Proc Indian Natn Sci Acad **79** No. 4 December 2013 Spl. Issue, Part A, pp. 547-552

DOI: 10.16943/ptinsa/2013/v79i4/47990

Research Paper

View on a Traditional Elastoplasticity Model[1]

ALBERT MEYERS, BOCHUM
Maischützenstr. 1, D-44805 Bochum, GERMANY

(Received 24 February 2013; Accepted 02 March 2013)

In recent years large deformation plasticity based on the additive decomposition of the stretching has become less popular. Constitutive relations of elastoplasticity have come up that are complicated in nature and mostly formulated in Lagrangean frame. Here it is shown that traditional Eulerian elastoplasticity may be a serious alternative. Its structure is simple and the formulation is straightforward.

Key Words : Elastoplasticity; Eulerian Formulation; Stretching Decomposition

1. Introduction

Midth of last century three-dimensional elastoplasticity got based on the additive decomposition of the additive decomposition of the stretching. This "traditional model" seemed to be a good foundation for the description of elastoplastic material behaviour. Since elastoplasticity deals with moderately large deformations, it was favourable that the formulation was in an Eulerian frame.

The disclosure of stress oscillations in simple shear problem [12, 19], however, evidenced that such model needed revision.

In sequence, various propositions were given and the matter started to grow in complexity. Among such proposals were the description in Langrangean frame, the multiplicative decomposition of the deformation gradient, the additive decomposition of the Lagrangean strain, the introduction of multiple yield surfaces or combinations of them. New internal parameters had to find physically motivated evolution laws. Some of them were accompanied by unwanted side effects like incompatibility, abandoning given definition range etc.

The challenge is to formulate constitutive laws that are as simple as possible, as complex as needed and mathematically flawless. In this sense it may be interesting to note that a small change in the original formulation, namely the consequent use of the logarithmic rate as objective rate for the Kirchhoff stress and the backstress, may revalue the traditional model. Then, it may be obsolete to look out for additional evolution laws that, sometimes, are hard to justify.

2. Kinematics of Large Deformations

Basically, large deformations of continua relate the rotation and lengthening of a line element in dx the actual state to its reference state d$\boldsymbol{X}$, i.e.

$$\mathrm{d}x = F(X,t)\mathrm{d}X \quad \text{with} \quad F = \frac{\partial x}{\partial \boldsymbol{X}} \tag{1}$$

The two-field deformation gradient has the determinant (Jacobian)

$$J = \det \boldsymbol{F}. \tag{2}$$

$\boldsymbol{F}$ may be multiplicatevily deomposed as

$$\boldsymbol{F} = \boldsymbol{V}\ \boldsymbol{R} = \boldsymbol{R}\boldsymbol{U},\ \ \boldsymbol{R}^T = \boldsymbol{R}^{-1},\ \ \boldsymbol{V} = \boldsymbol{V}^T,\ \ \boldsymbol{U} = \boldsymbol{U}^T \tag{3}$$

*Author for Correspondence: E-mails: meyers@web.de; meyboe@googlemail.com

[1]*Dedicated to Professor Narinder K. Gupta on his 70th Birthday*

$\boldsymbol{R}$ is the line element rotation tensor. The right stretch $\boldsymbol{U}$ characterises its lengthening in the non-rotated state (Lagrangean configuration), the left stretch $\boldsymbol{V}$ in the actually rotated state (Eulerian configuration). Since elastoplastic deformations may be moderately large, in the following the attention is focussed to the Eulerian configuration.

The Cauchy-Green tensor $\boldsymbol{B}$ is computed from $\boldsymbol{V}$ by

$$\boldsymbol{B} = \boldsymbol{V}^2 = \boldsymbol{F}\,\boldsymbol{F}^T. \tag{4}$$

$\boldsymbol{B}$ has $1 \le m \le 3$ distinct eigenvalues b_i. It shares the eigenprojections $\boldsymbol{B}_i$ with $\boldsymbol{V} \cdot \boldsymbol{B}$ may be decomposed according

$$\boldsymbol{B} = \sum_{i=1}^{m} b_i \boldsymbol{B}_i \tag{5}$$

The eigenprojections may be retrieved by Sylvester formula

$$\boldsymbol{B}_I = \begin{Bmatrix} I, & m=1, \\ \Pi_r^m \left(\Pi_{s=1,s\neq r}^m \dfrac{\boldsymbol{B} - b_s \boldsymbol{I}}{b_r - b_s} \right), & m>1, \end{Bmatrix} \tag{6}$$

