Subtract the arc in degrees from the degrees of the semi-circumference and multiplying the arc by the remainder, put down (the result) at two places, (at one place) subtract (the quantity) from 40500; by one-fourth of the ıemainder divide the quantity (at the second place) multiplied by the maximum value of the function; thus the value of the direct or reversed *R*sine of an arc and its complement is obtained wholly."[1]

If α be an arc of a circle of radius R in terms of degrees, then

$$jy\bar{a}\ \alpha = \frac{R(C/2-\alpha)\alpha}{\{40500-(C/2-\alpha)\ \alpha\}/4},$$

where C denotes the circumference of the circle in terms of degrees. Since $40500=(5/4)\times 180\times 180$, we can write the formula in the form

$$jy\bar{a}\ \alpha = \frac{4R(C/2-\alpha)\alpha}{5/4\ (C/2)^2-(C/2-\alpha)\ \alpha},$$

which is of course equivalent to

$$\sin\theta = \frac{4\ (\pi-\theta)\ \theta}{(5/4)\ \pi^2-(\pi-\theta)\ \theta}.$$

From a statement of Bhāskara I it appears that this formula was known to Ārya-bhaṭa I.[2]

The above formula has been restated by Brahmagupta (628) thus:

"Subtract the degrees of an arc or its complement from the semicircle (i.e. 180) and multiply (the remainder) by that; subtract one-fourth the product from 10125; divide the product by the remainder and multiply by the semi-diameter; (the result) is the *R*sine of that (arc or its complement)."[3]

$$jy\bar{a}\ \alpha = \frac{R\ (180-\alpha)\ \alpha}{10125-(180-\alpha)\ \alpha/4}.$$

[1] *MBh*, vii. 17ff.
[2] Bhāskara I's com. on *Ā*, i. 11, p. 40.
[3] *BrSpSi*, xiv. 23.

Almost in the same way Śrīpati (1039) says:

"Subtract the degrees of an arc or its complement from 180 and multiply (the remainder) by that; subtract one-fourth the product from 10125; multiply the product by the semi-diameter and divide by this remainder; thus the *R*sine of an arc or its complement can be found even without (a table of *R*sines)."[1]

Bhāskara II (1150) writes:

"Subtract an arc from the circumference and multiply (the remainder) by the arc; this product is called the 'first'. From five times the fourth part of the square of the circumference subtract the 'first', and by the remainder divide the 'first' multiplied by four times the diameter; the quotient will be the chord of the arc."[2]

If s denote the chord of an arc β of a circle, then

$$s = \frac{8R\,(C-\beta)\beta}{\frac{5}{4}C^2-(C-\beta)\beta}.$$

Now if $\beta=2\alpha$, then $s=2 jy\bar{a}\ \alpha$. So that on making the substitutions this formula will easily reduce to that of the elder Bhāskara.

This formula has been used by Gaṇeśa (1545) in his *Grahalāghava*.[3] Though it gives only a roughly approximate (*sthūla*) value of the *R*sine of an arc, observes Bhāskara II, it simplifies operations.

On putting $\theta = \pi/2-\phi$, in the above approximate formula, it becomes

$$\cos\phi = \frac{16\,(\pi/2+\phi)\,(\pi/2-\phi)}{5\pi^2-4\,(\pi/2+\phi)\,(\pi/2-\phi)}$$

$$= \frac{\pi^2-4\,\phi^2}{\pi^2+\phi^2}$$

$$= \left(1-\frac{4\phi^2}{\pi^2}\right)\left(1-\frac{\phi^2}{\pi^2}+\frac{\phi^4}{\pi^4}\right),$$

neglecting higher powers. Therefore, to the same order of approximation,

$$\cos\phi = 1-\frac{5\phi^2}{\pi^2}+\frac{5\phi^4}{\pi^4}.$$

[1] *SiŚe*, iii. 17.
[2] *L*, vs. 210, p. 21e. Also see *GK*, par 2, pp. 80-81.
[3] *GrL*, ii. 2 f.

If we put $\pi = \sqrt{10}$ approximately, we get

$$\cos\phi = 1 - \frac{\phi^2}{2} + \frac{\phi^4}{20}\,.$$

nearly. According to modern Trigonometry, to the same order of approximation,

$$\cos\phi = 1 - \frac{\phi^2}{2} + \frac{\phi^4}{24}\,.$$

Again putting $\phi = \pi/n$ in Bhāskara I's formula, where n is an integer, we get

$$\sin\pi/n = \frac{16\,(n-1)}{5n^2 - 4(n-1)}$$

whence we have

$\sin\pi/7 = \cdot 4343\ldots,\quad \sin\pi/8 = \cdot 3835\ \ldots,\qquad \sin\pi/9 = \cdot 3431\ \ldots,$
which are correct up to two places of decimals, the third figure in every case being too large.

Inverse Formula of Brahmagupta

Brahmagupta gave the following rule for finding approximately the arc corresponding to a given *R*sine function:

"Multiply 10125 by the given *R*sine and divide by the quarter of the given *R*sine plus the radius; subtracting the quotient from the square of 90, extract the square-root and subtract (the root) from 90; the remainder will be in degrees and minutes; thus will be found the arc of the given *R*sine without the table of *R*sines."[1]

If α be the arc corresponding to the given *R*sine function m, then the rule says that

$$\alpha = 90 - \sqrt{8100 - \frac{10125m}{(m/4+r)}}\,.$$

This result follows easily on reversing the approximate formula for the *R*sine and was very likely obtained in the same way.

$$m = jy\bar{a}\,\alpha = \frac{R(180-\alpha)\,\alpha}{10125-(180-\alpha)\,\alpha/4}\,.$$

Then,

$$\alpha^2 - 180\alpha + \frac{10125m}{(m/4+r)} = 0.$$

[1]*BrSpSi*, xiv. 25-6.

Therefore,

$$\alpha = 90 - \sqrt{8100 - \frac{10125m}{(m/4+r)}} .$$

The negative sign of the radical being retained, since α is supposed to be less than 90°.

Śrīpati describes the inverse formula thus:

"Multiply 10125 by the given *R*sine and divide by the quarter of the given *R*sine plus the radius; then subtract the quotient from the square of 90; ninety degrees lessened by the square root (of the remainder) will be the arc (determined) without the table of *R*sines."[1]

Bhāskara II writes:

"By four times the diameter added with the chord divide the square of the circumference multiplied by five times a quarter of the chord; the quotient being subtracted from the fourth part of the square of the circumference, and the square-root of the remainder being diminished from half the circumference, the result will be the arc."[2]

That is:

$$\beta = \frac{C}{2} - \left\{ \frac{C^2}{4} - \frac{5s\,C^2}{4(8R+s)} \right\}^{\frac{1}{2}}$$

which follows at once from his form of the approximate formula for the chord s.

5. Trigonometrical Tables

Twenty-four Sines

The Hindus generally calculate tables of trigonometrical functions for every arc of 3°45′, or what they call twenty-four *R*sines in a quadrant. In the choice of 24, they seem to have been led by an ancient observation that "the ninety-sixth part of a circle looks (straight) like a rod". Thus Balabhadra (*c.* 700 A.D.) observes, "If anybody asks the reason of this, he must know that each of these *Kardajat* is 1/96 of the circle = 225 minutes (= $3\frac{3}{4}$ degrees). And if we reckon its *R*sine, we find it also to be 225 minutes."[3]

The origin of this idea again lies in the impression that the human eye-sight reaches to a distance of 1/96th part of the circumference of the earth which appears

[1]*SiŚe*, iii. 18.

[2]*L*, vs. 212, p. 216.

[3]Quoted by Al-Bīrūnī in his *India* (Sachau, *Alberuni's India*, I, p. 275). Balabhadra's works are now lost. According to Chambers' *Mathematical Tables*, we find sin (3° 45′) = ·0654031, tan (3° 45′) = ·0655435, radian (3° 45′) = ·0654498, so that the assumption is fairly accurate.

flat.[1] A more plausible hypothesis about the choice of 3°45′ as the unit will be this: Having determined previously a very accurate value of π it was simply natural for the Hindus to choose the radius of the circle of reference to be 3438′. They also knew that *R*sin 30°=semi-radius=1719′. Starting with this they began to calculate the function of the semi arcs 15°, 7° 30′, 3°45′ with the help of the well known formulae and in so doing they soon found that 3° 45′ is the first whose *R*sine contains the same number of minutes as the arc. So they chose this arc.[2]

Sūrya-Siddhānta

The earliest known Hindu work to contain a table of trigonometrical functions is the *Sūrya-Siddhānta* (*c.* 300 A.D.). It has a table of *R*sines and versed *R*sines for every arc of 3°45′ of a circle of radius 3438′. The method of computation has been indicated to be as follows:

"The eighth part of the number of minutes in a sign (i.e. 225′) is the first *R*sine. It is divided by itself and then diminished by the quotient; the remainder added with the first *R*sine gives the second *R*sine.

"(Any) *R*sine is divided by the first *R*sine and then diminished by the quotient. The remainder added to the difference of that *R*sine and the preceding *R*sine will give the next *R*sine. Thus can be obtained the 24 *R*sines, which are as follows."[3]

Let the arc PQ=arc $QR=\alpha$

Then,

$$RK = OL = OS + SL = QT + SL,$$

and $$NS - SL = 2\,DS = 2\,\frac{OS}{OQ}\,QC$$

Now $$CP^2 = QC\,(2\,OQ - QC);$$

$$\therefore \quad QC = \frac{QP^2}{2\,OQ}.$$

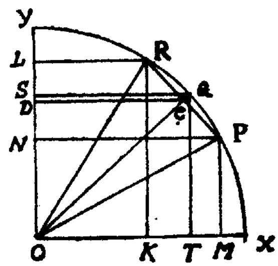

Fig. 11

Hence,

$$NS - SL = OS\left(\frac{QP}{OQ}\right)^2$$

[1]Balabhadra says, "Human eyesight reaches to a point distant from the earth and its rotundity the 96th part of 5000 *yojana*, i.e. 52 *yojana* (exactly $52\frac{1}{12}$). Therefore a man does not observe its rotundity, and hence the discrepancy of opinions on the subject". This remark of Balabhadra has been quoted by Al-Bīrūnī (*India*, I, p. 273).

[2]Cf. *Nouv. Ann. Math.*, xiii (1854), p. 390.

[3]*SūSi*, ii. 15-6.

