BINOMIAL THEOREM IN ANCIENT INDIA

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The binomial theorem for positive integral exponents was discovered in Europe in the sixteenth century. The triangular array formed by the binomial coefficients undoubtedly played a very important role in the development. The array was known as Pascal triangle (+A.D. 1665) in Europe. It appeared originally in the work of Apianus (+1527), Stifel (+1544), Scheubel (+1545), Tartaglia (+1556), Bombelli (+1572) and others. The same array was known in China as the 'Old method chart of seven multiplying squares' and appeared at least two centuries earlier in the work of Chu Shih-Chieh (+1303), Yang Hui (+1261) and Chia Hsien (+1100). The paper, apart from early discovery of the theorem in India, shows that the same triangular array was known as *meru-prastāra* in India and occurs several centuries earlier than that of China.

In the early history of the development of binomial theorem for positive integral exponents the discovery that the binomial coefficients can be arranged in a triangular array has naturally played a very important part. In Europe this triangular array came to be known as Pascal's Triangle from after the posthumous publication of Pascal's *Traité du triangle arithmétique* in 1665. The triangle however appeared in Europe more than a hundred years before the *Traité* on the title page of the arithmetic of Apianus (+A.D. 1527) and in the works of Stifel (+1544), Scheubel (+1545), Tartaglia (+1556), Bombelli (+1572) and others.[1] Similar triangular array appeared in Chu Shih-Chieh's *Ssu Yuan Yü Chien* (+1303). Earlier versions of the triangle in China have been traced to the works of Yang Hui's *Hsiang Chieh Chiu Chang Suan Fa* (+1261) and Chia Hsien's works (+1100).[2] Singh[3] pointed out the existence of an identical triangular array of binomial coefficients, known as *meru-prastāra*, in Piṅgala's *Chandaḥsūtra* (200 B.C.), which received only a passing notice in a footnote in Needham's admirable review[4] with the remark, based on Luckey,[5] that it 'has nothing to do with binomial coefficients.' It is proposed to show that, apart from very early discovery in India of the triangular arrangement of binomial coefficients, the technique really appeared in association with the expansion of a binomial with positive integral exponents, posed by metrical problems.

To appreciate how the binomial problem and the method of determining the binomial coefficients arose from metrical consideration, a few remarks about the peculiarities of metre in the Sanskrit literature may be necessary.

Intended for a musical recital, metres (*chandas*) are closely allied to music. The main varieties of music are: (1) the music of voice-modulation (or the *svara-saṅgîta*); (2) the music of sound variation (or the *varṇa-saṅgîta*); (3) the music of time-regulated accent (or the *tāla-saṅgîta*).

The first depends upon the modulation, i.e. the raising and the lowering of the human voice, so as to produce different tones; the second is produced by a pleasant variation of short and long sounds employed in the composition of metrical line, where a syllable, whether short or long, is considered as a unit for metrical scanning in a prosody and, regardless of its quantity, forms the basis of a metrical line. The third variety is different from the first two; in it the music is produced by varying the voice or sound after the lapse of definite periods measured by time moments.

The early Vedic metres were based mainly on *svara-saṅgîta* where the time element plays no important role. Its chief representatives are the *anuṣṭubh* (containing 8 syllables in a line), the *triṣṭubh* (containing 11 syllables in a line) and the *jagatî* (containing 12 syllables in a line).[6] A *jagatî* line probably developed from *triṣṭubh* by the addition of a single syllable in order to break the monotony of the two long syllables at the end of the *triṣṭubh* through the introduction of a penultimate short. Here perhaps was the beginning of the consciousness of a new type of the musical rhythm which brought into existence the classical *varṇa-saṅgîta* which could be produced by the alteration of short and long syllables. By the end of the Saṃhitā period, the earlier metrical music based on the modulation of voice to different *pitches* and *tunes* was generally replaced by the new kind of music based on the alteration of short and long sounds. The three main Vedic metres, viz. *anuṣṭubh, triṣṭubh* and *jagatî* gradually gave place to 33 or more metres at the hands of the post-Vedic composers.[7] In theory, however, a very large number of different kinds of metres were possible. In actual practice, the poets adopted only a few of each of these three classes. A few types and the number of syllables contained in each are given below:

Name of the metre (Sanskrit)	Number of syllables
Uktā	1
Atyuktā	2
Madhyā	3
Pratiṣṭhā	4
Supratiṣṭhā	5
Gāyatrî	6
Uṣṇik	7
Anuṣṭubh	8
Bṛhatî	9

Name of the metre (Sanskrit)	Number of syllables
Paṅkti	10
Triṣṭubh	11
Jagatî	12
Atijagatî	13
......	..

