BHĀSKARA I'S APPROXIMATION TO SINE

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The *Mahābhāskarīya* of Bhāskara I (*c.* A.D. 600) contains a simple but elegant algebraic formula for approximating the trigonometric sine function. It may be expressed as

$$\sin a = \frac{4a(180-a)}{40500-a(180-a)},$$

where the angular arc a is in degrees.

Equivalent forms of the formula have been given by almost all subsequent Indian astronomers and Mathematicians. To illustrate this, relevant passages from the works of Brahmagupta (A.D. 628), Vaṭeśvara (A.D. 904), Śrīpati (A.D. 1039), Bhāskara II (twelfth century), Nārāyaṇa (A.D. 1356) and Gaṇeśa (A.D. 1520) are quoted.

Accuracy of the rule is discussed and comparison with the actual values of sine is made and also depicted in a diagram.

In addition to the two proofs given earlier by M. G. Inamdar (*The Mathematics Student*, Vol. XVIII, 1950, p. 10) and K. S. Shukla, three more derivations are included by the present author.

We are not aware of the process by which Bhāskara I himself arrived at the formula which reflects a high standard of practical Mathematics in India as early as seventh century A.D.

Introduction

Indians were the first to use the trigonometric sine-function represented by half the chord of any arc of a circle. Hipparchus (second century B.C.) who has been called the 'Father of Trigonometry' dealt only with chords and not the half-chords as done by Hindus. So also Ptolemy (second century A.D.), who is much indebted to Hipparchus and has summarized all important features of Greek Trigonometry in his famous *Almagest*, used chords only. The history[1] of the word 'sine' will itself tell the story as to how the Indian Trigonometric functions sine, cosine and inversesine were introduced into the Western World through the Arabs.

Among the Indians, the usual method of finding the sine of any arc was as follows: First a table of sine-chords (i.e. half-chords), or their differences, was prepared on the basis of some rough rule, apparently derived geometrically, such as given in Āryabhaṭīya (A.D. 499) or by using elementary trigonometric identities, as seems to be done by Varāhmihir (early sixth century). To avoid fractions the *Sinus Totus* was taken large and the figures were rounded off. Most of the tables prepared gave such values of 24 sine-chords in the first quadrant at an equal interval of $3\frac{3}{4}$ degrees assuming the whole circumference to be represented by 360 degrees. For getting sine corresponding to any other arc simple forms of interpolation were used. In Bhāskara I we come across an entirely different method for computing sine of any arc approximately. He gave a simple but elegant algebraic formula with the help of which any sine can be calculated directly and with a fair degree of accuracy.

The Rule

The rule stating the approximate expression for the trigonometric sine function is given by Bhāskara I in his first work now called *Mahābhāskarīya*. The relevant Sanskrit text is:—

मख्यादि रहितं कर्म वक्ष्यते तत्समासतः ।
चक्रार्धांशक समूहाद्विशोध्या ये भुजांशका ।। १७ ।।
तत्छेष गुणिता द्विष्ठाः शोध्याः खाभ्रेषुखाब्धितः ।
चतुर्थांशेन शेषस्य द्विष्ठमन्त्य फलं हतम् ।। १८ ।।
बाहु कोट्योः फलं कृत्सनं क्रमोत्क्रम गुणस्य वा ।
लभ्यते चन्द्रतीक्ष्णांश्वोस्ताराणां वापि तत्वतः ।। १९ ।।

(*Mahābhāskarīya*, VII, 17–19)[2]

Dr. Shukla[3] translates the text as follows:

'17–19. (Now) I briefly state the rule (for finding the *bhujaphala* and the *koṭiphala*, etc.) without making use of the Rsine-differences, 225, etc. Subtract the degrees of the *bhuja* (or *koṭi*) from the degrees of half a circle (i.e. 180 degrees). Then multiply the remainder by degrees of the *bhuja* (or *koṭi*) and put down the result at two places. At one place subtract the result from 40500. By one-fourth of the remainder (thus obtained) divide the result at the other place as multiplied by the *antyaphala* (i.e. the epicyclic radius). Thus is obtained the entire *bāhuphala* (or *koṭiphala*) for the sun, moon or the star-planets. So also are obtained the direct and inverse Rsines.'

