SECOND ORDER INTERPOLATION IN INDIAN MATHEMATICS UP TO THE FIFTEENTH CENTURY

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The computational abilities of ancient Indian mathematicians are well known. The paper deals with the second order interpolation schemes found in a few astronomical works of India. The earliest one is the rule of Brahmagupta (c. A.D. 625) for equal intervals, which resembles the modern Newton-Stirling interpolation formula up to the second order. Later on (A.D. 665) Brahmagupta also gave a modified form of his rule to cover the case of unequal intervals. Then we come across a peculiar set of rules for second order interpolation in a work of Govindasvāmi (c. A.D. 800–850). The famous Bhāskara II (c. A.D. 1150) gave an empirical derivation of Brahmagupta's rule for equal knots. Next are described the Indian forms of the second order Taylor series approximations which are attributed to Mādhava (A.D. 1350–1410). Finally are given the forms of various rules quoted by Parameśvara (c. first quarter of the fifteenth century A.D.).

Symbols

a—the argument, circular arc measured in angular units; anomaly

a_1, a_2, etc.—successive unequidistant values of a

h—equal (common) arcual interval; elemental arc

h_1, h_2, etc.—unequal arcual intervals (*gatis*),

$$h_1 = a_1;\ h_2 = a_2 - a_1;\ h_3 = a_3 - a_2;\ \text{etc.}$$

R—*sinus totus* (radius)

$R \sin a$, $R \cos a$,

R versin a—Indian sine, cosine and versed sine of the arc a

$f(a)$—the functional value of sine, versed sine or certain astronomical function called 'equation' (*phala*)

p, q—positive integers

$x = p \,.\, h$ or a_p; arc passed over, such that $f(x)$ is known

θ—residual arc, such that $f(x+\theta)$ is required to be interpolated, θ being positive and less than h or h_{p+1}.

$n = \theta/h$

D_1, D_2, etc.—tabulated functional differences

$$D_1 = f(a_1) \text{ or } f(h)$$

$$D_2 = f(a_2) - f(a_1) \text{ or } f(2h) - f(h)$$

$$D_3 = f(a_3) - f(a_2) \text{ or } f(3h) - f(2h), \text{ etc., etc.}$$

Δ—first order forward difference operator

$$\Delta f(a) = f(a+h) - f(a); \quad \Delta f(a_q) = f(a_{q+1}) - f(a_q)$$

$$\Delta f(x) = D_{p+1}$$

Δ^2—second order difference operator

h_p, h_{p+1}—argumental intervals just passed over (last or *bhukta gati*) and yet to be passed over (current or *bhogya gati*) respectively

D_p, D_{p+1}—the corresponding tabulated functional differences passed over (*bhukta khaṇḍa* or *gatiphala*) and to be passed over (*bhogya khaṇḍa* or *gatiphala*) respectively

D_t—the envisaged true (*sphuṭa*) value of the functional difference to be passed over

Z_p—'adjusted' value of the functional difference passed over in case of unequal intervals.

1. Introduction

Tabular values of the trigonometric functions R sin a, R versin a or their differences and of certain astronomical functions are found almost in every work on astronomy of ancient and medieval India. Various numerical values of R and h were taken by the Indians. For computing the functional values corresponding to the intervening values of the argument the ordinary method used was that of linear proportion, i.e. first order interpolation. For better results more elegant techniques using second order interpolation schemes are also found in few Hindu works. Below we give the methods described by few Indian astronomer-mathematicians starting with Brahmagupta (seventh century) who taught, 'for the first time in the History of Mathematics, the improved rules for interpolation by using the second differences'.[1]

2 Brahmagupta's Rule for Equal Intervals

It is well known[2] that Brahmagupta composed *Brāhmasphuṭa Siddhānta* in A.D. 628 and *Khaṇḍakhādyaka* in A.D. 665. The famous couplet containing Brahmagupta's interpolation rule for equal intervals is found in the *uttara* (supplementary) part of *Khaṇḍakhādyaka*. However, an earlier reference is worth noting. The same couplet occurs in Brahmagupta's earliest known work, the *Dhyāna-Graha-adhikāra* or *Dhyāna-Graha-adhyāya* or the *Dhyāna-grahopadeśādhyāya*. That the *Dhyānagraha* was written earlier than the *Brāhmasphuṭa Siddhānta* is concluded on the ground that the latter (XXIV, 9) quotes the former.[3] Hence the invention of the second order interpolation formula given by Brahmagupta should be placed near the beginning of the second quarter of the seventh century A.D., if not earlier.