Or,

$$SL = NS - OS\left(\frac{QP}{OQ}\right)^2$$

$$= (QT - PM) - OS\left(\frac{QP}{OQ}\right)^2$$

Therefore,

$$RK = QT + (QT - PM) - QT\left(\frac{QP}{OQ}\right)^2$$

Now suppose the arc $XQ = n\alpha$; then arc $XP=(n-1)\alpha$; $XR=(n+1)\alpha$;

further $QP = 2jy\bar{a}\,\frac{\alpha}{2}$. Hence

$$jy\bar{a}\,(n+1)\alpha = jy\bar{a}\; n\alpha + \{\,jy\bar{a}\; n\alpha - jy\bar{a}(n-1)\alpha\,\} - jy\bar{a}\; n\alpha \times \left(\frac{2jy\bar{a}\,\alpha/2}{R}\right)^2,$$

which is equivalent to

$$\sin(n+1)\,\theta = \sin n\theta + \{\sin n\theta - \sin(n-1)\theta\} - \sin n\theta\left(2\sin\frac{\theta}{2}\right)^2.$$

It is also probable that the formula was obtained trigonometrically thus:

$$jy\bar{a}(\,\xi\pm\eta\,) = \frac{1}{R}(jy\bar{a}\;\xi\; kojy\bar{a}\;\eta \pm kojy\bar{a}\;\xi\; jy\bar{a}\;\eta\,).$$

Then,

$$jy\bar{a}\;(\,\xi+\eta\,) - jy\bar{a}\;\xi = \frac{1}{R}(kojy\bar{a}\;\xi\; jy\bar{a}\;\eta - jy\bar{a}\;\xi\; utjy\bar{a}\;\eta\,),$$

and,

$$jy\bar{a}\;\xi - jy\bar{a}\;(\,\xi-\eta\,) = \frac{1}{R}(kojy\bar{a}\;\xi\; jy\bar{a}\;\eta + jy\bar{a}\;\xi\; utjy\bar{a}\;\eta\,).$$

Hence;

$$jy\bar{a}\;(\xi+\eta) - jy\bar{a}\;\xi = jy\bar{a}\;\xi - jy\bar{a}\;(\xi-\eta) - \frac{2\,jy\bar{a}\;\zeta\; utjy\bar{a}\;\eta}{R}$$

$$= jy\bar{a}\;\xi - jy\bar{a}\;(\xi-\eta) - jy\bar{a}\;\xi\left(\frac{2\,jy\bar{a}\,\frac{\eta}{2}}{R}\right)^2$$

Now put $\eta=\alpha$, $\xi=n\,\alpha$; so that the formula becomes

$$jy\bar{a}\;(n+1)\,\alpha - jy\bar{a}\; n\,\alpha = jy\bar{a}\; n\,\alpha - jy\bar{a}\;(n-1)\,\alpha - jy\bar{a}\; n\,\alpha\left(\frac{2\,jy\bar{a}\,\frac{\alpha}{2}}{R}\right)^2$$

So far the formula is mathematically accurate. According to the *Sūrya-siddhānta*

$$\alpha = 3°45' = 225', jy\bar{a}\ \alpha = 225', R = 3438'$$

Therefore

$$\left(\frac{2\, jy\bar{a}\ \alpha/2}{R}\right)^2 = \left(\frac{jy\bar{a}\ \alpha}{R}\right)^2 \text{ approximately}$$

$$= \left(\frac{225}{3438}\right)^2 = \left(\frac{1}{15{\cdot}28}\right)^2 = \frac{1}{225} \text{ approximately.}$$

Hence we get

$$\sin (n+1)\,\theta = \sin n\theta + \{\sin n\theta - \sin\ (n-1)\ \theta\} - \frac{\sin n\theta}{225},$$

where

$$\theta = 3°45' \text{ and } n = 1, 2, \ldots, 24.$$

According to modern calculation, the divisor in the last term will be slightly different. For

$$\left(2 \sin \frac{\theta}{2}\right)^2 = (2 \sin 1°52'30'')^2 = {\cdot}00428255 = \frac{1}{233{\cdot}506}, \text{ nearly.}$$

This little discrepancy, however, does not make much difference in the values of the *R*sine functions calculated on the basis of that formula. They are indeed fairly accurate even according to modern calculations except in a few instances.[1]

About this method of constructing the tables of *R* sines, Delambre remarks: "The method is curious; it indicates a method of calculating the table of sines by means of their second differences."[2] He then goes on: "This differential process has not up to now been employed except by Briggs who himself did not know that the constant factor was the square of the chord $\triangle A$ (=3° 45′) or of the interval, and who could not obtain it except by comparing the second differences obtained in a different manner. The Indians also have probably done the same; they obtained the method of differences only from a table calculated previously by a geometric process. Here then is a method which the Indians possessed and which is found neither amongst the Greeks, nor amongst the Arabs."[3]

We do not understand what valid grounds were there for Delambre to suppose that the Hindus discovered the above theorem of *R*sines by inspection after having calculated the table of *R*sines by a different method. For there is absolutely no doubt

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[1] *Vide infra.*
[2] Delambre, *Histoire de l'Astronomie Ancienne*, t. 1. Paris, 1817, p. 457.
[3] *Ibid*, p. 459 f.

that the ancient Hindus were in possession of necessary and sufficient equipments to derive it in either of the ways indicated above. It is noteworthy that that theorem has an important geometrical foundation. If there be three arcs of a circle in arithmetical progression the sum of the sines of the two extreme arcs is to the sine of the middle arc as the sine of twice the common difference is to the sine of that difference. For

$$jy\bar{a}\ (\xi+\eta)\ jy\bar{a}\ (\xi-\eta) \equiv 2\,jy\bar{a}\ \xi - \frac{2 jy\bar{a}\ \xi\, utjy\bar{a}\ \eta}{R}$$

$$= \frac{2\,jy\bar{a}\ \xi\ kojy\bar{a}\ \eta}{R}.$$

Hence,

$$\frac{jy\bar{a}\ (\xi+\eta) + jy\bar{a}\ (\xi-\eta)}{jy\bar{a}\ \xi} = \frac{2\ kojy\bar{a}\ \eta}{R} = \frac{jy\bar{a}\ 2\ \eta}{jy\bar{a}\ \eta}.$$

This very remarkable property of the circle was discovered in Europe by Vieta (1580)[1].

Āryabhaṭa

The trigonometrical table of Āryabhaṭa I (499) contains the differences between the successive *R*sines for arcs of every 3° 45′ of a circle of radius 3438′.[2] His first method of computing it, which is rather cryptic, seems to be the same as that followed by Varāhamihira (*infra*). The other is practically the same as that of the *Sūrya-siddhānta*, though put in a different form. He says:

"Divide a quarter of the circumference of a plane circle (into as many equal parts as desired). From (right) triangles and quadrilaterals (can be obtained) the *R*sines of equal arcs, as many as desired, for (any given) radius."[3]

What is meant by the author is very probably this: If *P* be any point on the arc of the quadrant, draw the perpendiculars *PM* and *PN*; also join *PX*. So that corresponding to *P* we have a rectangle *PMON* and a right-angled triangle *PMX*. Now having given the *R*sine (*PM*) of the arc *XP* (=α), we can determine from the rectangle *PMON* the side *PN* which is the sine of the arc (90° −α). Having found *PN*, we can calculate *MX*, which is equal to *R*−*jyā* (90° − α). Then in the right-angled triangle *PMX*, we can determine the chord PX. Half of this is *jyā* α/2. Again

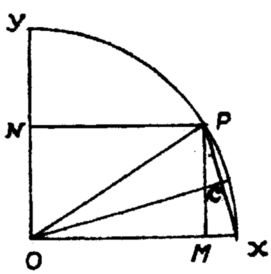


Fig. 12

[1] Playfair, J "Observations on the Trigonometrical Tables of the Brahmins", *Trans. Roy. Soc. Edin*, iv (1798), pp. 83-106; compare also *Asiatic Researches*, iv. p. 165.
[2] *Ā*, i. 12.
[3] *Ibid*, ii. 11.

from a similar set of a rectangle and a right-angled triangle corresponding to the half arc, we can calculate *jyā* $(90°-\alpha/2)$ and *jyā* $\alpha/4$. Proceeding thus we can compute the *R*sines of as many equal arcs as we please and it is clear that in so doing the quadrant will be broken up into a system of right-angled triangles and rectangles, as contemplated in the rule.

This is the interpretation of Āryabhaṭa's rule by his ancient commentators, like Sūryadeva Yajvā and Parameśvara (1430). Another interpretation will be this: The quadrant is trisected by the inscribed equilateral triangle and bisected by the inscribed square. The length of the arc between these points is 15° (=45°—30°). One-fourth of this is 3° 45′. So that the rule under discussion indicates how to divide the quarter of the circumference into portions of 3° 45′ each. If this interpretation is right[1], which is rather forced, then it will have to be said that Āryabhaṭa I gave only one method of computing the trigonometrical table.[2]

The *second* method of Āryabhaṭa I is this:

"The first *R*sine divided by itself and then diminished by the quotient will give the second difference (of tabular *R*sines). For computing any other difference, (the sum of) all the preceding differences is divided by the first *R*sine and the quotient is subtracted from the preceding difference. Thus, all the remaining differences (can be calculated)."[3]

Let $\triangle_1, \triangle_2, \ldots\ldots, \triangle_n$ denote successive differences of the tabular *R*sines, such that, α being equal to 3° 45′,

$$\triangle_1 = jy\bar{a}\ \alpha - jy\bar{a}\ 0,$$
$$\triangle_2 = jy\bar{a}\ 2\alpha - jy\bar{a}\ \alpha,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\triangle_n = jy\bar{a}\ n\alpha - jy\bar{a}\ (n-1)\alpha$$

Then $jya\ n\alpha = \triangle_1 + \triangle_2 + \ldots + \triangle_n$.

The rule says:

$$\triangle_{n+1} = \triangle_n - \frac{\triangle_1 + \triangle_2 + \ldots + \triangle_n}{jy\bar{a}\ \alpha}.$$

On substituting the values, this formula will be found to be equivalent to

$$\{jy\bar{a}\ (n+1)\alpha - jy\bar{a}\ n\alpha\} = \{jy\bar{a}\ n\alpha - jy\bar{a}\ (n-1)\alpha\} - \frac{jya\ n\alpha}{jy\bar{a}\ \alpha}.$$

[1]This interpretation has been suggested by Rodet, Kaye and Sengupta.

[2]In this connection, the reader is referred to "*Āryabhaṭīya* of Āryabhaṭa," edited with English translation by K. S. Shukla and K. V. Sarma, INSA, New Delhi, 1976, pp. 45-51.

[3]*Ā*, ii. 12.

It is also noteworthy that the text also admits of the following interpretation:

"The first *R*sine is divided by itself and then diminished by the quotient; the result with the first *R*sine will give the second *R*sine. For (computing), any of the remaining *R*sines, the sum of all the *R*sines preceding it is divided by the first *R*sine and the quotient is subtracted from the first *R*sine, and the result added to the preceding *R*sine."

$$jy\bar{a}\ (n+1)\alpha = jy\bar{a}\ n\alpha + jy\bar{a}\ \alpha - (jy\bar{a}\ \alpha + jy\bar{a}\ 2\alpha + \ldots + jy\bar{a}\ n\alpha)/jy\bar{a}\ \alpha.$$

If $\triangle_1, \triangle_2, \ldots$ be the tabular differences as before, then

$$\triangle_1 - \triangle_2 = \frac{2jy\bar{a}\ \alpha\ (R - kojy\bar{a}\ \alpha)}{R}$$

$$\triangle_2 - \triangle_3 = \frac{2jy\bar{a}\ 2\alpha\ (R - kojy\bar{a}\ \alpha)}{R}$$

....

$$\triangle_n - \triangle_{n+1} = 2\,jy\bar{a}\ n\alpha\ \frac{R - kojy\bar{a}\ \alpha}{R}$$

Adding up, we get

$$\triangle_1 - \triangle_{n+1} = \frac{2(R - kojy\bar{a}\alpha)}{R}\,(jy\bar{a}\ \alpha + jy\bar{a}\ 2\alpha + \ldots + jy\bar{a}\ n\alpha)$$

$$\text{Now } \frac{2(R - kojy\bar{a})\ \alpha}{R} = \left(\frac{2\,jy\bar{a}\ \frac{\alpha}{2}}{R}\right)^2$$

$$= \left(\frac{jy\bar{a}\ \alpha}{R}\right)^2 \text{ approximately}$$

$$= \frac{1}{225} \text{ approximately.}$$

Therefore

$$jy\bar{a}\ (n+1)\ \alpha = jy\bar{a}\ n\ \alpha + jy\bar{a}\ \alpha - (jy\bar{a}\ \alpha + jy\bar{a}\ 2\alpha + \ldots + jy\bar{a}\ n\ \alpha)/225$$

Also

$$\triangle_{n+1} = \triangle_n - \frac{jy\bar{a}\ n\alpha}{225}$$

$$= \triangle_n - \frac{\triangle_1 + \triangle_2 + \ldots + \triangle_n}{225}.$$

Of these two interpretations the first has been given by the commentator Parameśvara and the second by the commentators Prabhākara, Sūryadeva (b. 1191), Yallaya (1480) and Raghunātharāja (1597).