where $\boldsymbol{I}$ is the second order identity tensor. The velocity gradient is defined by

$$\boldsymbol{L} = \frac{\partial \dot{\boldsymbol{x}}}{\partial \boldsymbol{x}} = \dot{\boldsymbol{F}}\,\boldsymbol{F}^{-1}. \tag{7}$$

and is an Eulerian tensor. The decomposition holds

$$\boldsymbol{L} = \boldsymbol{D} + \boldsymbol{W}, \begin{Bmatrix} \boldsymbol{D} = (\boldsymbol{L} + \boldsymbol{L}^T)/2, \\ \boldsymbol{W} = (\boldsymbol{L} - \boldsymbol{L}^T)/2. \end{Bmatrix} \tag{8}$$

$\boldsymbol{D}$ and $\boldsymbol{W}$ are known as the Eulerian stretching and the vorticity tensor. $\boldsymbol{D}$ characterises the deformation rate.

The material time derivatives of obective tensors generally are not objective. Numerous objective time derivatives have been proposed in the past. Hill's class of objective rates [9] is defined for $p \in (-\infty, \infty)$

$$\mathring{A}^{(\mathrm{Hill})} = \dot{\boldsymbol{A}} + \boldsymbol{A}\boldsymbol{W} - \boldsymbol{W}\,\boldsymbol{A} - p(\boldsymbol{A}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{A}), \tag{9}$$

where $\boldsymbol{A}$ is an arbitrary objective Eulerian tensor. E.g., this class includes the upper and lower Oldroyd rates [20], i.e.

$$\mathring{A}^{(\mathrm{Ol})} = \dot{\boldsymbol{A}} + \boldsymbol{A}\,\boldsymbol{L} + \boldsymbol{L}^T\,\boldsymbol{A}, \tag{10}$$

$$\mathring{A}^{(\mathrm{Ou})} = \dot{\boldsymbol{A}} - \boldsymbol{A}\,\boldsymbol{L} - \boldsymbol{L}^T\,\boldsymbol{A}, \tag{11}$$

and a number of corotational rates. They have the structure

$$\mathring{A}^{(\mathrm{cor})} = \dot{\boldsymbol{A}} + \boldsymbol{A}\,\Omega - \Omega\,\boldsymbol{A} \tag{12}$$

where Ω is a (skew-symmetric) spin tensor, e.g.

$$\Omega^{\mathrm{J}} = \boldsymbol{W}, \qquad \text{Jaumann [10]} \tag{13}$$

$$\Omega^{\mathrm{GN}} = \boldsymbol{R}\,\boldsymbol{R}^{\mathrm{T}}, \quad \text{Green/Nagdhi [18]} \tag{14}$$

$$\Omega^{\log} = \boldsymbol{W} + \left(\frac{1 + b_r/b_s}{1 - b_r/b_s} + \frac{2}{\ln(b_r/b_s)} \right) \boldsymbol{B}_r\,\boldsymbol{D}\,\boldsymbol{B}_s, \quad \text{logarithmic [13, 23]} \tag{15}$$

The choice of the most appropriate objective time derivatives has been most controversary discussed in the past. Recommendations for the Kichhoff stress and backstress rates are given shortly later.

3. Traditional Model

The specific stress rate is defined by

$$\dot{w} = \tau\,\boldsymbol{D}. \tag{16}$$

Herein, $\boldsymbol{\tau}$ is the Kirchhoff stress. It is related to the Cauchy-true-stress σ by

$$\tau = J\sigma. \tag{17}$$

A decomposition of $\dot{w}$ into recoverable and irrecoverable parts may lead to a decomposition of the stretching as

$$\dot{w} = w^{\mathrm{e}} + w^{\mathrm{p}} = \tau\,(\boldsymbol{D}^{\mathrm{e}} + \boldsymbol{D}^{\mathrm{p}}) \rightarrow \boldsymbol{D} = \boldsymbol{D}^{\mathrm{e}} + \boldsymbol{D}^{\mathrm{p}} \tag{18}$$

It seems that the decomposition of $\boldsymbol{D}$ into a recoverable (elastic) and irrecoverable (plastic) parts has first been proposed by Hill in 1958 [8] and shortly later by Lehmann [11].

Upon attribution of elastic and plastic constitutive laws to $\boldsymbol{D}^{e}$ and $\boldsymbol{D}^{p}$ in (18), this decomposition may be considered as foundation of elastoplastic constitutive relations.

