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Research Paper

Nonlinear Models of Growing Solids

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The basic fundamentals of the mathematical theory of growing solids are under consideration.We focus on the surface growth when the joining of the material occurs at the boundary of a growing solid. The constitutive equations are formulated with respect to the complete distortion tensorial field, which may be represented as decomposition of initial (implant) distortion and compatible deformation gradients. The initial distortion induces the linear connection on the material manifold which becomes a flat space of affine connectivity with nontrivial torsion. The torsion of this connection describes locally the deviation from homogeneity and it is called the inhomogeneity in the given material uniformity. It is shown that one can obtain the initial (implant) distortion tensor field and therefore the torsion as the solutions of the boundary-value problem for the equilibrium of the growing boundary. The inhomogeneity causes residual stresses that are observable phenomena for the growing solids.

Key Words: Growing Solids; Accretion; Distortion; Material Manifold; Fibre Bundle; Residual Stresses

1. Introduction

A vast majority of objects around us arise from some growth processes. Many natural phenomena such as growth of biological tissues, glaciers, blocks of sedimentary and volcanic rocks, and space objects may serve as examples. Similar processes determine specific features of many industrial processes which include crystal growth, laser deposition, melt solidification, electrolytic formation, pyrolytic deposition, polymerization and concreting technologies. Recent researches indicates that growing solids exhibit properties dramatically different from those of conventional solids, and the classical solid mechanics cannot be used to model their behaviour. The old approaches should be replaced by new ideas and methods of modern mechanics, mathematics, physics, and engineering sciences.Thus, there is a new emerging area of solid mechanics that deals with the construction of adequate models for solid growth processes.

The methods of continuum mechanics can adequately simulate the processes of growth. Variety of processes, which are rather dissimilar in their physical nature, like electroplating, chemical vapour deposition and photopolymerization, have much in common from viepoint of continuum mechanics; these are growing solids that are deformed during the process of growth. Mechanical features of such solids differ significantly from those of permanent composition. Residual sresses must occur in the growing bodies, which lead to undesirable consequences, such as distortion of its geometry, local discontinuity and loss of stability. In particular, the estimation (and minimization) of possible distortions in stereo lithography, the analysis of stability of epitaxial thin-walled structures applicable in micro-electromechanical systems (MEMS) are essential.

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Deformation processes in a solid whose composition varies due to the accretion of new material to the boundary were studied in many works (see, e.g. [1, 2]). In the majority of papers which deal with the mechanics of growing solids the theory is constructed as some special replicas of the theory of deformable solids in three-dimensional Euclidean space. Nevertheless the geometric properties of Euclidean space are not enough to describe the stress-strain state of a body which was formed by the continuous joining of pre-stressed parts. It is extremely important that the growing body can be considered as a special class of inhomogeneous body, in which inhomogeneity arises because of nonholonomic distortion, caused by the joining of incompatible stressed parts. From this point of view the mechanics of growing solids have much in common with the theory of defects, in particular with the geometric theory of continuously distributed dislocations. In this context geometric concepts such as *connection, curvature, torsion, parallelism* are among the basic concepts of the general theory of inhomogeneous solids [3, 4, 5].

Apparently, the first who applied the methods of Cartan geometry in continuum mechanics was K Kondo (1955). These ideas have been developed in the works of Bilby *et al*., E Kröner and A Seeger in which relations between the incompatibility of deformations, the density of defects, and non-trivial geometry of manifolds with non-zero torsion and curvature were established. In this context geometric concepts such as connection, curvature, torsion, parallelism are among the basic concepts of the general theory of inhomogeneous solids. This theory was developed by W Noll [3] and CC Wang [4], as well as in the series of papers by M Epstein and G Maugin [5].