It should be observed that Āryabhaṭa I does not appear to have used this formula consistently to calculate the whole table. For as will be found from the accompanying table, certain values actually recorded by Āryabhaṭa differ from the values calcu-

TABLE

Differences Δ_n, $n=$	Calculated according to the formula	Recorded by Āryabhaṭa	Calculated according to the modern method
1	225	225	224·856
2	224	224	223·893
3	222·005	222	221·971
4	219·018	219	219·100
5	215·045	215	215·289
6	210·089	210	210·557
7	204·156	205	204·923
8	198·245	199	198·411
9	191·36	191	191·050
10	182·512	183	182·872
11	173·694	174	173·909
12	163·245	164	164·202
13	153·196	154	153·792
14	142·512	143	142·724
15	130·876	131	131·043
16	118·294	119	118·803
17	105·745	106	106·053
18	92·289	93	92·850
19	78·88	79	79·248
20	64·527	65	65·307
21	50·240	51	51·087
22	36·014	37	36·648
23	21·849	22	22·051
24	6·752	7	7·361

lated by the formula. Probably he corrected the calculated values in those cases by comparison with the known values of the sines of 30°, 45°, 60°; or what is much more likely employed the formula only to calculate the *R*sines of intermediate arcs. Other plausible explanations of the discrepancy have been furnished by Krishnaswami Ayyangar[1] and Naraharayya.[2]

Varāhamihira and Lalla

Varāhamihira's (d. 587) table contains the *R*sines for every 3° 45′ and the successive differences of the tabular *R*sines for the radius 60.[3] His method of computation is this:[4] Starting with the known values of *R*sine 30°, *R*sine 45° and *R*sine 60°, by the repeated and proper application of the formulae

$$\sin\frac{\theta}{2} = \tfrac{1}{2}\sqrt{\sin^2\theta + \text{versin}^2\theta}$$

$$\sin\frac{\theta}{2} = \sqrt{\tfrac{1}{2}\text{versin}\,\theta},$$

says he, the other *R*sines may be computed. Lalla[5] gives a table of *R*sines and versed *R*sines for the radius 3438′. His method of computation is the same as that of Āryabhaṭa I and the *Sūrya-siddhānta*. He has also a shorter table of *R*sines and their differences for intervals of 10° of arcs of a circle of radius 150.[6]

Brahmagupta

Brahmagupta (628) takes the radius quite arbitrarily to be 3270. His explanation[7] for this departure from the usual practice is unsatisfactory.[8] He has, however, indicated two methods of computation.[9] One is *graphic* and the other *mathematical.*

Graphic Method. "Starting from the joint of two quadrants, mark off successively (on either directions) portions of arcs equivalent to the eighth part of a sign (30°). Join two and two of these marks by threads. Half of them (lengths of threads) will be the *R*sines."[10]

Mathematical Method.[11] In this method Brahmagupta employs the trigonometrical formulae

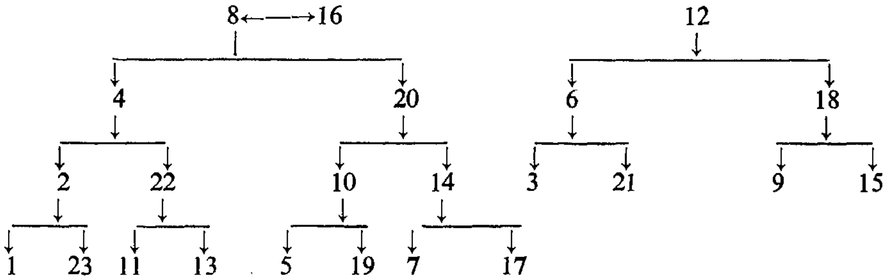
[1]*JIMS*, xv, (1924), pp. 121-6.
[2]*Ibid*, pp. 105-13 of "Notes and Questions."
[3]*PSi*, iv. 6-11, 12-15.
[4]*Ibid*, iv. 2-5.
[5]*ŚiDVṛ*, ii. 1-8.
[6]*ŚiDVṛ*, xiii. 2-3.
[7]*BrSpSi*, xxi. 16.
[8]Datta, Bibhutibhusan, "Hindu Values of π", *JASB*, N. S., Vol. 22 (1926), pp. 25-42; see particularly p. 32, footnote 1.
[9]His table will be found in *BrSpSi*, ii. 2-9.
[10]*BrSpSi*, xxi. 17.
[11]*Ibid*, xxi. 20-21; compare also the verse 23.

$$(a)\ \sin\frac{\theta}{2} = \tfrac{1}{2}\sqrt{\sin^2\theta + \text{versin}^2\theta}$$

$$(b)\ \sin\left(90^\circ-\frac{\theta}{2}\right)=\sqrt{1-\sin^2\frac{\theta}{2}}$$

From the known value of the *R*sine of 8α, that is, of 30°, α being equal to $3^\circ\,45'$, we can calculate, by (*a*), the *R*sines of 4α, 2α, α. Then by (*b*) will be obtained the *R*sine of 20α, 22α, 23α Again from the first two of the latter results, we shall obtain, by (*a*), the *R*sines of 10α and 11α; and thence by (*b*) the *R*sines of 14α and 13α. Continuing similar operations, we can compute the *R*sines of 5α and 19α, 7α and 17α. Again starting with the *R*sine of 12α, we shall obtain on proceeding in the same way, successively the values of the *R*sines of 6α and 18α; 3α and 21α; 9α and 15α. Thus the values of all the twenty-four *R*sines are computed.

It is perhaps noteworthy that *R*sin $n\alpha$ is called by Brahmagupta as the *n*th *R*sine. The successive order in which the various *R*sines have been obtained above can be exhibited as follows:



Brahmagupta then observes: "In this way (can be computed) the *R*sines in greater or smaller numbers, having known first the *R*sines of the sixth, fourth and third parts of the circumference of the circle."[1] He further remarks that the *R*sine of the semi-arc can be more easily calculated by the second formula of Varāhamihira.[2] Brahmagupta has also another table giving differences of *R*sines for every 15° of a circle of radius 150.[3]

Āryabhaṭa II and Śrīpati

Āryabhaṭa II (950) gives the same table as that of the *Sūrya-siddhānta*.[4] But his method of computation is entirely different.[5] He takes recourse to the formulae

$$\sin\tfrac{1}{2}(90^\circ \pm \theta) = \sqrt{\tfrac{1}{2}(1 \pm \sin\theta)}$$

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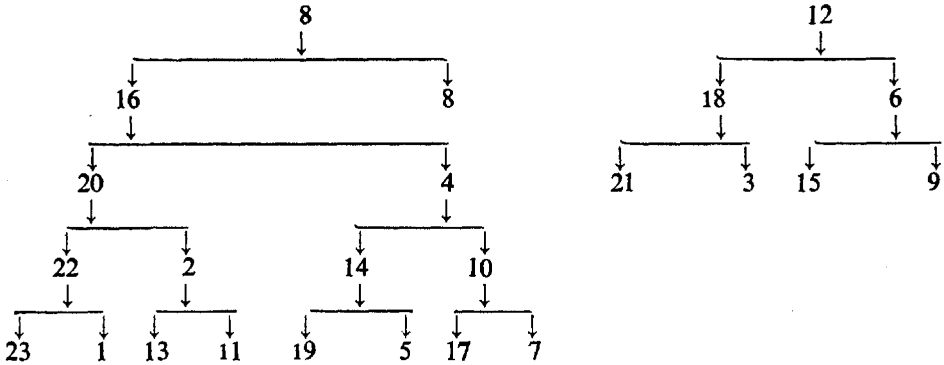
[1] *BrSpSi*, xxi. 22.
[2] *Ibid*, xxi. 23.
[3] *KK*, Part I, iii. 6; *DhGr*, 16.
[4] *MSi*, iii. 4-8.
[5] *MSi*, iii. 1-3.

Beginning with the known values of *R*sin 30° and *R*sin 45°, like Brahmagupta, the successive order in which the *R*sines will come out in the course of computation, can be best exhibited thus:



The table of Śrīpati (*c.* 1039) gives the *R*sines and versed *R*sines for every 3° 45′ of a circle of radius 3415.[1] His *first* method of computing it is the same as the graphic method of Brahmagupta. He says:

"Place marks at the eighth parts of a sign (30°); then (starting) from the joint of two quadrants, following up these marks, join two and two of them successively by means of threads; half of them will be the *R*sines."[2]

The *Second* method followed by Śrīpati is identical with the mathematical method of Brahmagupta.[3]

Bhāskara II

The table of Bhāskara II (1150) contains the *R*sines and versed *R*sines as well as their differences for every 3° 45′ of a circle of radius 3438′. He has indicated several methods of computing it. The *first* is practically the same as Brahmagupta's *graphic method*. He says:

"For computing the *R*sines, take any optional radius. On a plane ground describe a circle by means of a piece of thread equal to that radius. On it mark the cardinal points and 360 degrees; so in each quadrant of the circle there will be 90 degrees. Then divide every quadrant into as many equal parts as the number of *R*sines to be computed and put marks of these divisions. For instance, if it be required to calculate 24 *R*sines, there will be 24 marks. Then beginning from any of the cardinal points, and proceeding either ways, the threads connecting the successive points will be the chords. There will be thus 24 chords. Halves of these will be the *R*sines (required). So these half-chords should be measured and the results taken as the *R*sines."[4]

[1] *SiŚe*, iii. 3-10.
[2] *SiŚe*, xvi. 9.
[3] *SiŚe*, xvi. 14ff.
[4] *SiŚi*, *Gola*, v. 2-6 (Gloss).

The *second* is again a reproduction of Brahmagupta's *theoretical method:*

"When twenty-four *R*sines are required (to be computed), the *R*sine of 30° is the eighth element; its *R*cosine is the sixteenth; and *R*sin 45° is the twelfth. From these three elements, twenty-four elements can be computed in the way indicated. From the eighth we get the *R*sine of its half, that is, the fourth (element), its *R*cosine is the twentieth. Similarly from the fourth, the second and the twenty-second; from the second, the first and the twenty-third. In the same way from the eighth are obtained the tenth and fourteenth, fifth and nineteenth, seventh and seventeenth, eleventh and thirteenth. Again from the twelfth follow the sixth and eighteenth, third and twenty-first, ninth and fifteenth. The radius is the twenty-fourth *R*sine."[1]

The *third* method of computing trigonometrical tables described by Bhāskara II is the same as that of Āryabhaṭa II. The speciality of this method, as also of the two following, is, says Bhāskara II, that it does not employ the versed *R*sine function. As for the successive order of derivation, he points out that "from the eighth *R*sine (will be obtained) the sixteenth; from the sixteenth, the fourth and the twentieth; from the fourth, the tenth and fourteenth. In this way all the rest may be deduced."[2]

The *fourth* method of Bhāskara II is based on the application of the formula

$$R\sin\frac{1}{2}(\theta-\phi)=\frac{1}{2}\{(R\sin\theta-R\sin\phi)^2+(R\cos\theta-R\cos\phi)^2\}^{\frac{1}{2}},$$

"so that knowing any two *R*sines others may be derived. For instance, let one be the fourth *R*sine and the other eighth *R*sine. From them is derived the second *R*sine. From the second and fourth, the first; and so on."[3]

The *fifth* method depends on the formula

$$R\sin(45°-\theta)=\sqrt{\frac{1}{2}(R\cos\theta-R\sin\theta)^2}\ .$$

"Thus, for instance, take the eighth *R*sine; its *R*cosine is the sixteenth *R*sine. From these the fourth is derived; and so on."[4]

All the theoretical methods described above require the extraction of the square-root. So Bhāskara II propounds a new method (the *sixth*) in which that will not be necessary. It is based on the employment of the formula

$$R\cos 2\theta=R-\frac{2\,(R\sin\theta)^2}{R}$$

$$\text{or}\ \cos 2\theta=1-2\sin^2\theta.$$

[1] *SiŚi, Gola,* v. 2-6 (Gloss); xiv. 10-11 (Gloss).
[2] *SiŚi, Gola,* xiv. 12 (Gloss).
[3] *SiŚi, Gola,* xv. 13 (Gloss).
[4] *SiŚi, Gola,* xiv. 14 (Gloss).