3.1 Considerations Concerning $\boldsymbol{D}^{e}$

Very often, elasticity constitutive laws are formulated as stress-strain relations (Green elasticity) or are derived from a strain potential $W(\varepsilon)$ or, complementary, from a stress potential $\bar{W}(\tau)$ (hyperelasticity), i.e.

$$\tau = \frac{\partial W}{\partial \varepsilon}, \quad \varepsilon = \frac{\partial \bar{W}}{\partial \tau} \tag{19}$$

Both are not directly delivering an expression for $\boldsymbol{D}^{e}$; this may be of importance, see later.

Truesdell [22] proposed hypoelastic laws of the form

$$\overset{o}{\tau} = g(\boldsymbol{D}^{e}). \tag{20}$$

Herein, $\overset{o}{\tau}$ is a not specified objective Kirchhoff stress rate. Complementary, this law is

$$\boldsymbol{D}^{e} = \bar{g}(\overset{o}{\tau}, \tau), \tag{21}$$

which seems to conform to the purposes here. E.g., the hypoelastic law of grade zero is

$$\boldsymbol{D}^{e} = \frac{1}{2\mu}\left(\overset{o}{\tau} - \frac{\nu}{1+\nu}\operatorname{tr}(\overset{o}{\tau})\boldsymbol{I}\right). \tag{22}$$

Herein $\boldsymbol{I}$, μ, ν, and are second order identity tensor, Lamé's compression module and Poisson's ratio, respectively. 1979, Dienes [6] discloses that none of the commonly used objective rates can make relation (21) exactly integrable, i.e. conform to Bernstein's integrability conditions [1]. In [24] it is shown that objective Kirchhoff stress rates in (20), (21) may not be of arbitrary type. A necessary condition for the integrability of the hypoelastic relation is the use of the logarithmic Kirchhoff stress rate $\overset{o\,(\log)}{\tau}$

$$\boldsymbol{D}^{e} = \bar{g}\left(\overset{o\,(\log)}{\tau}, \tau\right). \tag{23}$$

Then, hypoelasticity and hyperelasticity may be brought together. This applies, e.g., for the hypoelastic relation of grade zero

$$\boldsymbol{D}^{e} = \frac{1}{2\mu}\left(\overset{o\,(\log)}{\tau} - \frac{\nu}{1+\nu}\operatorname{tr}(\overset{o\,(\log)}{\tau})\boldsymbol{I}\right). \tag{24}$$

3.2 Considerations Concerning $\boldsymbol{D}^{p}$

The plastic flow starts whenever the elastic domain is left. Traditionally, the delimiting surface in stress space is described by the yield function f

$$f(\tau, \kappa, \alpha) = 0. \tag{25}$$

Herein, κ is the (scalar) isotropic hardening parameter, and α the (tensorial) kinematic hardening parameter. A widely used yield function for metals, e.g., is due to von Mises and is expressed by

$$f = \frac{1}{2}(\tilde{\boldsymbol{\tau}} - \alpha) : (\tau - \alpha) - \tau_0^2, \tag{26}$$

where the tilde denotes the deviator, i.e.

$$\tilde{\boldsymbol{\tau}} = \tau - \frac{1}{3}\operatorname{tr}(\tau)\,\boldsymbol{I}. \tag{27}$$

The yield shear stress τ_0 may depend on κ.

It is assumed that the plastic work rate is equal to the rate of the isotropic hardening variable [7]. Then, its evolution law may be given by

$$\dot{\kappa} = \tau : \boldsymbol{D}^{p}. \tag{28}$$

A particular evolution law for α is proposed by Prager [21]

$$\overset{o}{\alpha} = c\boldsymbol{D}^{p}, \tag{29}$$

where $c(\kappa)$ is the anisotropic hardening modulus.

Apparently, the choice of the objective rate type $\overset{o}{\alpha}$ seems to be free. Xiao et aliter show [5] that from a weakened form of Ilyushin's postulate follows the necessity of using the same objective rate for the Kirchhoff stress τ and the backstress α. In virtue of the conclusion in the last paragraph of subsection 3.1

both should be of logarithmic type. Additionally [5], the convexity of the yield surface in Kirchhoff stress space as well as the normality rule have to be observed.