The continuous joining is a process of continuous adherence of infinitesimal areas to a body, i.e. areas of infinitesimal measure. One can consider an infinitely thin layers, threads, and points (Fig. 1). Since such infinitesimal field are continuous bodies of various dimensions, they can carry the stress-strain state corresponding to their dimension. In this regard, the distribution of stresses in a continuously growing body

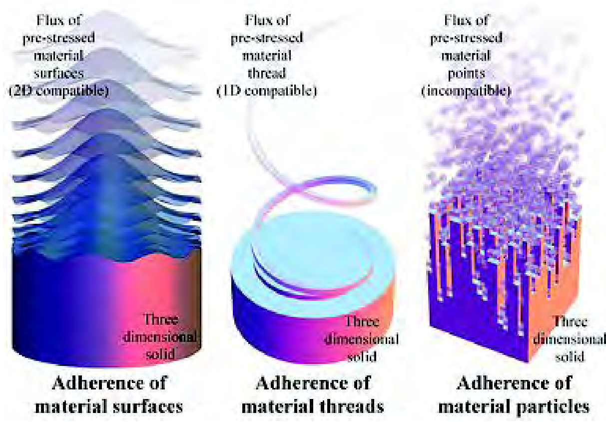

Fig. 1: Classification of growing processes

depends on geometric class of joined infinitesimal areas which implies the construction of different versions of the theory of growing solids. In this paper we consider a body which grows due to continuous adherence of two-dimensional layers considered as material surfaces [6].

The theory of fiber bundles of differentiable manifolds [7] is taken as the geometric foundation of mathematical theory of growing solids. Analytic properties of differentiable manifolds are determined without utilization of prescribed connection. This allow to formulate a boundary value problem in terms of quiet general geometrical properties and determine the particular type of connectivity *a posteriori* taking into account specific kinematic and static characteristics of the accretion process. Above all the geometrical concept of a fiber bundle corresponds to a key features of the growing solid whose growth is modeled as a continuous adhesion of deformed material areas. Such an assembly generates a nontrivial fiber bundle of material manifold. The structure of this bundle is completely determined by the scenario of accretion.

2.Bodies

We use the concept of *absolute physical space*, as it is customary in classical physics. This space $\mathcal{E}$ has the structure of three-dimensional affine (point) space. Elements of $\mathcal{E}$ we call *spatial points*. The translational space associated with the affine space $\mathcal{E}$ is denoted by V. It is a three-dimensional Euclidian space. The elements u, v, ... of V are called *spatial vectors*. The translation which carries a point $a \in \mathcal{E}$ to a point $b \in \mathcal{E}$ is represented by vector $v = b - a \in V$ and $a + v$ denotes the point into which point a is carried by the translation v.

We will employ the concept of a *body* $\mathfrak{B}$ which is connected open subset of an abstract topological space such that its image in the physical space is an region with a regular boundary. Moreover we consider a body as a *differentiable manifold*. Its elements $\mathfrak{p} \in \mathfrak{B}$ are called particles. We assume that the particles of the body are *simple particles* (in sense of W. Noll [3]).

The set $\mathfrak{B}$ endowed with a structure defined by the class $\mathfrak{C}$ of mapping.

$$\varkappa : \mathcal{B} \to \mathcal{E}.$$

Element of $\varkappa \in \mathfrak{C}$ is called *configuration*. The image of $\mathfrak{p} \in \mathfrak{B}$ under the mapping $\varkappa$, namely $\varkappa(\mathfrak{p}) \in \mathcal{E}$, called *place* of the material point $\mathfrak{p}$ in the configuration $\varkappa$. The image of set $\mathfrak{B}$ otherwise the collection of places of all material points, is called *shape*. Usually it is preferable to fix one of the shapes and refer to it as the reference shape. If γ, $\varkappa \in \mathfrak{C}$ are two configurations then the composition

$$\lambda = \varkappa \quad \gamma^{-1} : \gamma(\mathfrak{B}) \to \varkappa(\mathfrak{B})$$

is called the *deformation* of $\mathfrak{B}$ from the configuration γ into the configuration $\varkappa$. One can see vivid explanation of these concepts in Fig. 2.

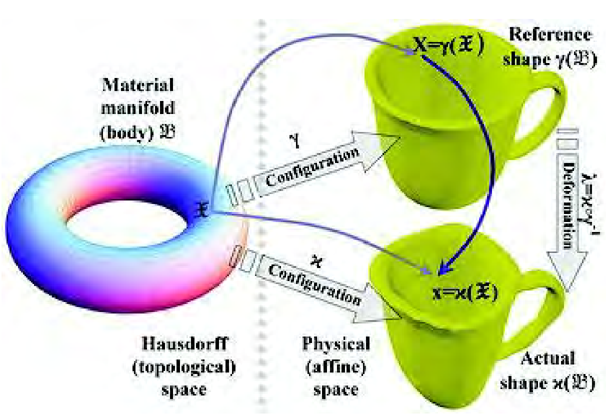

Fig. 2: Configurations and deformations

Since each shape is embedded in affine space with Euclidian structure it has natural parametrization, for example, by means of Cartesian coordinates. Hence the class $\mathfrak{C}$ endow subset $\mathfrak{B}$ with the structure of manifold. Smoothness of this manifold depends on differential properties of functions belongs to class $\mathfrak{C}$.