But this method is defective in as much as "only certain elements of a table of *R*sines can be calculated thus,"[1] but not the whole table. This defect is present in a sense in the previous methods, for no one of the trigonometrical formulae employed in them suffices alone for the computation of a table containing more *R*sines (*vide infra*).

The *seventh* method of Bhāskara II for calculating a table of twenty-four *R*sines, has been described thus:

"Multiply the *R*cosine by 100 and divide by 1529; diminish the *R*sine by its $\frac{1}{467}$ part. The sum of these two results will give the next *R*sine and their difference the previous *R*sine. Here 225 less 1/7 is the first *R*sine. And by this rule can be successively calculated the twenty-four *R*sines."[2]

$$jy\bar{a}\,(n\alpha \pm \alpha) = \left(jy\bar{a}\, n\alpha - \frac{jy\bar{a}\, n\alpha}{467} \right) \pm \frac{100}{1529}\, kojy\bar{a}\, n\alpha,$$

where $n=1, 2, \ldots, 24$; $\alpha = 3^\circ\, 45'$; and

$$jy\bar{a}\,\alpha = 225 - \frac{1}{7}.$$

The *rationale* of this formula is as follows:

By the Addition and Subtraction Theorems,

$$jy\bar{a}\,(n\alpha \pm \alpha) = \frac{1}{R}\,(jy\bar{a}\, n\alpha.\ kojy\bar{a}\,\alpha \pm kojy\bar{a}\, n\alpha.\ jy\bar{a}\,\alpha),$$

$$= jy\bar{a}\, n\alpha.\ \frac{kojya\,\alpha}{R} \pm kojy\bar{a}\, n\alpha.\ \frac{jy\bar{a}\,\alpha}{R}$$

Now

$$\frac{1}{R}\, jy\bar{a}\,\alpha = \frac{1}{3438}\left(225 - \frac{1}{7}\right) = \frac{787}{12033} = \frac{1}{15{\cdot}289707\ldots.}$$

$$= \frac{100}{1528{\cdot}9707\ldots} = \frac{100}{1529} \text{ nearly}$$

and $\frac{1}{R}\, kojy\bar{a}\,\alpha = \frac{1}{R}\sqrt{R^2 - (jy\bar{a}\,\alpha)^2} = \sqrt{1 - \left(\frac{jya\,\alpha}{R}\right)^2}$,

$$= \sqrt{1 - \frac{1}{233{\cdot}775\ldots}} = 1 - \frac{1}{467{\cdot}550\ldots}$$

[1] *SiŚi, Gola*, xiv. 15 (Gloss).
[2] *SiŚi, Gola*, xiv. 18-20.

$$= 1 - \frac{1}{467} \text{ nearly}$$

and hence the rule. This formula is very nearly accurate. For according to the modern values

$$jy\bar{a}\ (3°\ 45') = 224{\cdot}856\ldots.$$

Therefore $\frac{1}{R}\ jy\bar{a}\ (3°\ 45') = \frac{224.856}{3438} = \frac{1}{15{\cdot}28978\ldots.} = \frac{100}{15{\cdot}28978\ldots.}$

Bhāskara II has indicated how to compute a table of *R*sines for every 3° of a circle of radius 3438′. He writes:

"For instance if (it be required to compute) thirty *R*sines in a quadrant, half the radius is the tenth *R*sine, its *R*cosine is the twentieth *R*sine. *R*sin 45° is the fifteenth *R*sine; *R*sin 36° is the twelfth and *R*cos 36° the eighteenth. The *R*sine of 18° is the sixth and its *R*cosine is the twenty-fourth. Then by the rule for deriving the *R*sine of the half arc from the square-root of the sum of the squares of the *R*sine and versed *R*sine of an arc, as stated before, from the tenth (is derived) the fifth; its *R*cosine is the twenty-fifth. In that way from the twelfth (is calculated) the sixth and twenty-fourth; from the sixth, the third and twenty-seventh; from the eighteenth, the ninth and twenty-first. These are the only elements (of the table) of *R*sines which can be calculated in this way. So it has been observed that 'only certain elements etc'. Next the formula for the *R*sine of half the difference of two arcs should be employed. Let the fifth be the one *R*sine and the ninth the other. From them will follow the second; its *R*cosine is the twenty-eighth *R*sine. From these two again by employing the (previous) rule for the *R*sine of semi-arcs from the square-root of the sum of the squares of the *R*sine and versed *R*sine, the first and fourteenth (are obtained). The remaining fourteen *R*sines can also be computed in the same way."[1]

Bhāskara II has further given a rule for computing a trigonometrical table for every degree. So it is called *Pratibhāgika-jyakā-vidhi* ("The rule for the *R*sine of every degree").

"Deduct from the *R*sine of any arc its 6567th part; multiply its *R*cosine by 10 and then divide by 573. The sum of these two results is the next *R*sine and their difference the preceding *R*sine. Here the first *R*sine (i.e. *R*sin 1°) will be 60′ and other *R*sines may be successively found. Thus in a circle of radius equal to 3438′, will be found 90 *R*sines."[2]

$$jy\bar{a}\ (\theta \pm 1°) = \left(jy\bar{a}\ \theta - \frac{jy\bar{a}\ \theta}{6567}\right) \pm \frac{10}{573}\ kojy\bar{a}\ \theta$$

where $\theta = 1°, 2°, \ldots\ldots, 89°$; given $jy\bar{a}\ 1° = 60'$.

[1] *SiŚi, Gola*, xiv. 15 (Gloss).
[2] *SiŚi, Gola*, xiv. 16-8.

The *rationale* of this rule can be easily found : For by the Addition and Subtraction Theorems,

$$jy\bar{a}\,(\theta \pm 1^\circ) = \frac{1}{R}\,(jy\bar{a}\ \theta.\,kojy\bar{a}\ 1^\circ \pm kojy\bar{a}\ \theta.\,jy\bar{a}\ 1^\circ).$$

Now it is stated that R=3438′ and $jy\bar{a}$ 1° = 60′. Therefore

$$\frac{1}{R}\,jy\bar{a}\ 1^\circ = \frac{60}{3438} = \frac{10}{573}$$

$$\frac{1}{R}\,kojy\bar{a}\ 1^\circ = \sqrt{1-(jya\ 1^\circ/R)^2} = \left\{1-\left(\frac{10}{573}\right)^2\right\}^{1/2}$$

$$= 1 - \frac{100}{2 \times 328329} = 1 - \frac{1}{6566{\cdot}58}$$

$$= 1 - \frac{1}{6567}$$

$$= {\cdot}999847723\ldots.$$

The denominator wrongly appears as 6569 in Bapu Deva's edition of the *Siddhānta-Śiromaṇi*.[1]

The short table of Bhāskara II contains differences of *R*sines for intervals of 10° in a circle of radius 120.[2]

Posterior Writers

Amongst the writers posterior to Bhāskara II (1150) who have given tables of trigonometrical functions, the most notable are Mahendra Sūri (1370) and Kamalākara (1658). The latter has a table of *R*sines and their differences for every degree of arc of a circle of radius 60, while the former gives tables of *R*sines and versed *R*sines together with their differences for every degree of the arc of a circle of radius 3600. Mahendra Sūri has furnished also some other tables for ready reckoning in Astronomy. It is noteworthy that Mahendra Sūri's work, *Yantrarāja*[3], is admittedly based upon some Arabic work. We are informed by his commentator Malayendu Sūri, a direct disciple of the author and who wrote his commentary only 12 years after the text, that the author was the court astrologer of some potentate of the name of Firoz, who is probably the famous Sultan Firoz Shah Tughluk of Delhi (1351-88 A.D.). The illustrations chosen will agree with this date.

[1]The text given by Bapu Deva runs as "*Svago'ṅgeṣusaḍaṁśena....*". It will be"*Svāgāṅgeṣuṣaḍaṃśena....*"

[2]*SiŚi, Graha*, ii. 13.

[3]Mss of the text with the commentary of *Yantrarāja* are available in the libraries of India Office, London (Nos. 2906-8), Benares Sanskrit College (No. 2905), Bikaner Palace (No. 760) and also at other places. Our copy has been procured from Benares.

The following Table gives the relevant details of the various *R*sine-Tables constructed in India from time to time.

Constructor of Table	Radius chosen	Interval taken	Sexagesimal places calculated
Author of *SūSi*[1]	3438′	225′	1 (minutes only)
Āryabhaṭa[2]	3438′	225′	1 (same Table as in *SūSi*)
Varāhamihira[3]	120	225′	2 (minutes and seconds)
Brahmagupta (1)[4]	3270	225′	1
(2)[5]	150	15°	1
Deva[6]	300	10°	1
Lalla (1)[7]	3438′	225′	1 (same Table as in Ā)
(2)[8]	150	10°	1
Sumati[9]	3438′	1°	1
Govinda Svāmi[10]	3437′44″19‴	225	3
Vaṭeśvara[11]	3437′44″	56′15″	2
Mañjula[12]	8°8′	30°	2 (degrees and minutes)
Āryabhaṭa II[13]	3438′	225′	1
Srīpati[14]	3415′	225′	1
Udayadivākara[15]	12375859‴ or 3437′44″19‴	225′	1 (thirds only)
Bhāskara II[16] (1)	3438′	225′	1
(2)	120	10°	1
Brahmadeva[17]	120	15°	1
Vṛddha Vaśiṣṭha[18]	1000	10°	1
Malayendu Sūri[19]	3600	1°	2
Madanapāla[20]	21600	1°	2
Mādhava[21]	3437′ 44″ 48‴	225′	3
Parameśvara[22]	3437′ 44″	225′	2
Munīśvara[23]	191	1°	4
Kṛṣṇa-daivajña[24]	500	3°	1
Kamalākara[25]	60	1°	5
Jagannātha Samrāṭa[26]	60	30′	5

[1] *SūSi*, ii. 17-22(a-b).
[2] *Ā*, i. 12.
[3] *PSi*, iv. 6-12.
[4] *BrSpSi*, ii. 2-5.
[5] *KK*, Part 1, iii. 6; *DhGr*, 16.
[6] *KR*. i. 23.
[7] *ŚiDVṛ*, I, ii. 1-8.
[8] *Ibid*, xiii. 3.
[9] *SMT* and *SK*.
[10] His com. on *MBh*, iv. 22.
[11] *VSi*, II, i. 2-26.
[12] *LMā*, ii. 2(c-d).
[13] *MSi*, iii. 4-6(a-b).
[14] *SiŚe*, iii. 3-6.
[15] His com. on *LBh*, ii. 2-3.
[16] *SiŚi*, *Gaṇita*, ii. 3-6; 13.
[17] *KPr*, ii. 1.
[18] *VVSi*, ii. 10-11.
[19] *YR*, i. 5, commentary.
[20] His com. on *SūSi* xii. 83.
[21] See Nīlakaṇṭha's com. on *Ā*, ii. 12.
[22] His com. on *LBh*, ii. 2(c-d)-3(a-b).
[23] *SiSā*, ii. 3-18.
[24] *KKau*, ii. 4-5.
[25] *SiTVi*, ii. pp. 244-5 (Lucknow Edition).
[26] *Siddhānta-samrāṭ*, ii. beginning.