In current mathematical symbols the rule implied can be put as

$$R \sin \phi = R\, \phi(180-\phi)/\tfrac{1}{4}\{40500-\phi(180-\phi)\} \quad \ldots \quad \ldots \quad (1)$$

i.e.

$$\sin \phi = \frac{4\phi(180-\phi)}{40500-\phi(180-\phi)} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where ϕ is in degrees.

Equivalent Forms of the Rule from Subsequent Works

Many subsequent authors who dealt with the subject of finding sine without using tabular sines have given the rule more or less equivalent to that of Bhāskara I, who seems to be the first to give such rule. Below we give few instances of the same.

(i) *Brāhmasphuṭa Siddhānta*

The fourteenth chapter has the couplets:

भुजकोट् यंशोन गुणा भार्धांशास्तच्चतुर्थ भागोनैः ।
पञ्चद्वीन्दु खचन्द्रैर्विभाजिता व्यासदल गुणिता ।। २३ ।।
तज्ज्ये परमफलज्या संगुणितास्तत्फले विना ज्याभिः ।
इष्टोच्चनीच वृत्त व्यासार्धं परमफल जीवा ।। २४ ।।

(*Brāhma Sphuṭa Siddhānta*, XIV, 23-24)[4]

'Multiply the degrees of the *bhuja* or *koṭi* by degrees of half a circle diminished by the same. (The product so obtained) be divided by 10125 lessened by the fourth part of that same product. The whole multiplied by the semi-diameter gives the sine . . .'

i.e.

$$R \sin \phi = \frac{R\phi(180-\phi)}{10125-\frac{1}{4}\phi(180-\phi)}$$

which is equivalent to Bhāskara's rule.

The *Brāhmasphuṭa Siddhānta* was composed by Brahmagupta in the year A.D. 628 according to what the author himself says in the work at XXIV, 7-8.[5]

Dr. Shukla[6] points out that Bhāskara I's commentary on the Āryabhaṭīya was written in A.D. 629 and his *Mahābhāskarīya* was written earlier than this date. Thus Bhāskara I seems to be a senior contemporary of Brahmagupta. Kuppanna Sastri[7] even asserts that the statements of Pṛthudaka Svāmī (A.D. 860), the commentator of *Brāhma Sphuṭa Siddhānta,* imply that Bhāskara I's works must have been known to Brahmagupta.

(ii) *Vaṭeśvara Siddhānta*

In the fourth *adhyāya* of the *Spaṣṭādhikāra* of *Vaṭeśvara Siddhānta* the rule occurs in two forms as follows:

चक्रार्धांशा भुजांशैर्विरहित निहतास्तद्विहीनैर्विभक्ताः
खव्योमेष्वभ्रवेदैः सलिल निहताः पिंडराशिः प्रदिष्टः ।
षड्भांशघ्ना भुजांशा निजकृति रहितास्तत्तुरीयांशहीनैर्भक्ताः
स्यात्पिंड राशिर्विशख नयन भूव्योम शीतांशुभिर्वा ।।

(*Vaṭeśvara Siddhānta, Spaṣṭādhikāra*, IV, 2)[8]

'Multiply degrees of half the circle less the degrees of *bhuja* (by the degrees of *bhuja*). Divide (the product so obtained) by 40500 less that product. Multiplied by four is obtained the required sine. *Or* the *bhuja* in degrees be multiplied by 180 degrees and (the result) be lessened by its own square. Fourth part of the quantity (so obtained) be subtracted from 10125, and by this (new) result the first quantity be divided. The sine is obtained.' Thus we have the two forms as

$$\sin \phi = \frac{\phi(180-\phi)\times 4}{40500-\phi(180-\phi)}$$

and

$$\sin \phi = \frac{\phi \, . \, 180 - \phi^2}{10125 - \frac{1}{4}(\phi \, . \, 180 - \phi^2)}$$

Both being equivalent to Bhāskara I's rule.