It should be noted that in general the set $\mathfrak{B}$ has a topological structure that differs from the topological structure of corresponding image in physical space $\mathcal{E}$. Actually according to the Hahn-Mazurkiewicz theorem locally connected continuum can be represented as a continuous image of an interval of real line. In this case the topological dimension of a body $\mathfrak{B}$ may be equal to unity, while the topological dimension of its image may be equal to two or three. One can see on Fig. 3 steps of well known space filling process (Peano, Hilbert) when a line (one dimensional manifold) fills three dimensional region. However such mapping may not be bijective, i.e., the same point of the image can correspond to several different prototypes in the body interval. One can see here the contradiction with typically accepted postulate of inpenetrability. Even so by means of a special choice of mapping function such multiple points can be converted into points of contact that are quite acceptable in mechanics. Vividly speaking body $\mathfrak{B}$ that occupy in $\mathcal{E}$ a spatial region may in fact represent a thread which topological dimension is equal to one. This manner of body formation can not be ignored in common, but in present paper for simplicity we restrict our consideration and assume that the topological dimension of set $\mathfrak{B}$ coincides with the topological dimension of its images in $\mathcal{E}$. Such an assumption imposes a restriction on the class of mappings of $\mathfrak{B}$ in space $\mathcal{E}$: these maps should be *homeomorphisms* i.e. one-one continuous mappings whose inverses are also continuous (recall that the map of topological space onto a topological space is continuous if the preimage of any open set is open).

In classical mechanics solids are treated as invariant sets. Mechanics of growing solids admit the *evolution* of the set $\mathfrak{B}$ that represents body and treat it as *variable* set. In following we will refer to the bodies of invariable composition as to *permanent*

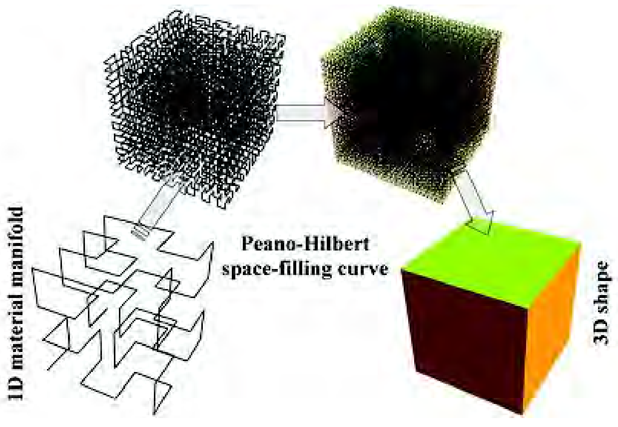

Fig. 3: Space filling process

bodies. If a permanent body has more than one configuration (up to rigid motion), i.e. undergoes deformation in physical space, we call it *permanent solid*. We will distinguish between the permanent and growing bodies, and respectively call them permanent solids and growing solids. Now according to W Noll [3] we introduce the definition of continuous permanent body. The body $\mathfrak{B}$ is *continuous body of class* C^p, ($p \geq = 1$), if a class of configurations $\mathfrak{C}$ satisfies the following axioms:

B1. Every configuration $\varkappa \in \mathfrak{C}$ is homeomorphism, and its range is an open subset of ε, which is called *shape* occupied by the body $\mathfrak{B}$ configuration $\varkappa$.

B2. If γ, $\varkappa \in \mathfrak{C}$ are configurations, then their composition $\lambda = \gamma \circ \varkappa^{-1}$ belongs to class C^p, i.e. functions λ are continuous and their derivatives up to order p are continuous also.

B3. If $\varkappa \in \mathfrak{C}$ is a configuration and if $\lambda(\mathfrak{B}) \rightarrow \varepsilon$ is a deformation of class C^p, then $\lambda \circ \varkappa \in \mathfrak{C}$.