6. Interpolation

Function of any arc

For finding the trigonometrical functions of an arc, other than those whose values have been tabulated, the Hindus generally follow the principle of proportional increase. Thus the *Sūrya-siddhānta* says:

"Divide the minutes (into which the given arc is first reduced) by 225; the quotient will indicate the number of tabular *R*sines exceeded; (the remainder) is multiplied by the difference between the (tabular) *R*sine exceeded and that which is still to be reached and then divide by 225. The result thus obtained should be added to the exceeded tabular *R*sine; (the sum) will be the (required) direct *R*sine. This rule is applicable also to the case of (determining) the versed *R*sine."[1]

Brahmagupta states:

"Divide the minutes by 225, the quotient (will indicate) the number of tabular *R*sines (exceeded); the remainder is multiplied by the (next) difference of *R*sines and divided by the square of 15; the result is added to the (tabular) *R*sine coresponding to the quotient. Such (is the method) for finding the *R*sine."[2]

Such rules appear also in other astronomical works.[3]

The method of Mañjula (932) is very simple though it yields results only roughly approximate. He says:

"The sum of the signes (in the given arc successively) multiplied by 4, 3 and 1 will give the degrees in the *R*sines and *R*cosines (to be found); such are the minutes."[4]

This rule though it appears to be cryptic has been fully explained by the commentators; Praśastidhara (968), Parameśvara (1430) and Yallaya (1482). We shall explain it with the help of a simple illustrative example: Suppose it is necessary to find the *R*sine of the angle 76° 30′. This angle can be written as 2 signs 16° 30′. Now the rule says that for the first sign take 4 and for the second sign 3. For the third sign of 30° we are to take 1, so for a portion 16° 30′ of that sign we should take $(16\frac{1}{2}\times 1)/30$. The sum of these 4, 3 and 33/60 will be the degrees in the required *R*sine; and they will also be the minutes of the required *R*sine. Therefore

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[1] *SūSi*, ii. 31-2.
[2] *BrSpSi*, ii. 10; compare also *KK*, Part I, iii. 6.
[3] *MBh*, iv. 3-4; *LBh*, ii. 2-3;, *ŚiDVr*, ii. 12; *MSi*, iii. 10½; *SiŚi*, *Graha*, ii. 10½; *SiTVi* ii. 171; *SiŚe*, iii. 15.
[4] "*Catustrekaghnarāśyaikyaṁ doḥkoṭyoraṁśakāḥ kalāḥ*."—*LMā*, ii. 2.

Jyā (76° 30′) =(4+3+33/60) degrees+(4+3+33/60) minutes
=7° 40′ 33″

The *rationale* of this rule which has also been given by the earlier commentators, is this: Mañjula considers the circle of reference to be of radius 488′ or in sexagesimal notation 8° 8′; and his Table is very short:

Arcs	*R*sines	Differences
0°	0° 0′	
		4° 4′
30°	4° 4′	
		3° 3′
60°	7° 7′[1]	
		1° 1′
90°	8° 8′	

To find the *R*sine of any intermediate arc he applies the principle of proportional increase. And this at once leads to the rule.

Arcs of Functions

The Hindus employed the principle of proportional increase also for the inverse problem of finding the arc which has a given trigonometrical function different from those tabulated. The *Sūrya-siddhānta* says:

"Subtract the (nearest smaller tabular) *R*sine (from the given *R*sine) multiply the remainder by 225 and divide by that difference (i.e. the difference corresponding to the interval in which the given *R*sine lies); the quotient added to the number (corresponding to the tabular *R*sine subtracted) multiplied by 225 will give the arc (required)."[2]

Brahmagupta writes:

"Subtract the (nearest smaller tabular) *R*sine (from the given *R*sine); the remainder is multiplied by 225 and divided by the (tabular) difference of *R*sines; the

[1]Accurately speaking
jyā 60° $= 488 \times \frac{\sqrt{3}}{2} = 7° \, 2'.608\ldots;$
Mañjula takes the value to be 7° 7′ obviously with the purpose of simplifying his rule.
[2]*SūSi*, ii. 33.

result should then be added to the product of the number corresponding to the subtracted *R*sine and the square of 15: (this will be) the arc (required)."[1]

Similarly in other works.[2]

Second Difference

The process explained above for calculating the trigonometrical functions of a given arc or the arc having given trigonometrical functions, will yield results correct only to a first degree of approximation in as much as the first difference alone of the tabular *R*sines has been employed.[3] More accurate results will be obtained by taking into consideration also the second (and higher) differences. The earliest Hindu writer to do so was Brahmagupta. It is perhaps noteworthy that this more correct method of interpolation does not occur in his bigger work, *Brāhma-sphuṭa-siddhānta*, which was composed in 628 A.D. but in his earlier monograph *Dhyānagrahopadeśa* as well as in his later work *Khaṇḍa-Khādyaka* written in 665 A.D. These latter works, as has been stated before, contain a table of differences of *R*sines for every arc of 15° in a circle of radius equal to 150. He says:

"Half the difference between the (tabular) difference passed over and that to be passed is multiplied by (residual) minutes and divided by 900; half the sum of those differences plus or minus that quotient according as it is less or greater than the (tabular) difference to be passed, will be the (corrected) value of the difference to be passed over."[4]

Suppose it is required to calculate the function—*R*sine, *R*cosine or versed *R*sine—of an arc α'. Let α_1, α_2, α_3 be the three consecutive values of the argument in the table such that $\alpha_3 > \alpha_2 > \alpha_1$.

Values of the argument α	Values of the function $f(\alpha)$	Differences of functions
α_1	f_1	
		Δ_1
α_2	f_2	
		Δ_2
α_3	f_3	

[1] *BrSpSi*, ii. 11; compare also *KK*, iii. 12; *DhGr*, 70.
[2] *MBh*, viii. 6; *ŚiDVṛ*, ii. 13; *MSi*, iii. 12; *SiŚe*, iii. 16; *SiŚi*, *Graha*, ii. 11f; *SiTVi*, ii. 172-3.
[3] The roughness of the result is due also to other causes. Bhāskara II observes: "As much large the radius of the circle is and into as many large number of (equal) arcs its quadrant is divided, so much accurate will be the *R*sines (calculated). Otherwise (the result) will be rough (*sthūla*)". ——*SiŚi*, *Graha*, ii. 15 (*Gloss*).
[4] "*Gatabhogyakhaṇḍakāntaradalavikalavadhācchatairnavabhirāptyā*
Tadyutidalaṁ yutonaṁ bhogyādūnādhikaṁ bhogyam" *KK*, Part 2, i,4; *DhGr*, 17.
See Sengupta, P. C., "Brahmagupta on Interpolation," *BCMS*, xxii, 1931.

Now if $\alpha_3 > \alpha' > \alpha_2$ for the calculation of $f(\alpha')$, f_2 will be technically called "the function exceeded," f_3 "the function to be reached," Δ_1 "the difference passed over" and Δ_2 "the difference to be passed." Let $\alpha' - \alpha_2 = r$.

Now suppose that $\alpha_3 - \alpha_2 = \alpha_2 - \alpha_1 = h$, say. Then according to the rules stated by all Hindu astronomers,

$$f(\alpha') = f_2 + \frac{r}{h} \Delta_2;$$

which is correct up to the first order of approximation. To get more accurate results, says Brahmagupta

$$\frac{\Delta_1 + \Delta_2}{2} \pm \frac{r}{h} \left(\frac{\Delta_1 \sim \Delta_2}{2} \right)$$

should be taken as the value of "the difference to be passed", instead of Δ_2; the positive or negative sign being taken, according as

$$\frac{\Delta_1 + \Delta_2}{2} < \text{or} > \Delta_2 .$$

Therefore, according to the method of Brahmagupta

$$f(\alpha') = f_2 + \frac{r}{h} \left\{ \frac{\Delta_1 + \Delta_2}{2} \pm \frac{r}{h} \left(\frac{\Delta_1 \sim \Delta_2}{2} \right) \right\} .$$

In the rule h is stated to be 900, as it was formulated with a view at the table of the *Khaṇḍa-Khādyaka*, in which the interval between the consecutive values of the argument, is 15° or 900′. This equation can be written in the form

$$f(\alpha') = f_2 + \frac{r}{h} \Delta_2 + \frac{r}{2h} \left(1 \pm \frac{r}{h} \right) (\Delta_1 \sim \Delta_2);$$

which agrees with the formula method of interpolation, correct up to the second degree.[1]

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[1]It may be mentioned here that the formula

$$f(\alpha') = f_2 + \frac{r}{h} \Delta_1 - \frac{r}{2h}\left(1 + \frac{r}{h}\right) (\Delta_1 - \Delta_2)$$

has been stated by Vaṭeśara (904) in his *Siddhānta* (Ch. 2, sec. 1, vss. 65-6) and the formula

$$f(\alpha') = f_2 + \frac{r}{h} \Delta_2 + \frac{r}{2h}\left(1 - \frac{r}{h}\right) (\Delta_1 - \Delta_2)$$

by Govindasvāmi (8th century) in his com. on *Mahā Bhāskarīya* (iv. 22) and by Parameśvara (1408) in his com. on *Laghu Bhāskarīya* [ii. 2(c-d)-3(a-b)]. Govindasvāmi has, however, prescribed it for the second sign only.

As has been observed by Bhāskara *II*[1] in the above formula one has to take the negative sign in calculating the *R*sine functions and the positive sign for the versed *R*sines. For in case of *R*sine functions, the first difference continuously decreases as the argument increases, while contrary is the case with the versed *R*sine functions.

Therefore the mean value of any two differences $\left(\text{i.e. } \frac{\triangle_1 + \triangle_2}{2}\right)$ is greater than the succeeding one ($\triangle_2$) in case of *R*sine functions and less in case of versed *R*sine functions.

Brahmagupta's method of interpolation appears also in the works of Mañjula (932) thus:

"(Find) half the sum of the tabular difference passed over and that to be passed; half their difference is multiplied by the (remaining) degrees etc. and divided by 30; half the sum minus this quotient will be the corrected value of the difference of (tabular) *R*sines to be passed in the (calculations of the *Laghu*)—*Mānasa*."[2]

The divisor is stated to be 30 in this rule, as Mañjula's table of *R*sines contains values at intervals of 30° each.

Bhāskara II (1150) writes:

"The difference of the (tabular) difference passed over and that to be passed is multiplied by the remaining degrees and divided by 20; half the sum of the (tabular) difference passed over and that to be passed minus or plus that quotient will be the corrected value of the difference to be passed over in calculation here for *R*sines and versed *R*sines."[3]

In formulating this rule Bhāskara II had in view a table calculated at intervals of 10°. The *rationale* of the rule has been explained by him thus:

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[1] "....*ūnaṁ kriyate yataḥ kramajyākaraṇe khaṇḍānyapacayena vartante. Utkramajyākaraṇe tūpacayenātastatra yutamityupapannam.*"—*SiŚi, Graha*, ii. 16 (Gloss).

[2] "*Gataiṣyakhaṇḍayogārdhamantarārdhena saṁguṇāt*
Bhāgādeḥ khāgnilabdhonaṁ bhogyajyā Mānase sphuṭāḥ."

There is a bit of uncertainty about the authenticity of this verse. In the Calcutta University Collection, there are three manuscripts of the *Laghu-Mānasa* and four commentaries which contain also the text. The commentary of Praśastidhara (958) is "copied from Ms. No. B 583 and compared with other Mss in the Oriental Library, Mysore." That by Parameśvara (1430) is "copied from a palm leaf manuscript in Malayalam character belonging to the office of the curator for the publication of Sanskrit Mss., Trivandrum." The source of the commentary of Yallaya (1482) is not mentioned. The above verse appears in the commentaries of Praśastidhara and Yallaya. But in the latter it has been attributed to Mallikārjuna-sūri. Now this writer flourished about 1180 A.D. Thus he is posterior to Praśastidhara by more than two centuries. So it is not possible for the latter to borrow anything from the former. I think the mistake has been made by Yallaya. The verse in question seems to me to be due in fact to Mañjula, and is more particularly from his *Bṛhat-mānasa*, which is now lost. Praśastidhara has quoted copiously from that work in his commentary of the *Laghu-mānasa* without, however, expressly mentioning it.