According to a verse[9] of the work itself, the *Vaṭeśvara Siddhānta* was composed in A.D. 904.

(iii) *Siddhānta Śekhara*

In this astronomical treatise the rule is given as:

दोः कोटिभाग रहिताभिहताः खनाग-
चन्द्रास्तदीय चरणोन शरार्कदिग्भिः ।
ते व्यास खण्ड गुणिता विहृताः फलेतु
ज्याभिविनैव भवतो भुजकोटि जीवे ॥ १७ ॥

(*Siddhānta Śekhara* III, 17)[10]

'The degrees of *doh* or *koṭi* multiplied to 180 less degrees of *doh* or *koṭi*. Semi-diameter times the product (so obtained) divided by 10125 less fourth part of that product becomes the *bhuja* or *koṭiphala*'

i.e.

$$R \sin \phi = \frac{R\phi(180 - \phi)}{10125 - \frac{1}{4}\phi(180 - \phi)}$$

This form is exactly the same as that of Brahmagupta. Sengupta[11] gives A.D. 1039 as the date of *Siddhānta Śekhara* which was composed by Śrīpati.

(iv) *Līlāvatī*

The concerned stanza is:

चापोननिघ्न परिधिः प्रथमाह्वयः स्यात्
पञ्चाहतः परिधिवर्ग चतुर्थभागः ।
अद्योनितेन खलु तेन भजेच्चतुर्घ्न-
व्यासाहतं प्रथममाप्तमिह ज्याका स्यात् ॥

(*Līlāvatī*, *Kṣetravyavahāra*, *śloka* No. 48)

'Circumference less (a given) arc multiplied by that arc is *prathama*. Multiply square of the circumference by five and take its fourth part. By the quantity so obtained, but lessened by *prathama*, divide the *prathama* multiplied by four times the diameter. The result will be chord (i.e. *pūrṇa jyā* or double-sine) of the (given) arc'

i.e.

$$(360 - 2\phi) \, . \, 2\phi = \textit{prathama}$$

$$2R \sin \phi = \frac{4 \times 2R \times (\textit{prathama})}{\frac{1}{4} \times 5 \times 360^2 - (\textit{prathama})}$$

$$= \frac{8R(360 - 2\phi) \, . \, 2\phi}{\frac{1}{4} \times 5 \times 360^2 - (360 - 2\phi) \, . \, 2\phi}$$

giving

$$\sin \phi = \frac{4\phi(180-\phi)}{40500-\phi(180-\phi)}$$

which is mathematically equivalent to the rule of Bhāskara I. *Līlāvatī* was composed by Bhāskara II in the first half of the twelfth century.

It should be noted that Bhāskara II himself accepted the rule to be very approximate. He says in his *Jyotpatti*[12] (an appendix to *Golādhyāya*):

स्थूलं ज्यानयनं पाट्यामिह तन्नोदितंमया

'The rough (crude) rule for finding sine given in my arithmetic (*Līlāvatī*) is not discussed here.'