By virtue of axioms B1 - B3 the class of mapping $\mathfrak{C}$ gives rise on the body $\mathfrak{B}$ the structure of C^p manifold. Since all configuration are homeomorphisms all their images, as well as the body $\mathfrak{B}$ itself, are topologically indistinguishable. Thus the class of admissible configurations defines the topological structure of solid.

The evolution of the set $\mathfrak{B}$ may be represented by the family of sets

$$\mathfrak{G} = \{\mathfrak{B}_\alpha\}_{\alpha \in \mathfrak{T}},$$

where $\mathfrak{T}$ is the set of indices that may be finite, countable or continuum. We introduce the *total body* $\mathfrak{B}^*$ and *initial body* $\mathfrak{B}_*$ as follows:

$$\mathfrak{B}^* = \bigcup_{\alpha \in \mathfrak{T}} \mathfrak{B}_\alpha, \qquad \mathfrak{B}_* \bigcap_{\alpha \in \mathfrak{T}} \mathfrak{B}_\alpha.$$

We give the definition of growing body in additional axiom B4. This axiom may be formulated in various forms depending on type of growing process.

B4.1. *Discrete Growth*. The set $\mathfrak{G}$ is a finite sequence of nested bodies:

$$\mathfrak{G} : \mathfrak{B}_1 \subset \mathfrak{B}_2 \subset \cdots \subset \mathfrak{B}_N.$$

It is obvious that the cardinality of the set $\mathfrak{G}$ is equal to finite number N, i.e., $| \mathfrak{G} | = N$.

B4.2. *Generalized Discrete Growth*. The set $\mathfrak{G}$ is transfinite sequence of nested bodies:

$$\mathfrak{G} : \mathfrak{B}_1 \subset \mathfrak{B}_2 \subset \cdots \subset \mathfrak{B}_\omega \subset \mathfrak{B}_{\omega+1} \subset \cdots.$$

It follows from the theorem of Alexandrov and Urysohn that $| \mathfrak{G} | V = \aleph_0$.

B4.3. *Continuous Growth*. The set $\mathfrak{G}$ is a family of bodies over an open interval $\mathfrak{T} = (\alpha, \beta) \subset \mathbb{R}$. This family satisfies the following conditions. There are two-dimensional manifolds Ω_k and no more than a countable set of homeomorphisms Ψ_k, such that $\Psi_k : (\Omega_k; \alpha) \rightarrow \mathfrak{B}^*$, $\alpha \in \mathfrak{T}$.

$$\forall \alpha < \beta \;\; \mathfrak{B}_\alpha \subset \mathfrak{B}_\beta; \; \forall \alpha \; \exists k \; \partial \mathfrak{B}_\alpha$$

$$= \Psi_k : (\Omega_k; \alpha); \; \bigcup_k \;\; \Psi_k (\Omega_k X \;\mathfrak{T}_k, \alpha)$$

$$\text{x} \;\; \mathfrak{B}: \bigcup_k \;\; \mathfrak{T}_k = \mathfrak{T}.$$

Obviously $| \mathfrak{G} | = \aleph_1$.

So by axiom B4.3 in the case of continuous growth we introduce material manifold $\mathfrak{M}$ as a fibre bundle of a smooth three-dimensional manifold. As one knows from differential geometry, a fibre bundle is defined by base $\mathfrak{N}$ which is a smooth manifold itself, structure group G, and projection mapping π [7]. As mentioned above, we restrict present consideration to the case of topological equivalence of body and its shapes. Moreover we assume that growing body is bounded, i.e. all its shapes may be placed into sufficiently large ball with fixed finite diameter, and the process of growing is implemented as continuously adhering of material surfaces to the boundary of the body. Under this assumptions the dimension of the base $\mathfrak{N}$ is equal to one and material manifold $\mathfrak{M}$ is represented as bundle over interval (Fig. 3).

The material point $\mathfrak{p} \in \mathfrak{M}$ is defined by three coordinates specified in local map U which contains a neighborhood of $\mathfrak{p}$. The set of three coordinates denote gothical symbol $\mathfrak{X}$ that means an ordered triple of numbers $\{\mathfrak{X}^1, \mathfrak{X}^2, \mathfrak{X}^3\}$. The last element of the triple $\mathfrak{X}^3$ is highlighted with a semicolon in notation and determines the coordinates of the base $\mathfrak{N}$. The first two elements $\mathfrak{X}^1, \mathfrak{X}^2$ are the coordinates of the fibre. As each fibre is uniquely determined by the value $\mathfrak{X}^3$ the fibres will be denoted by $\mathfrak{M}_{\mathfrak{X}^3}$.