[3] *SiŚi, Graha*, ii.16.

"Half the sum of the (tabular) difference passed over and that to be passed will be the difference at the middle of those differences. But the difference to be passed is at the end of that interval to be passed. Hence proportion (should be taken) with their difference: If for an interval of 10°, we obtain half the difference of them, then what will be obtained for (an interval of) the remaining degrees? Thus by the rule of three, 20 will be the divisor of the product of the remaining degrees and the difference of the (tabular) difference passed over and that to be passed. By the quotient is then diminished half the sum of the (tabular) difference passed over and that to be passed; for in the calculations of *R*sines the differences are in the decreasing order. But in the calculations of versed *R*sines they are in the increasing order and hence the plus in this case. Thus (the rule) is proved."

This method of interpolation has been severely criticised by Kamalākara.[1] But he is wrong. Munīśvara (1646) attempted to modify this method by iterating the process but his process of iteration is in coriect as he has replaced the (tabular) difference to be passed by the instantaneous difference, at every stage.[2]

The *rationale* of the rule can be shown with the help of trigonometry to be as follows:

$$\triangle_1 = \sin\alpha_2 - \sin\alpha_1 = \sin\alpha_2 - \sin(\alpha_2 - h) = \sin\alpha_2\,(1-\cos h) + \cos\alpha_2 \sin h).$$

Since h is small we can expand $\cos h$ and $\sin h$ in powers of h; then neglecting powers higher than the second, we get

$$\triangle_1 = h\cos\alpha_2 + \frac{h^2}{2}\sin\alpha_2\,.$$

Similarly, $\triangle_2 = h\cos\alpha_2 - \frac{h^2}{2}\sin\alpha_2\,.$

Therefore, $\frac{\triangle_1 + \triangle_2}{2} = h\cos\alpha_2$ and $\frac{\triangle_1 - \triangle_2}{2} = \frac{h^2}{2}\sin\alpha_2\,.$

Now, if $\alpha' = \alpha_2 + r$, up to the second order of approximation, we have

$$\sin\alpha' = \sin(\alpha_2 + r) = \sin\alpha_2\left(1 - \frac{r^2}{2}\right) + r\cos\alpha_2\,,$$

Therefore,

$$\sin\alpha' = \sin\alpha_2 + \frac{r}{h}\left(\frac{\triangle_1+\triangle_2}{2} - \frac{r}{h}\,\frac{\triangle_1-\triangle_2}{2}\right).$$

[1] *SiTVi*, ii. 175-83.

[2] For Munīśvara's process of iteration, see Gupta R. C., "Munīśvara's modification of Brahmagupta's rule for second order interpolation," *IJHS*, vol. 14, no. 1, 1979, pp. 66-72.

Evidently in this case ,

$$\frac{\Delta_1+\Delta_2}{2} > \Delta_2$$

Hence, $\sin \alpha' = \sin \alpha_2 + \frac{r}{h}\left(\frac{\Delta_1 + \Delta_2}{2} - \frac{r}{h}\,\frac{\Delta_1 \sim \Delta_2}{2} \right)$

In case of versin functions

$$\Delta_1 = \text{versin}\, \alpha_2 - \text{versin}\, (\alpha_2 - h),$$

$$= \cos (\alpha_2 - h) - \cos \alpha_2 = h \sin \alpha_2 - \frac{h^2}{2} \cos \alpha_2;$$

$$\Delta_2 = \text{versin}\, (\alpha_2 + h) - \text{versin}\, \alpha_2,$$

$$= \cos \alpha_2 - \cos (\alpha_2 + h) = h \sin \alpha_2 + \frac{h^2}{2} \cos \alpha_2.$$

Therefore $\frac{\Delta_1+\Delta_2}{2} < \Delta_2$

and $\frac{\Delta_1+\Delta_2}{2} = h \sin \alpha_2,\ \frac{\Delta_1-\Delta_2}{2} = -\frac{h^2}{2} \cos \alpha_2.$

Now $\text{versin}\, \alpha' = 1 - \cos \alpha' = 1 - \cos (\alpha_2 + r),$

$$= \text{versin}\, \alpha_2 + r \sin \alpha_2 + \frac{r^2}{2} \cos \alpha_2.$$

Therefore,

$$\text{versin}\, \alpha' = \text{versin}\, \alpha_2 + \frac{r}{h}\left(\frac{\Delta_1+\Delta_2}{2} + \frac{r}{h} \cdot \frac{\Delta_1 \sim \Delta_2}{2} \right).$$

Combining these two results we get Brahmagupta's formula.

This method of interpolation has been applied also to the inverse problem of finding the arc having a given trigonometrical function.

Brahmagupta says:

"To find the arc, multiply the residue (after subtracting as many *R*sines from the given quantity as possible) by 900 and divide by the difference to be passed after having determined that difference by repeated operations. The degrees in the quotient will be the arc of the residue. Subtract (as many possible) *R*sines (from the given quantity), multiply the residue by 900 and divide by the (next) difference not subtracted; the quotient will be the second residue; multiply it by half the difference between the (tabular) difference passed over and that to be passed and then divide by 900. With this quotient proceed as before for the (adjusted) value of the (tabular) difference to be passed. Repeat the same operations with the residue until the result is obtained finally."[1] (By "*R*sines" in this rule is meant "tabular *R*sine-differences.")

The latter portion of this rule has become rather cryptic, as all the successive operations have not been fully described. But it appears from the explanations of the commentator Bhaṭṭotpala (966) that Brahmagupta has intended the same formula as has been clearly described by Bhāskara II. The latter says:

"Subtract the (tabular) differences (as many as possible from the given value); multiply half the remainder by the difference of the (tabular) difference passed over and that to be passed, and then divide by the (tabular) difference to be passed. Half the sum of the (tabular) difference passed over and that to be passed plus or minus the quotient is the adjusted value of the (tabular) difference to be passed, whence (will follow) the arc (required)."[2]

He then remarks as before that the negative sign should be taken in calculating the *R*sines and the positive sign for the versed *R*sines.

The *rationale* of this will be clear from the previous formula on inversion. There we shall now have $f(\alpha')$ known and $r\ (= \alpha' - \alpha_2)$ as unknown.

$$f = f_2 + \frac{r}{h}\left(\frac{\Delta_1 + \Delta_2}{2} \pm \frac{r}{h}\,\frac{\Delta_1 - \Delta_2}{2}\right)$$

or $$\frac{r}{h}\,\Delta_2 = f - f_2 + \frac{r}{h}\,\frac{\Delta_2 - \Delta_1}{2} \mp \frac{r^2}{h^2}\,\frac{\Delta_1 - \Delta_2}{2}.$$

Now let us take for the first approximation, as before

$$\frac{r}{h} = \frac{f - f_2}{\Delta_2}.$$

Substituting this value of $\frac{r}{h}$ in the neglected terms; we get for the second approximation

$$r = (f - f_2)\,\frac{h}{\Delta_2}\left(1 + \frac{\Delta_2 - \Delta_1}{2\Delta_2} \mp \frac{f - f_2}{\Delta_2^2}\cdot\frac{\Delta_1 - \Delta_2}{2}\right)$$

$$= (f - f_2)\,\frac{h}{\Delta}, \text{ say}$$

[1]*KK*, Part 2, i. 12. The printed text gives only the earlier part of the rule. The remaining portion has been taken from the text of Bhaṭṭotpala.

[2]*SiŚi Graha*, ii. 17.

so that $\triangle$ will be the adjusted value of the (tabular) difference to be passed. Then

$$\frac{1}{\triangle} = \frac{1}{\triangle_2}\left(1+\frac{\triangle_2-\triangle_1}{2\triangle_2} \mp \frac{f-f_2}{\triangle_2^2}\frac{\triangle_1-\triangle_2}{2}\right)$$

Therefore, $$\triangle = \triangle_2\left(1+\frac{\triangle_2-\triangle_1}{2\triangle_2} \mp \frac{f-f_2}{\triangle_2^2}\frac{\triangle_1-\triangle_2}{2}\right)^{-1}$$

$$= \triangle_2\left(1-\frac{\triangle_2-\triangle_1}{2\triangle_2} \pm \frac{f-f_2}{\triangle_2^2}\frac{\triangle_1-\triangle_2}{2}\right)$$

or $$\triangle = \frac{\triangle_1+\triangle_2}{2} \pm \frac{f_1-f_2}{2\triangle_2}(\triangle_1-\triangle_2),$$

as stated in the rule. But the more accurate result by inversion would have been

$$\triangle = \frac{\triangle_1+\triangle_2}{2} \pm \frac{f-f_2}{\triangle_1+\triangle_2}(\triangle_1-\triangle_2).$$

Bhāskara II was clearly aware of this. For he is found to have remarked:

"This improved formula for calculating the arc is a little rough (*sthūla*). Though rough, it has been adopted for its simplicity (*sukhārtha*). By other means such as finer calculations or repeated applications it can be made more accurate."[1]

Generalised Formula

Brahmagupta has extended his formula of interpolation so as to be applicable also to the case when the intervals between the consecutive tabular values of the argument are not equal. He says:

"Multiply the increase of the *Śīghra* anomaly to be passed by the degrees of the increase of the *Śīghra* equation passed over and divide by the increase of the *Śīghra* anomaly passed over; the quotient is the (adjusted) increase of the *Śīghra* equation in degrees. Multiply half the difference of that and the increase of the *Śīghra* equation to be passed by the residue of the anomaly and divide by the increase of the *Śīghra* anomaly to be passed; half the sum of these equations is decreased or increased by the (last) quotient, according as it (half the sum) is greater or less than the *Śīghra* equation to be passed; the result will be corrected *Śīghra* equation to be passed."[2]

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[1] "*Idam dhanuḥkhaṇḍasphuṭīkaraṇaṁ kiñcit sthūlam. Sthūlamapi sukhārthamangīkṛtam. Anyathā bījakarmaṇā vā' sphuṭaṁ kartuṁ yujyate*"—*Ibid* (Gloss)."

[2] *Bhuktagatiphalāṁśaguṇā bhogyagatirbhuktagatihṛtā labdhaṃ*
Bhuktagateḥ phalabhāgāstadbhogyaphalāntarārdhahataṁ
Vikalaṁ bhogyagatihṛtaṁ labdhenonādhikaṁ phalaikyārdhaṁ
Bhogyaphalādadhikonaṁ tadbhogyaphalaṁ sphuṭaṁ bhavati"—*KK*, Part 2, ii. 2-3.

That is, if $\alpha_3-\alpha_2\neq\alpha_2-\alpha_1$, let $\alpha_3-\alpha_2=h_2$ and $\alpha_2-\alpha_1=h_1$.