(v) *Gaṇita Kaumudī*

As in case of *Vaṭeśvara Siddhānta*, two forms of the rule occur here in the chapter called *Kṣetravyavahāra* as follows:

वृत्त्यर्धं धनुरुनितं स्वगुणितं तेनोनयुक्ते क्रमाद्
वृत्त्यर्धं च वृतिश्चते स्वगुणते तौ गुण्यहाराह्वयौ ।
व्यासे गुण्यहते हराङ्घ्रि विहृते ज्यास्यादथाद्य ज्यया-
ऽऽसन्ना ज्या रहिता ग्रहाख्य गणितेस्युर्व्यास खण्डानि च ।।
अथवा
वृत्तेधनूरहित निघ्न वृतिर्द्विधा तां
व्यासाहतां च विभजेदितराङ्घ्रिहीनैः ।
वृत्त्यङ्घ्रि वर्ग गुणितै विषयैश्च जीवा
स्यात् खेचराख्य गणितेऽप्युपयोग एषः ।।

(*Gaṇita Kaumudī*, *Kṣetravyavahāra*, 69-70)[13]

'Multiply to itself half the circumference less (a given) arc. The quantity so obtained when respectively subtracted from and added to the squares of half the circumference and circumference respectively gives the Numerator and Denominator. Multiply the diameter by the Numerator and divide by one-fourth of the Denominator to get the chord . . .'
Alternately,
'Multiply the circumference less the given arc by circumference and put the result in two places. In one place multiply it by the diameter and divide the result by five times the square of the quarter circumference less the quarter of the result in the other place. The final result is chord . . .'

In symbols these can be expressed as follows. Let c stand for circumference.
First form:

$$\text{Numerator, } N = (\tfrac{1}{2}c)^2-(\tfrac{1}{2}c-2\phi)^2$$

$$\text{Denominator, } D = c^2+(\tfrac{1}{2}c-2\phi)^2$$

then

$$\text{Chord} = \frac{N \times 2R}{\frac{1}{4} . D}$$

i.e.

$$2R \sin \phi = \frac{2R\{180^2 - (180 - 2\phi)^2\}}{\frac{1}{4}\{360^2 + (180 - 2\phi)^2\}}$$

Second form:

$$\text{Chord} = \frac{(c - 2\phi)2\phi \,.\, 2R}{5(\frac{1}{4}c)^2 - \frac{1}{4} \,.\, 2\phi(c - 2\phi)}$$

or

$$2R \sin \phi = \frac{2R(360 - 2\phi)2\phi}{40500 - \phi(180 - \phi)} .$$

On simplification both the forms reduce to the rule of Bhāskara I.

It is stated by Padmākara Dvivedī[14] that *Gaṇita Kaumudī* was composed by Nārāyaṇa Paṇḍita in A.D. 1356. *See* colophonic verse No. 5 of the work in the edition referred.

(vi) *Grahalāghava*

In this work the rule is given in various modified forms adopted for particular cases. The relevant text from *Ravicandrae-spaṣṭādhikāra* for one such case is:

विधो: केन्द्र दोर्भाग षष्ठोन निघ्ना:
खरामा: पृथक् तन्नखांशोनितैश्च ।
रसाक्षैर्हृतास्ते लवाद्यं फलं स्याद्रवीन्दू
स्फुटौ संस्कृतौ स्तश्च ताभ्याम् ।। ३ ।।

(*Grahalāghava*, II, 3)[15]

'Subtract the sixth part of the degrees of the *bhuja* of the moon from 30 and multiply the result by the same sixth part. Put the product in two places. By 56 minus the twentieth part of the product in one place divide the product of the other place. The result is the *mandaphala*.'

$$R \sin \phi = \frac{\left(30 - \frac{\phi}{6}\right) \times \frac{\phi}{6}}{56 - \frac{1}{20}\left(30 - \frac{\phi}{6}\right)\frac{\phi}{6}}$$

or

$$\sin \phi = \frac{20\phi(180 - \phi)}{R\{40320 - \phi(180 - \phi)\}}$$

$$= \frac{4 \,.\, \phi(180 - \phi)}{40320 - \phi(180 - \phi)}$$

since the value of the maximum *mandaphala* for moon, i.e. the value of R for moon, is 5 degrees approximately.[16]

Thus we see that the form of the rule given here is slightly modified. In place of the figure 40500 of Bhāskara I we have 40320 here.

Another modified form is given in the stanza preceding the one we have quoted above.