Now we quote the couplet:

गत भोग्य खण्डकान्तर दल विकल वधात् शतैर्नवभिराप्तैः ।
तद्युति दलं युतोनं भोग्यादूनाधिकं भोग्यम् ।।

(*Dhyāna-Graha-Upadeśa-adhyāya*, 17; *Khaṇḍa Khādyaka*, IX, 8, etc.)[4]

'Multiply half the difference of the tabular differences crossed over and to be crossed over by the residual arc and divide by 900′ (= h). By the result (so obtained) increase or decrease half the sum of the same (two) differences, according as this (semi-sum) is less or greater than the difference to be crossed over. We get the true functional differences to be crossed over.'

That is

$$D_t = \tfrac{1}{2}(D_p + D_{p+1}) \pm \tfrac{1}{2}(D_p \sim D_{p+1}) \frac{\theta}{h}, \qquad \ldots \qquad \ldots \qquad (1)$$

the upper or lower sign is to be taken according as $\frac{1}{2}(D_p + D_{p+1})$ is less than or greater than D_{p+1}, i.e. according as D_p is less or greater than D_{p+1}.
Then we have

$$f(x+\theta) = f(x) + \frac{\theta}{h} . D_t \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2)$$

Combining (1) and (2) and using $D_{p+1} = \Delta f(x)$ we easily obtain

$$f(x+nh) = f(x) + \frac{n}{2}\{\Delta f(x-h) + \Delta f(x)\} + \frac{n^2}{2}\{\Delta f(x) - \Delta f(x-h)\}$$

which may be regarded as the modern form of Brahmagupta's rule and is a particular case (up to second orders) of the more general Newton-Stirling interpolation formula of modern Mathematics.[5] From the context in the work *Dhyāna-Graha-Upadeśa-adhyāya* it is clear that Brahamagupta gave the rule for the interpolation of sine ($D_p > D_{p+1}$) and the versed sine ($D_p < D_{p+1}$). However, in the statement of the rule itself there is no such limitation on the scope of its use and the rule may be applied to other functions tabulated at equal intervals. In fact the commentators Pṛthūdaka (A.D. 864) and Āmarāja (1180) both explained its use in finding the equation of centre (*manda-phala*).[6] Brahmagupta's Rule is also found in the *Vaṭeśvara Siddhānta* (II, i, 62) of A.D. 904.

3. Bhaskara II's Form of Brahmagupta's Formula and its Rationale

In the *Grahagaṇita* part of his *Siddhānta Śiromaṇi*, Bhāskara II (A.D. 1150) gives the rule as

यातैष्ययोः खण्डकयोर्विशेषः
शेषांश निघ्नो नखहृत् तदूनम् ।
युतं गतैष्यैक्य दलं स्फुटं स्यात्
क्रमोत्क्रमज्या करणेऽत्र भोग्यम् ।। १६ ।।

(*Siddhānta Śiromaṇi* II (*Spaṣṭādhikāra*), 16)[7]

'Multiply the difference of the tabular differences passed over and to be passed over by the residual arc and divide by 20. By the result decrease or increase half the sum of (the differences) passed over and to be passed over to get the true difference to be passed over for interpolating sine and versed sine respectively.'

That is

$$D_t = \tfrac{1}{2}(D_p + D_{p+1}) \pm \frac{\theta}{20} \cdot (D_p \sim D_{p+1}),$$

where we have to take the negative and positive sign respectively for computing sine and versed sine.

Thus we see that the formula is same as that of Brahmagupta except that Bhāskara II takes $h = 10°$, instead of $h = 900'$ ($= 15°$).

The rationale (*upapatti*) of the rule given by Bhāskara II is as follows:[8]

D_{p+1} and $\frac{1}{2}(D_p + D_{p+1})$ are the tabular differences at the end and beginning of the current interval respectively. Take proportional part of their difference (to get the necessary correction to be made for finding the true difference corresponding to the intervening point).