Then

$$f(\alpha')=f_2+\frac{r}{h_2}\left\{\left(\tfrac{1}{2}\,\frac{\Delta_1\times h_2}{h_1}+\Delta_2\right)\pm\frac{r}{h_2}\cdot\tfrac{1}{2}\left(\frac{\Delta_1\times h_2}{h_1}-\Delta_2\right)\right\},$$

or $$f(\alpha')=f_2+\frac{r}{h_2}\,\Delta_2+\frac{r}{2h_2}\left(1\pm\frac{r}{h_2}\right)\left(\frac{\Delta_1\times h_2}{h_1}-\Delta_2\right),$$

where the upper or lower sign is to be taken according as

$$\tfrac{1}{2}\left(\frac{\Delta_1\times h_2}{h_1}+\Delta_2\right)<\text{ or }>\Delta_2.$$

7. Spherical Trigonometry

Solution of Spherical Triangles

From the use in the treatment of astronomical problems, we find that the Hindus knew how to solve spherical triangles, of oblique as well as of right-angled varieties. They do not seem to possess a general method of solution in this matter unlike the Greeks who systematically followed the method of Ptolemy (*c*. 150) based on the well-known theorem of Menelaus (90). Still with the help of the properties of similar plane triangles and of the theorem of the square of the hypotenuse, they arrived at a set of accurate formulae sufficient for the purpose. As has been proved conclusively by Sengupta[1], Braunmühl was wrong in supposing that in the matter of solution of spherical triangles the Hindus utilised the method of projection contained in the Analemma of Ptolemy.

Right-angled Spherical Triangle

The Hindus obtained the following formulae for the right-angled spherical triangle, right-angled at C:

(*i*) $\sin a=\sin c \sin A$,
(*ii*) $\cos c=\cos a \cos b$,
(*iii*) $\sin c \cos A=\cos a \sin b$,
(*iv*) $\sin b=\tan a \cot A$,
(*v*) $\cos A=\tan b \cot c$.

It should be particularly noted that as the tangent and cotangent functions are

[1]Sengupta, P. C., "Greek and Hindu Methods in Spherical Trigonometry", *Journ. Dept. Letters. Cal. Univ.*, Vol. xxi (1931).
[2]Braunmühl, *Geschichte*, pp. 38ff; compare also Heath, *Greek Math.*, II, p. 291.

not recognised in Hindu Trigonometry, the formulae (*iv*) and (*v*) are ordinarily written in the forms

$$(iv)\ \sin b = \frac{\sin a.\cos A}{\cos a.\sin A}, \qquad (v)\ \cos A = \frac{\sin b.\cos c}{\cos b.\sin c},$$

These formulae were obtained thus: (See Fig. 13)

Let ABC be a spherical triangle right-angled at C, lying on a sphere whose centre is at O. Produce the sides AC and AB to P and Q respectively, such that the arc AP=arc AQ=90°. Join PQ by an arc of the great circle on the sphere. Produce the arcs CB and PQ so as to meet at the point Z.

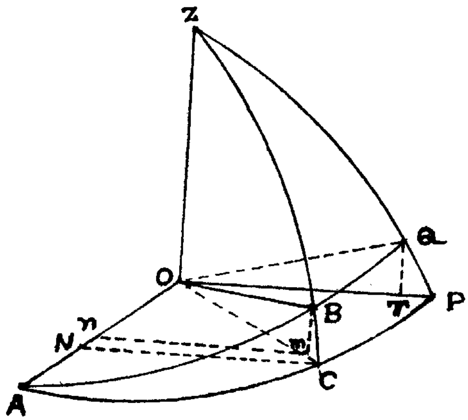

Fig. 13

Then clearly Z is the pole of the circle ACP as A is of the circle PQZ. Join A, B, C, P, Q, Z, with O. Draw QT perpendicular to OP, Bm to OC, Bn and CN to OA. Join mn.

From the similar triangles OQT and nBm

$$\frac{nB}{OQ} = \frac{nm}{OT} = \frac{Bm}{QT} \qquad (1)$$

and from the similar triangles Onm and ONC

$$\frac{On}{ON} = \frac{Om}{OC} = \frac{nm}{NC}. \qquad (2)$$

Substituting the values in terms of trigonometrical functions, we get

$$\frac{jy\bar{a}\ c}{R} = \frac{nm}{kojy\bar{a}\ A} = \frac{jy\bar{a}\ a}{jy\bar{a}\ A} \tag{1.1}$$

$$\frac{kojy\bar{a}\ c}{kojy\bar{a}\ b} = \frac{kojy\bar{a}\ a}{R} = \frac{nm}{jy\bar{a}\ b}. \tag{2.1}$$

Hence from (1.1), we get

$$jy\bar{a}\ a = \frac{jy\bar{a}\ c.\ jy\bar{a}\ A}{R},$$

which is of course equivalent to

$$\sin a = \sin c \sin A. \tag{3.i}$$

Similarly from (2.1)

$$kojy\bar{a}\ c = \frac{kojy\bar{a}\ a.\ kojy\bar{a}\ b,}{R}$$

or $$\cos c = \cos a \cos b. \tag{3ii}$$

Again equating the values of *mn* from (1.1) and (2.1), we get

$$jy\bar{a}\ c.\ kojy\bar{a}\ A = kojy\bar{a}\ a.\ jy\bar{a}\ b;$$

that is, $$\sin c \cos A = \cos a \sin b, \tag{3.iii}$$

Eliminating $\sin c$ between (3.i) and (3.iii), we get

$$\sin b = \tan a \cot A; \tag{3.iv}$$

and eliminating $\cos a$ between (3.ii) and (3.iii), we have

$$\cos A = \tan b \cot c. \tag{3.v}$$

As an illustration of the application of the above formulae let us take the problem of determination of the Sun's right ascension (α) when the Sun's longitude (λ) and declination (δ) are known. Let γM be the equator, γS the ecliptic and S the Sun. Then $\lambda(=\gamma S)$ denotes the sun's longitude, $\delta\ (=SM)$ the Sun's declination, and $\alpha(=\gamma M)$ the Sun's right ascension. If ϵ denotes the obliquity of the ecliptic, then by the formula (iii), we get

$$\sin \alpha = \frac{\sin \lambda.\cos \epsilon}{\cos \delta}$$

This result is stated by Āryabhaṭa[1], Brahmagupta[2] and others[3]. It is noteworthy that in this particular case the triangles *Bnm* and *OQT* are called technically *Krānti-Kṣetra* or "declination triangles" which shows definitely that they were actually drawn and the final result was actually obtained by the method stated above.[4]

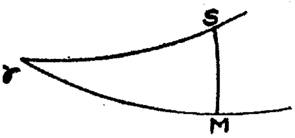

Fig. 14

Oblique Spherical Triangle

For the solution of an oblique spherical triangle the Hindus had equivalents of the following formulae:

(*i*) $\cos a = \cos b \, \cos c + \sin b \sin c \, \cos A$,

(*ii*) $\sin a \, \cos C = \cos c \, \sin b - \sin c \, \cos b \, \cos A$,

(*iii*) $\dfrac{\sin a}{\sin A} = \dfrac{\sin b}{\sin B} = \dfrac{\sin c}{\sin C}$

Let ABC be a spherical triangle lying on a sphere whose centre is at O. Produce the arc AC to S and arc CB to Q, such that the arc CS = arc CQ = 90°. Then C will be the pole of the great circle SQN. Join OA, OS and ON. Through B draw the small circle RVR' perpendicular to OA, intersecting the great circle SQN at D and D'. Join DD' intersecting SON at V. Let the diameter RVR' of the small circle cut OA at O'. Again through O draw the straight line WOE parallel to DVD'. Draw the great circle $KEK'W$ parallel to the small circle RBR' and the great circle EAW perpendicular to the latter and cutting it at F and F'. From B draw BT perpendicular to RVR', BH to DVD' and BM to OQ. Join MH cutting WOE at L. Draw NY perpendicular to OA, QQ' and MM' to OS. Let BH cut FF' at G.

From the similar triangles BMH and OYN, we get

$$\frac{BM}{OY} = \frac{MH}{NY} = \frac{HB}{NO}\,; \tag{1}$$

and from the similar triangles OVO' and ONY

$$\frac{OV}{ON} = \frac{VO'}{NY} = \frac{O'O}{YO}\,. \tag{2}$$

[1] *A*, iv. 25.
[2] *BrSpSi*, iii. 15.
[3] *SūSi*, iii. 41-43; *PS*, iv. 29; *SiŚi*, *Graha*, ii. 54-5 etc.
[4] Compare Sen Gupta, P.C., "*Papers on Hindu Mathematics and Astronomy*", Part I, Calcutta, 1916, pp. 46ff; *PS*, iv. 35 (comments).
[5] Braunmühl, *Geschichte*, I, p. 41; Kaye, G.R., "Ancient Hindu Spherical Astronomy", *JASB*, Vol. xv (1919), pp. 153ff; P. C. Sen Gupta, *Papers etc.*, pp. 57ff.

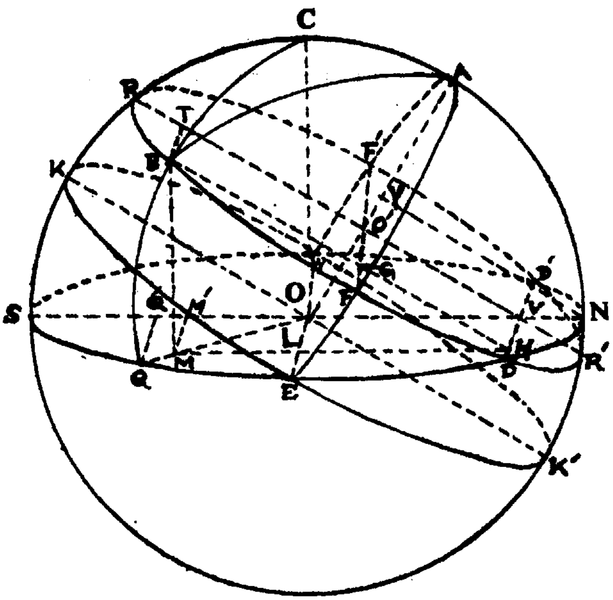

Fig. 15

Hence substituting the values

$$\frac{kojy\bar{a}\ a}{kojy\bar{a}\ (90^\circ - b)} = \frac{MH}{jy\bar{a}\ (90^\circ - b)} = \frac{HB}{R}$$

$$\text{and } \frac{OV}{R} = \frac{VO'}{jy\bar{a}\ (90^\circ - b)} = \frac{kojy\bar{a}\ c}{kojy\bar{a}\ (90^\circ - b)}\ ;$$

$$\text{whence } HB = \frac{R\ kojy\bar{a}\ a}{jy\bar{a}\ b},\quad MH = \frac{kojy\bar{a}\ a.\ kojy\bar{a}\ b}{jy\bar{a}\ b}, \tag{1.1}$$

$$\text{and } OV = \frac{R\ kojy\bar{a}\ c}{jy\bar{a}\ b},\quad O'V = \frac{kojy\bar{a}\ b.\ kojy\bar{a}\ c}{jy\bar{a}\ b} \tag{2.1}$$

$$\left.\begin{aligned} \text{Further } O'R &= jy\bar{a}\ c, \\ RT &= \frac{jy\bar{a}\ c\ utjy\bar{a}\ A}{R}, \\ ML &= \frac{jy\bar{a}\ \bar{a}.\ kojy\bar{a}\ (C - 90^\circ)}{R} \end{aligned}\right\} \tag{3}$$

Now $HB = VT = RO' + O'V - RT$.

Therefore substituting the values of the constituent elements on either sides of equations from (1.1), (2.1) and (3), we get

$$\frac{R\ kojy\bar{a}\ a}{jy\bar{a}\ b} = jy\bar{a}\ c + \frac{kojy\bar{a}\ b.\ kojy\bar{a}\ c}{jy\bar{a}\ b} - \frac{jy\bar{a}\ c\ .\ utjy\bar{a}\ A}{R},$$

$$= \frac{kojy\bar{a}\ b.\ kojy\bar{a}\ c}{jy\bar{a}\ b} + \frac{jy\bar{a}\ c\ kojy\bar{a}\ A}{R};$$

which is equivalent to

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Again $MH = ML + LH = ML + OV$.