In course of time this naturally posed the problem as to how many different kinds of *chandas* could be produced from one of 3 syllables, 4 syllables, 5 syllables, etc., by varying the long and short sounds within each syllable group. This was, doubtless, a kind of problem which lent itself to the foundation of algebraical rule to detect the quality as well as the shortcomings of the metres (*chandas*).

Piṅgala (200 B.C.) and possibly other specialists of the metre gave a hint to the computing of this technique. Piṅgala's cryptic statement *ādyantāvupajātayaḥ*[8] has been explained by his commentator, Halāyudha (+tenth century), thus: *ādyantāviti anantaroktau indravajropendravajrayoḥ pādāvāha, tau jadā vikalpena yatheṣṭaṃ bhavatastadopajātayaḥ prastāravacanāt*[9], that is, 'the beginning and ending must be mixed metres (*upajāti*) of *indravajrā* and *upendravajrā* and must be placed one after another. They (*indravajrā* and *upendravajrā*) must be mixed in all ways. These mixed metres are to be produced from the combination of (desired) syllables.'

From Piṅgala's own exposition of *indravajrā* and *upendravajrā* metre,[10] we know that if one considers a metre of 3 syllables, the intermediate metres which consist of *guru* and *laghu* sounds must be mixed. Thus a metre beginning with 3 *guru* sounds can end with 3 *laghu* sounds with mixed sounds in the middle as follows, where a represents *laghu* and b *guru*.

Expansion of the metre of three syllables (i.e., *Madhyā*)	Equivalent results	Explanation
$b\ b\ b$	$= b^3$	Here the number of arrangements with
$a\ b\ b$	$= ab^2$	... 3 gurus $= 1 = 1.b^3$
$b\ a\ b$	$= ab^2$	... 2 gurus $= 3 = 3ab^2$
$a\ a\ b$	$= a^2b$	... 1 guru $= 3 = 3a^2b$
$b\ b\ a$	$= ab^2$	... 0 guru $= 1 = 1.a^3$
$a\ b\ a$	$= a^2b$	The number of syllables is 3 and this is
$b\ a\ a$	$= a^2b$	formed by the variation of sounds
$a\ a\ a$	$= a^3$	a and b. This undoubtedly gives the expansion of $(a+b)^3$.

The *Piṅgala-chandaḥsūtra*[11] and *Vṛttajātisamuccaya*[12] describe clearly a rule of laying down the different *prastāras*. In Piṅgala's rule as explained by Halāyudha, the following steps are indicated:

1. First one should write down the line containing *gurus* only.
2. Then *laghu* should be written below the first guru and the subsequent places with the *gurus*.
3. If there be any space on the left-hand side of the *laghu* these must be filled up with the *gurus* only irrespective of how the corresponding spaces in the above line are filled.
4. This is to be continued till the line containing only *laghus* is arrived at.

The various possibilities of arrangement of *guru* and *laghu*, say in the *Pratiṣṭhā-chanda* containing 4 syllables, may be represented thus:

Arrangements	Equivalent results
$b\ b\ b\ b$	$= b^4$
$a\ b\ b\ b$	$= ab^3$
$b\ a\ b\ b$	$= ab^3$
$a\ a\ b\ b$	$= a^2b^2$
$b\ b\ a\ b$	$= ab^3$
$a\ b\ a\ b$	$= a^2b^2$
$b\ a\ a\ b$	$= a^2b^2$
$a\ a\ a\ b$	$= a^3b$
$b\ b\ b\ a$	$= ab^3$
$a\ b\ b\ a$	$= a^2b^2$
$b\ a\ b\ a$	$= a^2b^2$
$a\ a\ b\ a$	$= a^3b$
$b\ b\ a\ a$	$= a^2b^2$
$a\ b\ a\ a$	$= a^3b$
$b\ a\ a\ a$	$= a^3b$
$a\ a\ a\ a$	$= a^4$

Thus the number of arrangements with four *gurus* $= 1.b^4 = 1$

,,	,,	three	,,	$= 4.b^3a = 4$
,,	,,	two	,,	$= 6.b^2a^2 = 6$
,,	,,	one	,,	$= 4.ba^3 = 4$
,,	,,	no	,,	$= 1.a^4 = 1$

Since the number of syllables in the *Pratiṣṭhā-chanda* is four, the scheme gives the expansion of the binomial $(a+b)^4$ where a and b denote the short and long sounds respectively. The same method has been applied to metres of any syllable.