In revision of above conclusion, relation (27) needs revision, i.e.

$$\overset{\circ}{\alpha}{}^{(\log)} = c\boldsymbol{D}^{p} \tag{30}$$

Upon the assumption that the plastic potential is equal to the yield function [7], the associated flow rule is

$$\boldsymbol{D}^{\mathrm{p}} = \frac{\xi}{h}\left(\frac{\partial f}{\partial \tau} : \overset{\circ}{\tau}{}^{(\log)}\right)\frac{\partial f}{\partial \tau}. \tag{31}$$

ξ is the plastic indicator; it takes values of 0 and 1 for unloading and loading cases. The hardening parameter h is deduced from the stationarity of the yield function and results to

$$h = \frac{\partial f}{\partial \kappa}\left(\tau : \frac{\partial f}{\partial \kappa}\right) - c\frac{\partial f}{\partial \kappa} : \frac{\partial f}{\partial \alpha}. \tag{32}$$

3.3 Summarisation

The traditional model (18) is applicable in a facile form, if

- the hypoelastic relation (23) is taken for the recoverable part and is
- checked to fulfill Bernstein's integrability condition;
- relation (31) together with (28), (30) and (32) describe the irrecoverable part.

The notion of the strain is neither needed in $\boldsymbol{D}^{\mathrm{e}}$ nor in $\boldsymbol{D}^{\mathrm{p}}$. However, the strain may be evaluated from the stretching $\boldsymbol{D}$

$$\overset{\circ}{\boldsymbol{h}}{}^{(\log)} = \boldsymbol{D}. \tag{33}$$

$\boldsymbol{h}$ is the Hencky strain. Only the logarithmic rate of the Hencky strain can be identified with $\boldsymbol{D}$. Any other strain measure and any other objective rate are excluded [23].

A number of publications prove the validity of above model, e.g.

- Stress oscillations are not encountered in simple shear [3, 15].
- Hypoelastic closed deformation loops remain dissipationless [14, 17, 26].
- The large deformation Swift effect in torsion is verified [16, 25, 4].
- The bending of an elastoplastic strip with isotropic and kinematic hardening is examined [2].
- Strain recovery loops, as observed in memory alloys, may be described [28].

4. Some Remarks and Notes

Some notice should be given to the following details:

1. The logarithmic Kirchhoff stress rate assures integrability to a restricted subgroup of hyperelastic relations only. The integrability has to be checked for each hypoelastic ansatz.
2. Generally, the total strain is not separable into elastic and plastic parts, i.e.

$$\overset{\circ}{h}{}^{\mathrm{e}\,(\log)} \neq \boldsymbol{D}^{\mathrm{e}}, \quad \overset{\circ}{h}{}^{\mathrm{p}\,(\log)} \neq \boldsymbol{D}^{\mathrm{p}}, \tag{34}$$

   since the deformation gradient covering the total deformation is involved in the definition of the objective rate.
3. Chapter 3.6 of publication [27] presents the general treatment of initial material symmetry groups.
4. Thermodynamic laws, that are based on the stretching decomposition (18), are derived in [27].

Though being simple in design the traditional elastoplasticity theory presented here has to compete with a number of sophisticated recent developments. Some rudimentary considerations for two classes of them are as follows:

- The multiplicative decomposition of the deformation gradient into recoverable and irrecoverable parts is as follows

$$\boldsymbol{F} = \boldsymbol{F}^{o}\boldsymbol{F}^{p}. \tag{35}$$

This decomposition creates 15 additional internal variables with the need of defining a number of physically motivated evolution laws and/or the relation between them or other variables. Moreover, problems may occur whenever the yield surface no more embraces the stress free origin or when the decomposition gets incompatible.

- Another proposal is to deompose the Lagrangean strain as

$$\boldsymbol{E} = \boldsymbol{E}^{p} + (\boldsymbol{E} - \boldsymbol{E}^{p}). \tag{36}$$

Herein, $\boldsymbol{E}^{p}$ is a Lagrangean measure related to the plastic deformation. Due to the symmetry of $\boldsymbol{E}$, 6 additional internal variables are emerging. The challenge is: Reasonable physically motivated laws have to be formulated for a measure, that furthermore is Lagrangean, i.e. related to the undeformed body.

5. Acknowledgements

The author likes to thank Professor Heng Xiao and Professor Otto Bruhns for their diligence and patience in making him acquainted with a number of problems and discussing with him so many details. Also, the fruitful dialogue with Professor N K Gupta about related topics is thankfully acknowledged.

6. Conclusion

The traditional formulation based on the additive decomposition of the stretching into recoverable and irrecoverable parts may be considered as a powerful, though simple, instrument to describe moderately large deformations; moreover, it has the advantage of being formulated in an Eulerian frame, which is related to the deformed body. The use of the logarithmic backstress and Kirchhoff rate is essential.

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