By the rule of three this correction (change)

$$= [\tfrac{1}{2}(D_p + D_{p+1}) \sim D_{p+1}] \frac{\theta}{10}$$

$$= \frac{\theta}{20} \cdot (D_p \sim D_{p+1}).$$

This combined with $\frac{1}{2}(D_p + D_{p+1})$ (which has been taken above as the tabular difference at the beginning of the current interval) will give the required D_t. We have to add or subtract the correction term since tabular differences decrease (*apacaya*) for sine and increase (*upacaya*) in case of versed sine. Thus it is proved.

In the above proof of Bhāskara II, the assumption that $\frac{1}{2}(D_p + D_{p+1})$ is the tabular difference at the beginning of the interval considered is wrong since it is actually D_p there.* Kamalākara (A.D. 1658) attacked Bhāskara II on this point in his *Siddhānta Tattvaviveka* (II, 179).[9] Whether Brahmagupta argued in the same manner as Bhāskara II to arrive at his rule or had some other approach for it is difficult to say in the absence of specific evidence.

In case of sine function the following derivation of the rule will not be without interest here. We have

$$\tfrac{1}{2}(D_p + D_{p+1}) = \frac{R}{2}\{\sin x - \sin (x-h) + \sin (x+h) - \sin x\}$$

$$= R \cos x \sin h \qquad \ldots \qquad (3)$$

* However, these arguments are not adequate, since the true functional difference, corresponding to the residual arc, is $\frac{\theta}{h} . D_t$ (and not simply D_t) which will be zero (as it ought to be) when $\theta = 0$, whether D_t is taken D_p or $\frac{1}{2}(D_p + D_{p+1})$ there.

and

$$\tfrac{1}{2}(D_p - D_{p+1}) = \frac{R}{2}\{\sin x - \sin (x-h) - \sin (x+h) + \sin x\}$$

$$= R \sin x(1-\cos h). \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

If θ and h are small, we may assume

$$\frac{\sin \theta}{\sin h} = \frac{\sin (\frac{1}{2}\theta)}{\sin (\frac{1}{2}h)} = \frac{\theta}{h}. \qquad \ldots \quad \ldots \quad \ldots \quad (5)$$

Now the 'true' *bhogyaphala*, $\frac{\theta}{h} . D_t$

$$= R \sin (x+\theta) - R \sin x$$

$$= R \cos x \sin \theta - R \sin x \,(1-\cos \theta)$$

$$= R \cos x \sin \theta - R \sin x \,.\, 2 \sin^2 (\tfrac{1}{2}\theta)$$

$$= R \cos x \,.\, \frac{\theta}{h} \,.\, \sin h - R \sin x \,.\, 2\, \frac{\theta^2}{h^2} \sin^2 (\tfrac{1}{2}h) \quad \text{by (5)}$$

$$= R\, \frac{\theta}{h} \,.\, \cos x \,.\, \sin h - R \,.\, \frac{\theta^2}{h^2} \,.\, \sin x \,(1-\cos h)$$

$$= \frac{\theta}{h} \,.\, \tfrac{1}{2}(D_p + D_{p+1}) - \frac{\theta}{h} \,.\, \frac{\theta}{h} \,.\, \tfrac{1}{2}(D_p - D_{p+1}) \quad \text{by (3) and (4)}$$

hence we get the required expression for the 'true' *bhogyakhaṇḍa*, D_t.

4. Brahmagupta's Rule for Unequal Intervals

This rule is given by Brahmagupta in *Khaṇḍa Khādyaka* (A.D. 665) in connection with computing the *gatiphala* (change in the equation) correspond to any given *gati* (change in anomaly) by using the tabulated values of the *gatiphala* at unequal intervals. The relevant Sanskrit passage is

भुक्त गति फलांश गुणा भोग्य गतिर्भुक्त गति हृता लब्धम् ।
भुक्त गतेः फल भागास्तद् भोग्य फलान्तरार्ध हृतम् ।।
विकलं भोग्य गति हृतं लब्धेनोनाधिकं फलैक्यार्धम् ।
भोग्य फलादधिकोनं तद् भोग्य फलं स्फुटं भवति ।।

(*Khaṇḍa Khādyaka*, IX, 12–13, etc.)[10]

'Multiply the last *gatiphala* (in degrees) by the current *gati* and divide by the last *gati*; the result is the "adjusted" last *gatiphala* (in degrees). Multiply half the difference of the "adjusted" last *gatiphala* and the current *gatiphala* by the residual arc and divide by the current *gati*. By the new result decrease or increase half the sum of the "adjusted" last *gatiphala* and the current *gatiphala* when this half sum is more or less than the current *gatiphala*. The final result is the true current *gatiphala*, i.e. true functional difference to be passed cover.'