Grahalāghava was written by Gaṇeśa Daivajña in the year 1520. He dealt with the whole of astronomy contained in the work without using *Jya-gaṇita* (i.e. sine-chords). For sine, whenever needed, he used the rule equivalent to that of Bhāskara I after duly modifying it and rounding off the fractions.

Discussion of the Rule

The rule of Bhāskara I, mathematically expressed by relation (2), is a representation of the transcendental function sin ϕ, by means of a rational function, i.e. the quotient of two polynomials. This algebraic approximation is not only simple but surprisingly accurate remembering that it was given more than thirteen hundred years ago. The relative accuracy of the formula can be easily judged from Table I which gives the comparison of the values of sines obtained by using Bhāskara's rule with the correct values. Calculations have been made by using Castle's Five Figure Logarithmic and Other Tables (1959 ed.) from which the actual values of sin ϕ are also taken.

Table I

ϕ	sin ϕ by Bhāskara's rule	Actual value of sin ϕ
0	0·00000	0·00000
10	0·17525	0·17365
20	0·34317	0·34202
30	0·50000	0·50000
40	0·64183	0·64279
50	0·76471	0·76604
60	0·86486	0·86603
70	0·93903	0·93969
80	0·98461	0·98481
90	1·00000	1·00000

A glance at Table I will show that Bhāskara's approximation effects the third place of decimal by one or two digits. The maximum deviation in the table occurs at 10 degrees and is

$$= +\ 0{\cdot}00160$$

The corresponding percentage error can be seen to be less than one per cent.

Fig. 1 shows how nicely the curve represented by Bhāskara's formula approximates the actual sine curve. The deviations have been exaggerated so that the two curves may be clearly distinguished otherwise they agree so well that, on the size of the scale used, it was not practically possible to draw them distinctly.

The proper range of validity of the rule is from $\phi = 0$ to $\phi = 180$. It cannot be used directly for getting sine between 180 and 360. However with slight modification[17] it can yield $\sin \phi$ for any value of ϕ.

The last word *tattvatah* ('truly' or 'really') of the text, quoted for Bhāskara I's rule, indicates that the rule is to be taken as 'accurate'. Dr. Singh[18] used the word 'grossly' in his translation of the passage. As already pointed out earlier (*see* under (iv) *Līlāvatī*), Bhāskara II clearly stated his equivalent rule as *sthūla* ('rough' or 'gross'). But whether Bhāskara I also meant his rule to be so is doubtful.

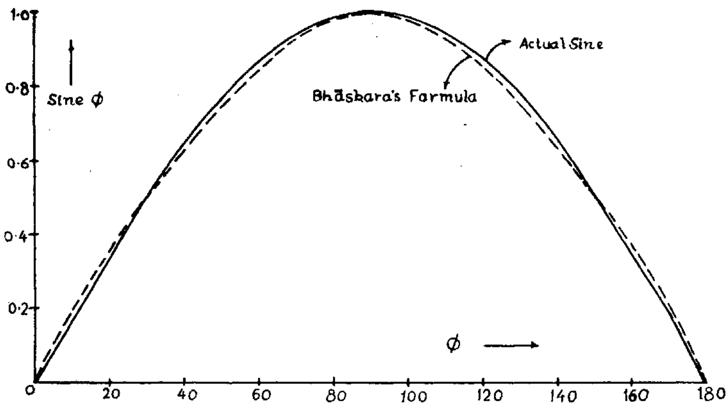

FIG. 1

DERIVATION OF THE FORMULA

The procedure through which Bhāskara I arrived at the rule is not given in *Mahābhāskarīya* which contains the rule. But this seems to be the general feature in case of most of the results in ancient Indian mathematics. This may be partly due to the fact that these mathematical rules are found mostly in works which do not deal exclusively with mathematics as they are treatises on astronomy and not on mathematics. Below we give few methods of arriving at the approximate formula.