In addition to the method of computing the binomial terms b^4, b^3a, b^2a^2, etc., in the foregoing example, Piṅgala gave his general *meru-prastāra* rule for determining such binomial coefficients, which is exactly the same as Pascal's 'triangular array', or Chu Shih-Chieh's 'the old method chart of the seven multiplying squares'. The rule has been explained by Halāyudha as follows:[13]

anena ekadvyādilaghukriyāsiddhyartham yāvadabhimatam prathamaprastāravat meruprastāram darśayati, upariṣṭādekam caturasrakoṣṭham likhitvā tasyā'dhastāt ubhayato'rddhaniṣkrāntam koṣṭhakadvayam likhet, tasyāpyadhastāttrayam tasyāpyadhastāccatuṣṭayamevam yāvadabhimatam sthānamiti meruprastāraḥ, tasya prathame koṣṭhe ekasaṃkhyāṃ vyavasthāpya lakṣaṇamidaṃ pravarttayet, tatra dvikoṣṭhāyāṃ paṅktāvubhayoḥ koṣṭhayorekaikamaṅka dadyāt, tatastṛtîyāyāṃ paṅktau paryantakoṣṭhayorekaikamaṅkaṃ dadyāt, madhyamakoṣṭhe tūparikoṣṭhadvayayāṅkamekîkṛtya pūrṇaṃ niveśayediti pūrṇaśabdārthaḥ, caturthyāṃ paṅktāvapi paryantakoṣṭhayorekaikamaṅkaṃ sthāpayet, madhyamakoṣṭhayostūparakoṣṭhadvayāṅkamekîkṛtya pūrṇaṃ trisaṃkhyārūpaṃ sthāpayet, uttaratrāpyevameva nyāsaḥ, tatra dvikoṣṭhāyāṃ paṅktāvekākṣarasya prastāraḥ, ... tṛtîyāyāṃ paṅktau dvyakṣarasya prastāraḥ, caturthyāṃ paṅktau tryakṣarasya prastāraḥ ...

The above may be rendered into English as follows:

'Here the method of pyramidal expansion (*meru-prastāra*) of the (number of) combinations of one, two, etc., syllables formed of short (and long sounds) are explained. After drawing a square on the top, two squares are drawn below (side by side) so that half of each is extended on either side. Below it three squares, below it (again) four squares are drawn and the process is repeated till the desired pyramid is attained. In the (topmost) first square the symbol for one is to be marked. Then in each of the two squares of the second line figure one is to be placed. Then in the third line figure one is to be placed on each of the two extreme squares. In the middle square (of the third line) the sum of the figures in the two squares immediately above is to be placed; this is the meaning of the term *pūrṇa*. In the fourth line one is to be placed in each of the two extreme squares. In each of the two middle squares, the sum of the figures in the two squares immediately above, that is, three, is placed. Subsequent squares are filled in this way. Thus the second line gives the expansion of combinations of (short and long sounds forming) one syllable; the third line the same for two syllables, the fourth line for three syllables, and so on' (Fig. 1).

The identity of the *meru-prastāra* with Pascal's triangle is thus fully established. It may be noted that the last line of Halāyudha quoted above distinctly points out that the second line is the expansion of a metre with one syllable, i.e. of $(a+b)^1$, 3rd line that of a metre with two syllables, that