One may define a permanent body $\mathfrak{B}$ as an open subset of a material manifold $\mathfrak{M}$ bounded by non-identical fibers. In this case the body $\mathfrak{B}$ can be represented as a union of fibers whose indices are in the open interval (α, β) i.e.

$$\mathfrak{B} = \mathfrak{B}(\alpha, \beta) = \bigcup_{\mathfrak{X}^3 \in (\alpha, \beta)} \mathfrak{M}_{\mathfrak{X}^3}.$$

A growing body can be defined as a parametric family of such sets

$$\mathfrak{G} = \{\mathfrak{B}_\gamma = \mathfrak{B}(\alpha, \gamma) | \gamma \in (\alpha, \beta)\},$$

where γ is a parameter of the family. While $\alpha \to \beta$

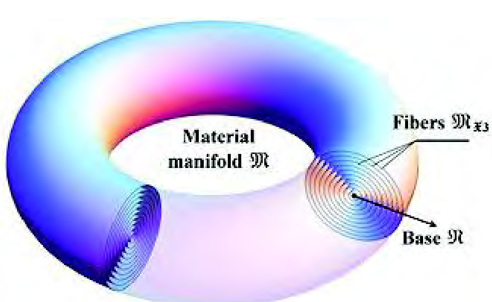

Fig. 4: Bundle of material manifold

the body degenerates into an infinitely thin ply or a point. Obviously this definition may be generalized to:

$$\mathfrak{G} = \{\mathfrak{B}_\gamma = \mathfrak{B}(\alpha_\gamma, \beta_\gamma) | (\alpha_\gamma, \beta_\gamma) \subset (\alpha, \beta)\},$$

where $(\alpha_\gamma, \beta_\gamma)$ are nested intervals.

According to the above definitions the boundary of growing solid should be topologically equivalent to the typical fiber which is smooth manifold itself and hence the growing boundary should be geometrically closed surface. If the growing boundary is topologically equivalent to the *manifold with edge* then a growing body can be defined as follows:

$$\mathfrak{G} = \{\mathfrak{B}_\gamma = \mathfrak{B}_0 \cap (\alpha_\gamma, \beta_\gamma) | (\alpha_\gamma, \beta_\gamma) \subset (\alpha, \beta)\}.$$

Here $\mathfrak{B}_0$ is fixed subset of $\mathfrak{M}$ with smooth boundary

3. Geometry of Material Manifold

Some facts about non-symmetric connection and geometry or Rieman-Cartan manifolds are reviewed in [7, 16]. We denote by symbol $T\mathfrak{B}$ the set of vector fields on $\mathfrak{B}$, and by symbol $C^\infty(\mathfrak{B})$ the set of smooth functions on $\mathfrak{B}$. A *linear* (*affine*) *connection* on manifold $\mathfrak{B}$ is an operator

$$\nabla : T\mathfrak{B} \times T\mathfrak{B} \to T\mathfrak{B},$$

such that $\forall\, f, f_1, f_2 \in C^\infty(\mathfrak{B}),\ \forall\, a_1, a_2 \in \mathbb{R}$:

i) $\nabla_{f_1\mathbf{X}_1 + f_2\mathbf{X}_2}\mathbf{Y} = f_1\nabla_{\mathbf{X}_1}\mathbf{Y} + f_2\nabla_{\mathbf{X}_2}\mathbf{Y}$,

ii) $\nabla_{\mathbf{X}}(a_1\mathbf{Y}_1 + a_2\mathbf{Y}_2) = a_1\nabla_{\mathbf{X}}\mathbf{Y}_1 + f_2\nabla_{\mathbf{X}}\mathbf{Y}_2$,

iii) $\nabla_{\mathbf{X}}(\mathrm{f}\mathbf{Y}) = \mathrm{f}\nabla_{\mathbf{X}}\mathbf{Y} + (\mathbf{X}\mathrm{f})\mathbf{Y}$.