(i) FIRST APPROACH *

Following Shukla,[19] let CA be the diameter of a circle of radius R, where the arc AB is equal to ϕ degrees, and

$$BD = R \sin \phi$$

* *See* M. G. Inamdar's paper in *The Mathematics Student*, vol. XVIII, 1950.

Now

$$\text{Area of the } \triangle ABC = \tfrac{1}{2}\, AB.BC$$

$$\text{and also} \qquad = \tfrac{1}{2}\, AC.BD$$

Therefore

$$\frac{1}{BD} = \frac{AC}{AB.BC}$$

so that

$$\frac{1}{BD} > \frac{AC}{(\text{arc } AB).(\text{arc } BC)} \quad \ldots \quad (3)$$

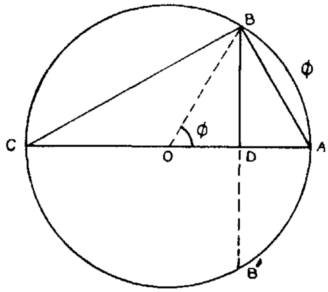

FIG. 2

After this Shukla assumes

$$\frac{1}{BD} = \frac{X.AC}{(\text{arc } AB).(\text{arc } BC)} + Y \quad \ldots \quad (4)$$

$$= \frac{2XR}{\phi(180-\phi)} + Y$$

so that

$$R \sin\phi = \frac{\phi(180-\phi)}{2XR + Y\phi(180-\phi)} \quad \ldots \quad (5)$$

By taking two particular values, viz. 30 and 90, for ϕ we get two equations in X and Y from (5). Solving these it is easily seen that

$$Y = -\frac{1}{4R}$$

and

$$2XR = \frac{40500}{4R}$$

Putting these in (5), Bhāskara's formula is readily obtained.

If X is greater than one and Y is positive the assumption (4) is justified by virtue of the inequality (3). But as noted above Y comes out to be negative. Hence some more investigation to justify (4) is needed which I give below.

Let a, b, p and q be four positive quantities all different from each other. If

$$a > b$$

then we can write

$$a = pb - q. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

that is

$$b = \frac{a+q}{p}$$

provided

$$a > \frac{a+q}{p}, \text{ since } a > b$$

That is to say we can write (6) if

$$p > \frac{a+q}{a}$$

that is

$$p > 1 + \frac{q}{a} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

In the above case

$$a = \frac{1}{R \sin \phi}$$

$$b = \frac{AC}{(\text{arc } AB) \,.\, (\text{arc } BC)}$$

$$p = X = \frac{40500}{8R^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

and

$$q = -Y = \frac{1}{4R}$$

therefore

$$1 + \frac{q}{a} = 1 + \frac{R \sin \phi}{4R}$$

Now greatest value of this

$$= 1 + \frac{1}{4} = \frac{5}{4}$$

Also, since

$$R = \frac{360}{2\pi},$$

from (8) we get

$$p = \frac{405 \,.\, \pi^2}{2592} > \frac{5}{4}$$

So that the condition (7) is satisfied and hence Shukla's assumption (4) is justified.

(ii) Second Approach

Let[20]

$$\sin \lambda = \frac{a + b\lambda + c\lambda^2}{A + B\lambda + C\lambda^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

where λ is in radians and corresponds to ϕ degrees.

Out of the six unknown coefficients a, b, c, A, B, C, only five are independent. These can be found by using the following five mathematical facts:

$$\lambda = 0, \quad \sin \lambda = 0$$

$$\lambda = \pi, \quad \sin \lambda = 0$$

$$\sin \lambda = \sin (\pi - \lambda)$$

$$\lambda = \tfrac{1}{6}\pi, \quad \sin \lambda = \tfrac{1}{2}$$

$$\lambda = \tfrac{1}{2}\pi. \quad \sin \lambda = 1.$$

Utilizing these five conditions we easily arrive at Bhāskara's formula. Perhaps the justification for supposition (9) is that the very form of Bhāskara's rule is of type (9).