In symbols

$$Z_p = D_p \,.\, \frac{h_{p+1}}{h_p}$$

and

$$D_t = \tfrac{1}{2}(Z_p + D_{p+1}) \pm \frac{\theta}{h_{p+1}}\, \tfrac{1}{2}(Z_p \sim D_{p+1}),$$

the upper or lower sign is to be taken according as $\frac{1}{2}(Z_p + D_{p+1})$ is less or greater than D_{p+1}, i.e. according as Z_p is less or greater than D_{p+1}.

The desired result is then given by

$$f(x+\theta) = f(x) + \frac{\theta}{h_{p+1}} \cdot D_t$$

where $x = a_p$

$= h_1 + h_2 + \ldots + h_p$

and $f(x) = D_1 + D_2 + \ldots + D_p$.

The numerical illustration of this rule given by Sengupta[11] in his paper is wrong. Instead of D_p ($= 12°$ in his example) he put θ ($= 14°$) in finding Z_p. However, this error was avoided by him while illustrating the rule in his translation of the *Khaṇḍa Khādyaka*.[12]

In case the intervals are equal, i.e. $h_p = h_{p+1}$, we will have $Z_p = D_p$ itself, and the rule will reduce, as should be expected, to his earlier rule for equal intervals.

5. Govindasvāmi's Rules for Second Order Interpolation

About two centuries after Brahmagupta we come across a set of peculiar rules of making second order interpolation to compute the intermediary functional values. These rules are found in Govindasvāmi's (*c.* A.D. 800–850)[13] commentary on *Mahābhāskarīya* of Bhāskara I (seventh century A.D.). The peculiar thing to note is that different formulae are laid down for different argumental intervals. The relevant text is as follows:

गच्छद्यात गुणान्तराहत वपुर्यातैष्यदिष्वासनच्छे-
दाभ्यास समूह कार्मुक कृति प्राप्तात्, त्रिभिस्ताडितात् ।
वैदे: षड्भिरवाप्तमन्त्य गुणजे राश्यो: क्रमाद्, अन्त्यभे
गन्तव्याहत वर्तमान गुणजाच्चापाप्तमेकादिभि: ।।
अन्त्यादुत्क्रमत: क्रमेण विषमै: विशेषै: क्षिपेद्
भङक्त्वाप्तं, यदि मौर्विका विधिरयं मख्या: क्रमाद् वर्तते ।
शोध्यं व्युत्क्रमतस्तथाकृत फलं, . . .

(Govindasvāmi's commentary on *Mahābhāskarīya*)[14]

'Multiply the difference of the last and the current sine differences by the two parts of the elemental arc (made by any intermediary point on it) and divide by the square of the elemental arc and further multiply by three. Now divide the result so obstained by four, in the first *rāśi*, or by six, in the second *rāśi*. The final result thus obtained should be added to the portion of the current sine difference (got by linear proportion).

In the last (third) *rāśi*, multiply the linearly proportional part of the current sine difference by the remaining part of the elemental arc and divide by the elemental arc. Now divide the result (so obtained) by the odd numbers 1, (3, 5,), etc., according as the current sine difference is first, (second, third), etc., when counted from the end in the reversed order. Add the final result thus obtained to the portion of the current sine difference (got by ordinary proportion).

These are the rules of computing true sine difference for (direct) sines. In case of versed sines apply the rules in the reversed order* and the above corrections are to be subtracted from the respective differences (got by linear interpolation).'