The expression $\nabla_{\mathbf{X}}\mathbf{Y}$ is called the covariant derivative of $\mathbf{Y}$ along $\mathbf{X}$. In a local chart $\{\mathfrak{X}^\alpha\}$

$$\nabla_{\partial_\alpha}\partial_\beta = \Gamma^\gamma_{\alpha\beta}\partial_\gamma.$$

Here $\Gamma^\gamma{}_{\alpha\beta}$ are Cristoffel symbols of the connection (connection symbols) and ∂_α are the natural basis for the tangent space corresponding to a coordinate chart $\{\mathfrak{X}^\alpha\}$. Suppose that a metric $\boldsymbol{G}$ is introduced on manifold $\mathfrak{B}$. A linear connection is said to be *compatible* with metric $\boldsymbol{G}$ of the manifold if

$$\mathbf{X}\langle \mathbf{Y}, \mathbf{Z}\rangle_G = \langle \nabla_{\mathbf{X}} \mathbf{Y}, \mathbf{Z}\rangle_G + \langle \mathbf{Y}, \nabla_{\mathbf{X}} \mathbf{Z}\rangle_G$$

where $\langle .,...\rangle_G$ is the inner product, induced by metric G. It can be shown that affine connection ∇ is is compatible with metric G if and only if ∇G. In following we consider a three-dimensional manifold $\mathfrak{B}$ with metric G and a G-compatible connection ∇. We will reffer to this geometrical object as $(\mathfrak{B}, \boldsymbol{G}, \nabla)$ and call it Riemann-Cartan manifold.

The torsion of the connection ∇ is an operation $\mathcal{T}$.

$$\mathcal{T}: T\mathfrak{B} \text{ x } T\mathfrak{B} \to T\mathfrak{B},\ \mathcal{T}(\mathbf{X}, \mathbf{Y}) = \nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{X}} \mathbf{X} - [X, Y].$$

In components in a local chart $\{\mathfrak{X}^\alpha\}$

$$\mathcal{T}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta}.$$

The connection is said to be symmetric if it is torsion-free, that is

$$\nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} = [X, Y].$$

It can be shown that on any Riemann manifold $(\mathfrak{B}, \boldsymbol{G})$ that is an unique linear connection ∇ that compatible with metric $\boldsymbol{G}$ and is torsion-free. It's Cristoffel symbols are

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} G^{\gamma\zeta} \left(\frac{\partial G_{\beta\zeta}}{\partial X^\alpha} + \frac{\partial G_{\alpha\zeta}}{\partial X^\beta} - \frac{\partial G_{\alpha\beta}}{\partial X^\zeta} \right).$$

The assosiated connection is called *Levi-Civita connection*.

In a manifold with a connection the Riemann curvature is an operation $\mathcal{R}\cdot$

$$\mathcal{R}: T\mathfrak{B} \text{ x } T\mathfrak{B} \text{ x } T\mathfrak{B} \to T\mathfrak{B},\ \mathcal{R}(\boldsymbol{X}, \boldsymbol{Y}) = \nabla_{\mathbf{X}} \nabla_{\mathbf{Y}} \mathbf{Z} - \nabla_{\mathbf{Y}} \nabla_{\mathbf{X}} \mathbf{Z} - \nabla_{[\mathbf{X},\mathbf{Y}]} \mathbf{Z}.$$

A metric-affine manifold is a manifold equipped with both a connection and a metric: $(\mathfrak{B}, \boldsymbol{G}, \nabla)$ If the connection is metric and compatible, the manifold is called a Rieman-Cartrn manifold. If the connection is torsion-free but has curvature then $\mathfrak{B}$ is called a Riemann manifold. If the curvature of the connection vanishes but it has torsion then $\mathfrak{B}$ is called a Weitzenböck manifold. If both torsion and curvature vanish, $\mathfrak{B}$ is a flat (Euclidean) manifold.