(iii) THIRD APPROACH

Shifting the origin through 90 degrees by putting

$$\phi = 90 + X$$

(2) becomes

$$\cos X = \frac{4(90+X)(90-X)}{40500-(90+X)(90-X)}$$

or

$$\cos X = \frac{4(8100-X^2)}{32400+X^2} \quad \ldots \quad (10)$$

By elementary empirical arguments we shall derive formula (10) which resembles the first form given by Nārāyaṇa Paṇḍita (*vide* under (v) *Gaṇita Kaumudī*).

Since the sine function, in the interval 0 to 180, is symmetrical about $\phi = 90$, it follows that the cosine function will be symmetrical about $X = 0$ in the interval -90 to $+90$.

In other words $\cos X$ is an even function of X, say

$$\cos X = f(X^2)$$

Now $\cos X$ decreases as X increases from 0 to 90. The simplest way of effecting this is to take

$$\cos X \propto \frac{1}{X^2}$$

But $\cos X$ also remains finite at $X = 0$ hence we should assume, instead of above,

$$\cos X \propto \frac{1}{X^2+a}, \text{ where } a \neq 0$$

Finally, remembering that $\cos X$ vanishes for finite values of X, we take

$$\cos X = \frac{C}{X^2+a} - k$$

k being positive.

It should be noted that the possibility of taking simply

$$\cos X = \frac{C}{X^2+a}$$

is ruled out since by this assumption we cannot make cos X to vanish for any finite value of X while we know cos 90 = 0.

The three unknowns a, c and k can be found by taking particular known simple rational values, e.g.

$$\cos 0 = 1$$
$$\cos 60 = \tfrac{1}{2}$$
$$\cos 90 = 0$$

Utilizing these we get (10) without difficulty.

(iv) Fourth Approach

Now we take the method of approximation by continued fractions. The general formula which we shall use is[21]

$$f(X) = a_0 + \frac{X-X_0}{a_1+}\;\frac{X-X_1}{a_2+}\;\frac{X-X_2}{a_3+}\quad \ldots \quad \ldots \quad \ldots \quad (11)$$

where

$$a_k = \phi_k[X_0, X_1, X_2 \ldots X_{k-1}, X_k]$$

The quantities ϕ_k's are called the 'inverted differences' and are defined as follows:

$$\phi_0[X] = f(X)$$

$$\phi_1[X_0, X] = \frac{X-X_0}{\phi_0[X]-\phi_0[X_0]} = \frac{X-X_0}{f(X)-f(X_0)}$$

$$\phi_2[X_0, X_1, X] = \frac{X-X_1}{\phi_1[X_0, X]-\phi_1[X_0, X_1]}$$

and so on.

Now we form the table of 'inverted differences' for the function $f(X) = \sin X$ by taking some simple rational values of the sines (*see* Table II).

Table II

X in degrees	$f(X) = \phi_0 = \sin X$	ϕ_1	ϕ_2	ϕ_3	ϕ_4
$X_0 = 0$	$0 = a_0$	—	—	—	—
$X_1 = 30$	$\tfrac{1}{2}$	$60 = a_1$	—	—	—
$X_2 = 90$	1	90	$2 = a_2$	—	—
$X_3 = 150$	$\tfrac{1}{2}$	300	$\tfrac{1}{2}$	$-40 = a_3$	—
$X_4 = 180$	0	∞	0	-45	$-6 = a_4$

Using (11) the successive convergents are:

First, $= a_0 = 0$

Second, $= a_0 + \frac{X-X_0}{a_1} = \frac{X}{60}$

Third, $= a_0 + \frac{X-X_0}{a_1+} \; \frac{X-X_1}{a_2}$

$$= \frac{2X}{X+90}$$

Fourth, $= a_0 + \frac{X-X_0}{a_1+} \; \frac{X-X_1}{a_2+} \; \frac{X-X_2}{a_3}$

$$= \frac{X(170-X)}{9000-20X}$$

Fifth, $= a_0 + \frac{X-X_0}{a_1+} \; \frac{X-X_1}{a_2+} \; \frac{X-X_2}{a_3+} \; \frac{X-X_3}{a_4}$

$$= \frac{4X(180-X)}{40500-X(180-X)}$$

which is the rule of Bhāskara I.