Let the true sine difference (*bhogyaphala*) desired,

$$R \sin (x+\theta)-R \sin x=\frac{\theta}{h} \cdot D_{p+1}+E, \text{ approximately}$$

where $\frac{\theta}{h} \cdot D_{p+1}$ is the portion of the current sine difference, D_{p+1}, obtained by the ordinary first order linear interpolation and E is the term got by second order interpolation. Then, according to the above rules, we have (assuming 24 equal divisions of the first quadrant)

$$E=\frac{1}{4} \cdot \frac{3\theta(h-\theta)}{h^2}(D_p-D_{p+1}), \quad \text{when } p=1 \text{ to } 7$$

$$E=\frac{1}{6} \cdot \frac{3\theta(h-\theta)}{h^2}(D_p-D_{p+1}), \quad \text{when } p=8 \text{ to } 15$$

and

$$E=\frac{(h-\theta)}{h} \cdot \frac{\theta}{h} D_{p+1} \cdot \frac{1}{(47-2p)}, \quad \text{when } p=16 \text{ to } 23.$$

Using the general functional notation and finite difference operator, the rule for the second *rāśi* (30° to 60°) may be put as

$$f(x+nh)=f(x)+n \Delta f(x)+\frac{n(n-1)}{2}\left\{\Delta f(x)-\Delta f(x-h)\right\},$$

which is the modern form of Govindasvāmi's rule and is a particular case (up to the second order) of the general Newton-Gauss interpolation formula.[15] Mathematically this rule of Govindasvāmi is equivalent to that of Brahmagupta for equal knots. But, unlike Brahmagupta, Govindasvāmi gives different rules for the three *rāśis* of the first quadrant.†

6. Mādhava's Taylor Series Approximation

Nīlakaṇṭha (1443–1545)[16] in his commentary on *Āryabhaṭīya* quotes the text of rules for computing sine and cosine functions, which are equivalent to modern Taylor series approximations up to the second order of small quantities. Nīlakaṇṭha attributes these rules to Mādhava (1350–1410),[17] who antedates

* That is the first, second, third *rāśi* rules of sines are to be used, respectively, for third, second, first *rāśi* in case of versed sines.

† The author of the present paper proposes to pubilsh a separate article about Govindasvāmi's computations of Indian sines.

Taylor by more than 300 years.[18] The same verses containing the same rules (but without mentioning Mādhava) are also included by Nīlakaṇṭha in his *Tantra-saṃgraha* (A.D. 1501), a 'compendium' on astronomy. The text is

तत्राह् माधव:
इष्ट दो: कोटि धनुषो: स्वसमीप समीरिते ।
ज्ये द्वे सावयवेऽन्यस्य कुर्याद्नाधिकं धनु: ।
द्विघ्नतल्लिप्तिकाप्तैकशरशैलशिखीन्दव: ।
न्यस्याच्छेदाय च मिथस्तत्संस्कार विधित्सया ।
छित्वैकां प्राक् क्षिपेञ्जह्यात् तद्धनुष्यधिकोनके ।
अन्यस्यामथ तां द्विघ्नां तथा स्यामिति संस्कृति: ।
इति ते कृत संस्कारे स्वगुणौ धनुषोस्तयो: ।

(Nīlakaṇṭha's commentary on *Āryabhaṭīya* (II, 12);[19] *Tantra-saṃgraha* (II, 10–13))[20]

'Thus spake Mādhava:

Placing the (sine and cosine) chords nearest to the arc whose sine and cosine chords are required get the arc difference to be subtracted or added. For making the correction 13751 should be divided by twice the arc difference in minutes and the quotient is to be placed as the divisor. Divide the one (say sine) by this (divisor) and add to or subtract from the other (cosine) according as the arc difference is to be added or subtracted. Double this (result) and do as before (i.e. divide by the divisor). Add or subtract the result (so obtained) to or from the first sine or cosine to get the desired sine or cosine chords.'

That is

$$\text{'divisor'} = \frac{13751}{2\theta} = D, \quad \text{say.}$$

Then

$$\sin(x+\theta) = \sin x + \left(\cos x - \frac{\sin x}{D}\right)\frac{2}{D}$$

and

$$\cos(x+\theta) = \cos x - \left(\sin x + \frac{\cos x}{D}\right)\frac{2}{D}$$

The *sinus totus* in this case is

$$R = \frac{21600}{2\pi} = 3437{\cdot}75 \text{ very nearly}$$

$$= \frac{13751}{4}$$

$$\therefore\ D = \frac{2R}{\theta}.$$

Using this, the above approximations can be easily put as

$$\sin(x+\theta) = \sin x + \frac{\theta}{R}\cdot\cos x - \frac{\theta^2}{2R^2}\cdot\sin x$$

and

$$\cos (x+\theta) = \cos x - \frac{\theta}{R} \cdot \sin x - \frac{\theta^2}{2R^2} \cos x$$

which are the particular cases of the well-known Taylor series,

$$f(x+\theta) = f(x) + \theta f'(x) + \frac{\theta^2}{2!} f''(x) + \dots ,$$

for sine and cosine respectively up to second power of small quantity (in radians when using the Taylor series).