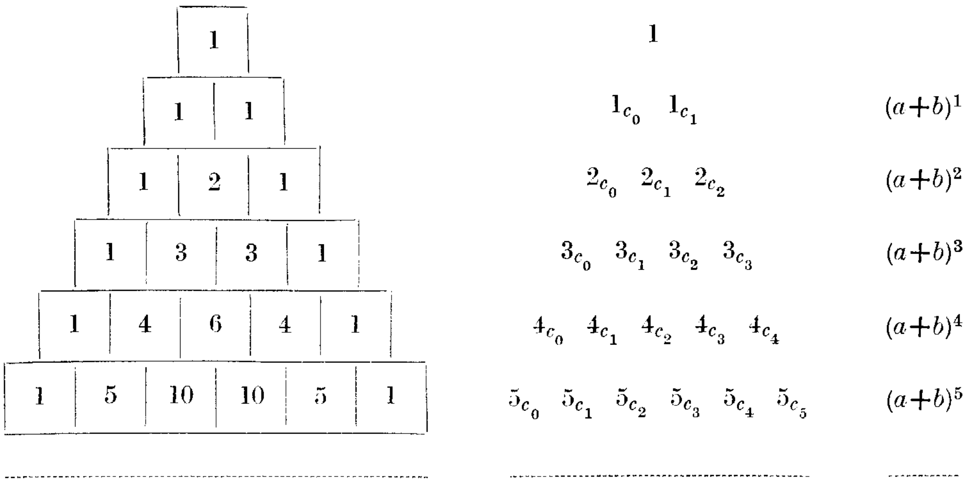

FIG. 1. Diagrammatical representation of *meru-prastāra.*

is, of $(a+b)^2$, the 4th line that of a metre with 3 syllables, that is, of $(a+b)^3$, and so on. In this way the general expansion

$$(a+b)^n = a^n + {}^nc_1a^{n-1}b + {}^nc_2a^{n-2}b^2 + \ldots\ldots + {}^nc_{n-1}ab^{n-1} + b^n$$

was readily obtained for a metre of n syllables. Thus Luckey's comment, '*meru-prastāra* technique concerns only prosodic combination and has nothing to do with the binomial coefficients', is not correct.

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4 Needham, J., Science and Civilization in China, **3**, p. 137.

5 Luckey, P., Ausziehung des n-term wurzel und der binomische Lehrsatz in der islamischen Mathematik', *Mathematischen Annalen*, band 120, p. 219, 1948.

6 *Jayadāman*, A collection of ancient texts on Sanskrit Prosody and classified list of Sanskrit metres with an alphabetical index, ed. by H. D. Velankar, p. 7; H. D. Velankar is a great scholar on metre. For dates of the *Chandaḥsūtras*, the author consulted the work *Jayadāman*.

7 *Ibid.*, p. 15.

8 *Chandaḥsūtra* of Piṅgalācārya, ed. by Sitanath Sarman, Chap. 6, 18.

9 *See* Halāyudha's commentary on *Piṅgala-chandaḥsūtra* on the relevant line, edited by Sitanath Sarman, Calcutta.

10 *Ibid.*, Chap. 6, 16, 17 and 18.

11 *Dvikau glau, miśrau ca, pṛthaglā miśrāḥ.* (*Piṅgala-chandaḥsūtra*, Ch. 8, 20–22.)

Explanation after Halāyudha's commentary:

ga stands for *guru* sound $= b$ (say)

la stands for *laghu* sound $= a$ (say)

$$\text{The expansion of one syllable} = \begin{pmatrix} b \\ a \end{pmatrix} = \begin{matrix} b \\ a \end{matrix}$$

$$\text{Expansion of two syllables} = \left.\begin{matrix} \begin{pmatrix} b \\ a \end{pmatrix} + b \\ \begin{pmatrix} b \\ a \end{pmatrix} + a \end{matrix}\right\} = \begin{matrix} bb \\ ab \\ ba \\ aa \end{matrix}$$

$$\text{Expansion of three syllables} = \left.\begin{matrix} \begin{pmatrix} bb \\ ab \\ ba \\ aa \end{pmatrix} + b \\ \begin{pmatrix} bb \\ ab \\ ba \\ aa \end{pmatrix} + a \end{matrix}\right\} = \begin{matrix} bbb \\ abb \\ bab \\ aab \\ bba \\ aba \\ baa \\ aaa \end{matrix}$$

and so on.

12 See *Vṛttajātisamuccaya* of Vīrahaṅka (*c.* ninth to tenth century A.D.), Ch. 5, Verse 21 (C/O. H. D. Velankar's edition, *Vṛttajātisamuccaya* of Vīrahaṅka, *Journal of the Bombay Branch of the Royal Asiatic Society*, 8 (new series), 1932, p. 3.

13 *Piṅgala-chandaḥsūtra,* Ch. 8, 34 (*see* Halāyudha's Commentary).