4. Equilibrium Equations

Recall some facts from classical theory of hyperelasticity [8, 9]. The elastic potential of the permanent hyperelastic solid $W_\varkappa^{\varkappa_0}$ may be represent as follows :

$$W_\varkappa^{\varkappa_0} = W_\varkappa(F_\varkappa^{\varkappa_0}, \mathfrak{X}),$$

where $F_\varkappa^{\varkappa_0}$ is a deformation gradient* which is the gradient of the mapping of stress free (natural) shape $\varkappa_0(\mathfrak{B})$ on actual shape $\varkappa(\mathfrak{B})$, i.e.

$$F_\varkappa^{\varkappa_0} = \nabla_{\varkappa_0} = \nabla(\varkappa_0 \cdot \varkappa_0^{-1}).$$

The distribution of stored energy that is determined by the elastic potential $W_\varkappa^{\varkappa_0}$ induces the stress field $T_\varkappa^{\varkappa_0}$ (first Piola-Kirchhoff stresses) in accordance with the constitutive relation

$$T_\varkappa^{\varkappa_0} = \frac{\partial W_\varkappa^{\varkappa_0}}{\partial \mathbf{F}_\varkappa^{\varkappa_0}}.$$

Here and in follows the subscript x indicates that considered field corresponds to the configuration x the shape $x(\mathfrak{B})$ and appropriate external fields that cause it.

In order to obtain particular approximations of $W_\varkappa^{\varkappa_0}$ one has to proceed the *gauge*. If the deformed solid possesses *natural* (i.e. free of stresses) shape then the gauge leads to the following conditions:

*We use the standard definition of gradient ∇ as linear operator which provides a linear approximation of the mapping $f: A \to B$:

$$f(x+h) = f(x) + \nabla f(x)[h] + o(\|h\|), x, x+h \in A,$$

$f(x), f(x+h) \in B, \nabla f(x) \in \mathcal{L}(A, B)$, where $\mathcal{L}(A, B)$ is the space of all linear mapping from A to B. The notation $\nabla_g f$ corresponds to relative gradient of mapping f with respect to mappin g, i.e. the linear approximation of composition $f \circ g^{-1}$.

$$W_{\varkappa}^{\varkappa_0}(\mathbf{1}, \mathfrak{X}) = 0, \quad \frac{\partial W_{\varkappa^0}}{\partial \mathbf{F}_{\varkappa}^{\varkappa_0}}\Big|_{\mathbf{F}_{\varkappa}^{0}=1} = \mathbf{0}.$$

The first condition expresses the fact that a stored elastic energy differs from zero only in a deformed (i.e., a non-natural) shape and the second condition shows that stresses in natural shape vanish.

In common case the natural shape may not exists. This lead to so-called theory of *inhomogeneous bodies* that was developed by W Noll [3], C C Wang [4] and others. So, one can find the configuration that relax neighbourhood of fixed point locally, but there is no configuration that carries the whole body to natural state. More precisely there is no such configuration with image imbedded in Euclidian space. At the same time there is possibility to find the configuration that maps the whole body to stress free shape if it permissible to embed this shape into Cartan space with nontrivial torsion. We refer to this procedure as *immersing* and illustrates it in Fig. 5. Explain it in more details.

Let the reference shape $\varkappa_R(\mathfrak{B})$ is the image of body $\mathfrak{B}$ under the mapping $\varkappa_R$. It is stressed. One can associate with each material point the deformation that carries the neighbourhood of the point to stress free state. However this deformations for different points in general corresponds to different configurations. One may construct the collection of such local deformations associated with all material points and obtain the tensor field $\mathbf{K}^{-1}$ over set $\mathfrak{B}$ that is not in general a gradient of any vector field. The corresponding tensor field of local configurations we denote by $\mathcal{K}$. It's clear that $\mathcal{K} = \mathbf{K}^{-1} \circ \varkappa_R$. We can determine the connection coefficients $\Gamma^{\beta}_{\gamma\alpha}$ on material manifold :

$$\Gamma^{\beta}_{\gamma\alpha} = \partial_{\alpha}(\mathbf{K}^{-1})^{\rho}_{\cdot\gamma} K^{\beta}_{\cdot\rho}.$$

The corresponding connection can be nontrivial, i.e. with non zero torsion. This geometrization of the manifold $\mathfrak{M}$ transforms it into a configuration that is not embedded in three-dimensional Euclidean space.