Inverted differences or rather the related quantities called 'reciprocal differences' were introduced by T. N. Thiele[22] in A.D. 1909. We do not know if any ancient Indian mathematician ever used the inverted or reciprocal differences, although Brahmagupta A.D. (665) has been credited for being the first to use 'direct' differences up to second order.[23]

(v) FIFTH APPROACH

It will be noted that the quantity

$$\phi(180-\phi) = p, \text{ say}$$

is an important function of ϕ in connection with sin ϕ. In fact it is a quarter of what Bhāskara II has called *prathama* (*see* under (iv) *Līlāvatī*). Now p can be written as

$$p = 8100-(\phi-90)^2.$$

It follows that the maximum value of p will be 8100 when $\phi = 90$. Also the function p vanishes for $\phi = 0$ and 180 and is symmetrical about $\phi = 90$. Thus the nature of p and sin ϕ resembles in certain points. Since the maximum value of sin ϕ is 1, we may take as a crude approximation

$$\sin\phi = \frac{p}{8100} = P, \text{ say}.$$

However Bhāskara I's formula is far better than the above simplest (linear) but very rough approximation. We now attempt to find a better relation between sin ϕ and P. Since $0 \times 0 = 0$ and $1 \times 1 = 1$, the product function $P \times$ sin ϕ will also have the same nature as P or sin ϕ. The simplest relation between P, sin ϕ and P sin ϕ will be a linear one. Therefore we assume

$$lP \sin \phi + mP + n \sin \phi = 0 \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

which is the general form of a linear relation. Taking $\phi = 90$ and $\phi = 30$ we get respectively

$$l + m + n = 0$$

and

$$\frac{5}{18} l + \frac{5}{9} m + \frac{1}{2} n = 0$$

Solving the above two equations we get

$$l = -\frac{1}{5} n$$

$$m = -\frac{4}{5} n$$

Using these the linear relation (12) becomes, after simplification (i.e. solving for sin ϕ),

$$\sin \phi = \frac{4P}{5-P} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

which is the formula of Bhāskara I in disguise. Form (2) of the rule can be obtained by substitution of the value of P, viz.

$$P = \frac{\phi(180-\phi)}{8100}$$

Relation (13) may be regarded as the simplest form of Bhāskara's rule and may be derived in many ways.

Conclusion

To Bhāskara I (early seventh century A.D.) goes the credit of being the first to give a surprisingly simple algebraic approximation to the trigonometric sine function. In its simplest form his rule may be expressed by the mathematical formula

$$\sin \phi = \frac{4P}{5-P}$$

where P may be called as the 'modified *prathama*' (accepting the definition of Bhāskara II). If ϕ is measured in terms of right angles (quadrants) then P will be given by

$$P = \phi(2-\phi).$$

The formula is fairly accurate for all practical purposes. A better formula, which will have the simplicity and practicability of Bhāskara I's formula, can hardly be given without introducing bigger rational numbers or irrational numbers and higher degree polynomials of ϕ. No such mathematical formula approximating algebraically a transcendental function seems to be given by other nations of antiquity. The formula as such, or its modified form has been used by almost all the subsequent authors, a few instances of which are given in this paper. Remembering that it was given more than thirteen hundred years ago, it reflects a high standard of mathematics prevalent at that time in India. 'How Indians arrived at the rule' may be taken as an open question.

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Note: In this edition few alternative readings of the text are also given but they do not substantially effect the meaning of the passage.

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