Shukla[21] interprets the text to yield the rule in the following mathematical form:

$$\sin (x+\theta) = \sin x + \frac{\theta}{2R} \{\cos x + \cos (x+h)\}.$$

But this interpretation does not seem to conform to the text closely. Our interpretation is justified by the commentator Śaṅkara Vāriar[22] (A.D. 1556) as well as by the exposition of the text in *Yukti-Bhāṣā*,[23] a work attributed to Jyeṣṭhadeva (*c*. 1475–1575) by K. V. Sarma.[24]

7. Forms of Various Rules found in the Works of Parameśvara

In Parameśvara's commentary (A.D. 1408)[25] on *Laghubhāskarīya* is found a second order interpolation rule described in the following text:

गतैष्य चापांशकयोः संवर्गेण समाहतम् ।
पूर्वापरोत्थ खण्ड ज्या विवरस्य दलं हरेत् ।।
चाप वर्गेण तत्राप्तमिष्ट ज्यासु विनिक्षिपेत् ।
यत्राधिका परा खण्ड जीवा तत्र तु विशोधयेत् ।।

(Parameśvara's commentary on *Laghubhāskarīya*)[26]

'By the product of the two parts of the elemental arc (made by any intermediary point on it) multiply half the differences of the last and the current sine differences and divide by the square of the elemental arc. By the quotient (so obtained) increase or decrease the sine desired for correction accordingly as the current sine difference is less or greater than the last sine difference.'

That is

$$E = \pm \frac{\theta(h-\theta)}{h^2} \cdot \tfrac{1}{2}(D_p \sim D_{p+1}).$$

Thus we see that this formula is same as that of Govindasvāmi for the argumental interval from 30 degrees to 60 degrees. However, unlike Govindasvāmi, Parameśvara recommends the use of this single rule for the whole of the first quadrant.

Exactly the same rule but described in different words is found quoted in Parameśvara's supercommentary (called *Siddhāntadīpikā*) on Govindasvāmi's commentary on the *Mahābhāskarīya*.[27] But here Parameśvara accepts that the rule is not his own, for the statement of the rule is found to be preceded by the words

केचिदाहुः or केचिदेवमाहुः

'Thus is said by some others' thereby clearly ascribing the rule to other persons.

Finally in the *Siddhāntadīpikā* mentioned above are also found the rules, again ascribed to others, for computing sines and cosines using central values but ultimately amounting to the use of Taylor series approximations up to the second order. The text quoted is

चाप खण्डस्य मध्योत्था या कोटिज्या तया हतात् ।*
चाप खण्डात् त्रिज्याप्तं तत्खण्डे दोर्गुणो भवेत् ।। 7 ।।
चाप खण्डस्य मध्योत्थ भुजज्या निहतात्तथा ।
चाप खण्डात् त्रिज्याप्तं तत्खण्डे कोटिका भवेत् ।। 8 ।।
चाप खण्डार्ध संभूत दोः कोटयोर्विधिरुच्यते ।
दोर्ज्यां चापान्तजां चाप खण्डार्धेन समाहताम् ।। 9 ।।
व्यासार्धेन विभज्याप्तं यत् स्यात्तेन विवर्जिता ।
चापान्तोत्था कोटिजीवा चाप खण्डार्धजा भवेत् ।। 10 ।।
तया कोटया हताच्चापखण्डार्धात् त्रिज्यया हृतम् ।
यत् स्यात्तेन संयुक्ता दोर्ज्या चापान्त संभवा ।। 11 ।।
चाप खण्डार्धजा सा स्याद्, भूयोऽप्येवं गुण द्वयम् ।
साध्यं परस्परं चाप खण्डोत्थ जीवया ।। 12 ।।
अविशिष्टं तु तद् द्वन्द्वं ज्या खण्डाप्तौ स्फुटं भवेत् ।
.. .. ।। 13 ।।

(*Siddāntadīpikā* of Parameśvara)[28]

'7. Multiply the cosine at the middle of the residual arc by the residual arc and divide by the radius. That becomes the sine difference for the residual arc.