Therefore by (1.1), (2.1) and (3)

$$\frac{kojy\bar{a}\ a,\ kojy\bar{a}\ b}{jy\bar{a}\ b} = \frac{jy\bar{a}\ a.\ jy\bar{a}\ (C - 90^\circ)}{R} + \frac{R\ kojy\bar{a}\ c}{jy\bar{a}\ b},$$

or $R^2\ kojy\bar{a}\ c = R\ kojy\bar{a}\ a.\ kojy\bar{a}\ b + jy\bar{a}\ a\ jy\bar{a}\ b\ kojy\bar{a}\ C$;

which is equivalent to

$$\cos c = \cos a \cos b + \sin a \sin b \cos C, \tag{i.1}$$

a formula similar to (*i*).

From the similar triangles OQQ' and OMM', we have

$$\frac{OQ}{OM} = \frac{QQ'}{MM'} = \frac{Q'O}{M'O}. \tag{4}$$

Therefore $OM.\ Q'O = OQ\ .\ OM' = OQ\ (MH - OV)$

$$= YN.\ HB - OV.\ R.\ \left[\because \frac{MH}{YN} = \frac{HB}{NO}\right]$$

Hence $OM.\ Q'O = YN\ (RO' + O'V - RT) - OV.\ R$;

or $jy\bar{a}\ a.\ jy\bar{a}\ (C - 90^\circ) = jy\bar{a}\ (90^\circ - b) \left(jy\bar{a}\ c + \frac{kojy\bar{a}\ b.\ kojy\bar{a}\ c}{jy\bar{a}\ b} - \frac{jy\bar{a}\ c.\ utjy\bar{a}\ A}{R}\right) - \frac{R^2\ kojy\bar{a}\ c}{jy\bar{a}\ b}$;

or $-jy\bar{a}\, a.\, kojy\bar{a}\, C = \frac{kojy\bar{a}\, b.\, jy\bar{a}\, c}{R}(R - utjy\bar{a}\, A) - \frac{kojy\bar{a}\, c}{jy\bar{a}\, b}\,[R^2 - (kojy\bar{a}\, b)^2]$,

or $jy\bar{a}\, a.\, kojy\bar{a}\, C = kojy\bar{a}\, c.\, jy\bar{a}\, b - jy\bar{a}\, c.\, kojy\bar{a}\, b.\, kojy\bar{a}\, A$;

which is equivalent to

$$\sin a \cos C = \cos c \sin b - \sin c \cos b \cos A.$$

Since $MM' = BT$, we get from (4)

$$\frac{OM}{OQ} = \frac{BT}{QQ'}$$

Hence $$\frac{jy\bar{a}\, a}{R} = \frac{jy\bar{a}\, c\ \ jy\bar{a}\, A}{R\, jy\bar{a}\, C}$$

or $$\frac{jy\bar{a}\, a}{jy\bar{a}\, A} = \frac{jy\bar{a}\, c}{jy\bar{a}\, C}\ .$$

Similarly it can be proved that

$$\frac{jy\bar{a}\, c}{jy\bar{a}\, C} = \frac{jy\bar{a}\, b}{jy\bar{a}\, B}\ .$$

These are of course equivalent to

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}\ .$$

As an illustration of the application of the above formulae we take up the problem of the determination of the relation between the zenith distance (z), azimuth (ψ) and hour angle (H) of a heavenly body of known declination (δ) at a station whose terrestrial latitude is ϕ. In Fig. 15 let $NEQS$ denote the horizon, $NACS$ the meridian circle, KEK' the equator, RBR' the diurnal circle of the heavenly body (B), AFE the six o'clock circle, A the north pole and C the zenith. Then $a=z$, $b=90°-\phi$, $c=90°-\delta$, $\angle A=H$, $C=\psi$.

Substituting these values in the formulae (i), (i.1) and (ii), we obtain

$$\cos z = \sin \delta \sin \phi + \cos \delta \cos \phi \cos H$$

$$\sin \delta = \cos z \sin \phi + \sin z \cos \phi \cos \psi,$$

and

$$\sin z \cos \psi = \sin \delta \cos \phi - \cos \delta \sin \phi \cos H.$$

These equations were obtained by most of the Hindu astronomers.[1] It should however be made clear that the final results were arrived at by successive stages. The straight line DVD', the line of intersection of the diurnal circle with the horizon, is technically called *udayāsta-sūtra* ("the thread through the rising and setting points"), BM is called *śaṅku* ("gnomon"); BH *cheda* or *iṣṭahṛti* ("optional divisor"), MH *śaṅkutala*, BG (=the *jyā* in the diurnal circle of the complement of the hour angle) *kalā*, GH (=the *jyā* of the arc of the diurnal circle intercepted between the horizon and the six o'clock circle) *kujyā* or *kṣitijyā* ("earth-sine"), HL(=*jyā* ED) *agrā*, ML *bāhu*, OM *dṛgjyā* ("*jyā* of the zenith distance"), the angle EAD *cara* ("the ascensional difference"). The existence of these technical terms proves conclusively that the Hindus actually made the constructions contemplated above. They recognise the angle MBH to be equal to the latitude of the observer's station.

The *Sūrya-siddhānta* says:

"The *R*sine of the declination multiplied by the *palabhā* (=12 tan ϕ) and divided by 12 gives the *kujyā* ("the earth-sine"); that multiplied by the radius and divided by the radius of the diurnal circle will give the *jyā* whose arc will be the *cara* ("ascensional difference")."[2]

$$kujy\bar{a} = \frac{jy\bar{a}\ \delta \times 12\, jy\bar{a}\ \phi}{12 \times kojy\bar{a}\ \phi}$$

$$carajy\bar{a} = \frac{kujy\bar{a} \times R}{kojy\bar{a}\ \delta}$$

Again,

"The radius plus the *carajyā* in the northern hemisphere, or minus it in the southern hemisphere is called *antyā*; subtract from it the versed *R*sine of the hour angle; (the remainder) multiplied by the radius of the diurnal circle and divided by the radius will be the *cheda*; that multiplied by the *R*sine of the co-latitude and divided by the radius will be the *śaṅku*; subtract the square of it from the square of the radius; the square-root of the remainder will be the *jyā* of the zenith distance."[3]

$$R \pm carajy\bar{a} = anty\bar{a},$$

$$\frac{(anty\bar{a} - utjy\bar{a}\ H) \times kojy\bar{a}\,\delta}{R} = cheda,$$

[1] *PSi*, iv. 42-4; *SūSi*, iii. 28-31, 34-5; *BrSpSi*, iii. 25-40, 54-6; *SiŚi*, *Graha*, iii. 50-52; etc.
[2] *SūSi*, ii. 61; also *Ā*, iv. 26; *ŚiDVṛ*, ii. 17; *BrSpSi*, ii. 57-60; *PSi*, iv. 26-7; *SiŚi*, *Graha*, ii. 48.
[3] *SūSi*, iii. 34-6.

$$\frac{cheda \times kojy\bar{a}\ \phi}{R} = \acute{s}a\dot{n}ku,$$

and $$\sqrt{R^2 - (\acute{s}a\dot{n}ku)^2} = jy\bar{a}\ z,$$

Therefore, in the northern hemisphere,

$$kojy\bar{a}\ z = \acute{s}a\dot{n}ku$$
$$= \frac{kojy\bar{a}\ \delta\ kojy\bar{a}\ \phi}{R^2}\left(R + R\frac{jy\bar{a}\ \delta\ jy\bar{a}\ \phi}{kojy\bar{a}\ \delta\ kojy\bar{a}\ \phi} - utjy\bar{a}\ H\right)$$

or $$R^2\ kojy\bar{a}\ z = R\ jy\bar{a}\ \delta \cdot jy\bar{a}\ \phi + kojy\bar{a}\ \delta \cdot kojy\bar{a}\ \phi \cdot kojy\bar{a}\ H;$$

which is of course equivalent to

$$\cos z = \sin \delta \sin \phi + \cos \delta \cos \phi \cos H.$$

Again it has been said that[1]

$$\acute{s}a\dot{n}kutala \mp b\bar{a}hu = agr\bar{a},$$

the negative or positive sign being taken according as the heavenly body is in the northern or southern hemisphere. Further[2]

$$agr\bar{a} = \frac{R\ jy\bar{a}\ \delta}{kojy\bar{a}\ \phi} \text{ and } b\bar{a}hu = -\frac{jy\bar{a}\ z\ kojy\bar{a}\ \psi}{R};$$

Also[3]

$$\acute{s}a\dot{n}kutala = \frac{\acute{s}a\dot{n}ku \times jy\bar{a}\ \phi}{kojy\bar{a}\ \phi}$$

Hence substituting the values

$$-\frac{jy\bar{a}\ z\ kojy\bar{a}\ \psi}{R} = \frac{\acute{s}a\dot{n}ku \times jy\bar{a}\ \phi}{kojy\bar{a}\ \phi} - \frac{R\ jy\bar{a}\ \delta}{kojy\bar{a}\ \phi}$$
$$= \frac{jy\bar{a}\ \phi\ kojy\bar{a}\ \delta}{R^2}\left\{R + R\frac{jy\bar{a}\ \phi \cdot jy\bar{a}\ \delta}{kojy\bar{a}\ \phi\ kojy\bar{a}\ \delta} - utjy\bar{a}\ H\right\} - \frac{R\ jy\bar{a}\ \delta}{kojy\bar{a}\ \phi}$$
$$= \frac{jy\bar{a}\ \phi\ kojy\bar{a}\ \delta\ kojy\bar{a}\ H}{R^2} - \frac{jy\bar{a}\ \delta \cdot kojy\bar{a}\ \phi}{R},$$

[1] *Ibid*, iii. 23–4
[2] *SūSi*, iii. 27; *PSi*, iv. 39; *Ā*, iv. 30; *BrSpSi*, xxi. 61.
[3] *Ā*, iv. 28, 29; *BrSpSi*, iii. 65, xxi. 63.

which is of course equivalent to

$$\sin z \cos \psi = \sin \delta \cos \phi - \cos \delta \sin \phi \cos H.$$

Expansion of Trigonometrical Functions

Remarkable work on the expansion of trigonometrical functions, $\sin \theta$, $\cos \theta$, $\tan^{-1}\theta$, etc., was done in India by the astronomers of Kerala in the fourteenth, fifteenth and sixteenth centuries A.D. It will be discussed in another article which will be devoted to the "Calculus".

ABBREVIATIONS

Ā	*Āryabhaṭīya*
ĀpŚlSū	*Āpastamba-Śulba-sūtra*
BCMS	*Bulletin of the Calcutta Mathemetical Society*
BrSpSi	*Brāhma-sphuṭa-siddhānta*
DhGr	*Dhyānagrahopadeśa*
GrL	*Graha-lāghava*
GSS	*Gaṇita-sāra-saṅgraha*
IJHS	*Indian Journal of History of Science*
JIMS	*Journal of the Indian Mathematical Society*
KK	*Khaṇḍa-khādyaka* (Bina Chatterjee's edition)
KKau	*Karaṇa-kaustubha*
KPr	*Karaṇa-prakāśa*
KR	*Karaṇa-ratna*
L	*Līlāvatī* (Ānandāśrama edition)
LBh	*Laghu-Bhāskarīya*
LMā	*Laghu-mānasa*
MBh	*Mahā-Bhāskarīya*
MSi	*Mahā-siddhānta*
PSi	*Pañca-siddhāntikā*
ŚiDVṛ	*Śiṣya-dhī-vṛddhida*
SiSā	*Siddhānta-sārvabhauma*
SiŚe	*Siddhānta-śekhara*
SiŚi	*Siddhānta-śiromaṇi*
SiTVi	*Siddhānta-tattva-viveka*
SK	*Sumati-karaṇa*
SMT	*Sumati-mahā-tantra*
SūSi	*Sūrya-siddhānta*
VSi	*Vaṭeśvara-siddhānta*
VVSi	*Vṛddha-Vaśiṣṭha-siddhānta*
	Yantra-rāja