8. Multiply the sine at the middle of the residual arc by the residual arc and divide by the radius. That becomes the cosine difference for the residual arc.

9. (Now) is described the method of finding the sine and cosine at the middle of the residual arc (needed above).
Multiply the sine of the arc passed over by half the residual arc,

10. and divide by the radius. By the result, so obtained, subtract the cosine of the arc passed over. This gives the cosine at the middle of the residual arc.

11. Multiply the cosine of the arc passed over by half the residual arc and divide by the radius. The result, so obtained, is to be added to the sine of the arc passed over.

12. That becomes the sine at the middle of the residual arc. Thus compute the sine and cosine differences (of the residual arc) mutually from the cosine and sine at the middle of the residual arc.

13. Combine the sine and cosine of the arc passed over respectively with the sine and cosine differences of the residual arc (got above), we get the true desired sine and cosine for any arc . . .'

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* We have changed the printed 'hṛtāt' to 'hatāt' for an obvious reason.

In symbols we can write the rules as follows:

$$\text{(i) } R\sin(x+\theta)-R\sin x = R\cos\left(x+\frac{\theta}{2}\right)\cdot\frac{\theta}{R}$$

$$\text{(ii) } R\cos(x+\theta)\sim R\cos x = R\sin\left(x+\frac{\theta}{2}\right)\cdot\frac{\theta}{R}$$

$$\text{(iii) } R\cos(x+\tfrac{1}{2}\theta) = R\cos x - R\sin x\,.\,\frac{\theta}{2R},$$

$$\text{(iv) } R\sin(x+\tfrac{1}{2}\theta) = R\sin x + R\cos x\,.\,\frac{\theta}{2R}.$$

Combining (i) with (iii) and (ii) with (iv) we get

$$R\sin(x+\theta) = R\sin x + \frac{\theta}{R}\,.\,R\cos x - \frac{\theta^2}{2R^2}\,.\,R\sin x$$

$$R\cos(x+\theta) = R\cos x - \frac{\theta}{R}\,.\,R\sin x - \frac{\theta^2}{2R^2}\,.\,R\cos x$$

which are the modern Taylor series approximations up to the second order and which are already ascribed to Mādhava by Nīlakaṇṭha as has been already pointed out. Also, as noted above, Parameśvara himself attributes the rules to others although he does not mention any specific names in this connection. But since Parameśvara (c. A.D. 1360–1460) is stated[29] to have studied under Mādhava (c. A.D. 1350–1410) in his young age, it is very likely that Mādhava was the source for Parameśvara for these rules involving values at the centre of the residual arc.

7. Concluding Remarks

The differences of the first order finite differences, that is the second order differences, were used as early as the fifth century A.D. by the Indian astronomer Āryabhaṭa I. His *Āryabhaṭīya* (II, 12)[30] contains a rule equivalent to the relation

$$\Delta^2 \sin x = -k \sin x$$

which was apparently used in finding tabular sine differences. But the use of the second order differences for interpolating intervening functional values appears in India in the early part of the seventh century in the works of Brahmagupta. In modern language Brahmagupta's technique of interpolating the functional value between a pair of tabular entries amounts to passing a parabola through the two functional values at the end points of the interval and the next preceding tabular value. It is stated[31] that Al-Bīrūnī (eleventh century A.D.) in his work *Canon Masudicus* employs a parabola through the same end points. The scheme found in the work Zīj-i-Khāqānī of Al-Kāshī (d. A.D. 1429) using second order differences is about interpolating planet's longitudinal speed denoted by the Persian-Arabic word '*buht*' which, as pointed out by Kennedy,[32] is from the Sanskrit word '*bhukti*'. It is not difficult to understand the Indian influence in general when we remember that both the important works of Brahmagupta, viz. *Brāhmasphuṭa Siddhānta* and *Khaṇḍakhādyaka*, were translated into Arabic at Baghdad under the titles '*Sind-Hind*' and '*Al-arkand*' as early as the eighth century of our era.[33]

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References and Notes

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3 *Brāhmasphuṭa Siddhānta* edited by R. S. Sharma and his team (in 4 vols.). Institute of Astronomical and Sanskrit Research. New Delhi, 1966, Vol. I, p. 320. *Also see* Diksit, S. B., *Bhārtīya Jyotiṣa, op. cit.* (Ref. 2), p. 301.

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