Note that material manifold is a bundle and each its fiber $\mathfrak{M}_{\mathfrak{X}^3}$ is associated with a material surface

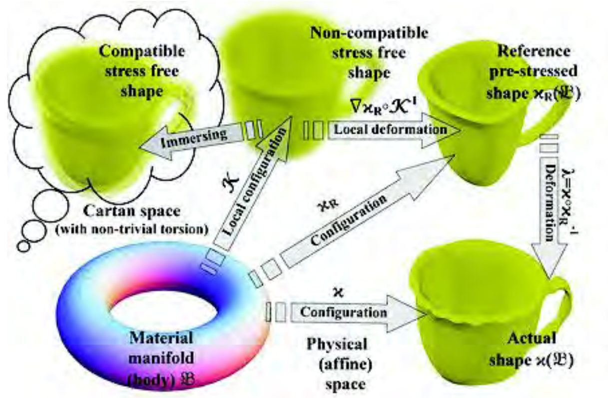

Fig. 5: Immersing of reference shape

strained before its adhesion to a body. From the mechanical point of view it means that each fiber was deformed from natural state before adhesion. So each individual fiber has a a global two-dimensional natural reference configuration.

Taking into account this, we introduce the tensor field $\mathbf{K}(\mathfrak{X}^1, \mathfrak{X}^2, \mathfrak{X}^3)$ as a local deformation associated with individual material surface $\mathfrak{M}_{\mathfrak{X}^3}$. It can be quantified as pre-deformation of a surface from natural reference onto configuration that match its position in reference shape. Thus, at each point of the material manifold tensor field $\mathrm{K}(\mathfrak{X})$ is defined. This field is continuously differentiable due to the continuity of accretion process. Moreover the field $\mathbf{K}^* \cdot \mathbf{K}$ specifies a metric on a surface embedded in three-dimensional Euclidean space on each fiber $\mathfrak{M}_{\mathfrak{X}^3}$. The complete distortion has the form $\mathbf{H} = \mathbf{K} \cdot \mathbf{F}$ and elastic potential $W_{\varkappa}^{\varkappa_0}(\mathbf{K}, \mathfrak{X})$ can be defined on the configuration $\varkappa_0$ only only locally. The stress tensor $T_{\varkappa}^{\varkappa_0}$ relevant to can be defined fiber-by-fiber as follows:

$$T_{\varkappa}^{\varkappa_0} = \frac{\partial W_{\varkappa}^{\varkappa_0}}{\partial \mathbf{H}}.$$

Elastic potential can be determined globally with respect to the configuration $\varkappa_R$. Introduce elastic potential $W_{\varkappa}^{\varkappa_0}$ that is an elastic energy per unit volume in the reference state $\varkappa_R$ and can be interpreted as a function of three arguments $\mathbf{F}$, $\mathbf{K}$, $\mathfrak{X}$ (see. [5]), i.e.

$$W_{\varkappa}^{\varkappa_0}(\mathbf{K}, \mathbf{F}, \mathfrak{X}) = J_{\mathbf{K}}^{-1} W_{\varkappa}^{\varkappa_0}(\mathbf{H}, \mathfrak{X})$$

$$= J_{\mathbf{K}}^{-1} W_{\varkappa}^{\varkappa_0}(\mathbf{H} \cdot \mathbf{F}, \mathfrak{X}).$$

Even if the functional $W_{\varkappa}^{\varkappa_0}$ is uniform (i.e., it does not depend explicit on material coordinates $\mathfrak{X}$), functional $W_{\varkappa}^{\varkappa_0}$ is not uniform because **K** depends on $\mathfrak{X}$. This yields the expression for Piola stress tensor $T_{\varkappa}^{\varkappa_0}$ field relevant to $\varkappa_R$ by formula

$$T_{\varkappa}^{\varkappa_0} = \frac{\partial W_{\varkappa}^{\varkappa_0}}{\partial \mathbf{F}} \cdot J_{\mathbf{K}}^{-1} \mathbf{K}^{*} \cdot \frac{\partial W_{\varkappa}^{\varkappa_0}}{\partial \mathbf{H}} \cdot$$

Let us assume now that the body $\mathfrak{B}$ is subject to the external body forces $\rho_{\varkappa}\mathbf{b}$. If the forces acting on every part of $\mathfrak{B}$ are balanced one can obtain the Cauchy equation of balance

$$\nabla_{\varkappa} \cdot \mathbf{T}_{\varkappa} + \rho_{\varkappa} \mathbf{b} = 0,$$

that is valid in configuration x. Here T_x is Cauchy stress tensor occurring in the shape $\varkappa(\mathfrak{B})$. If $\varkappa_R$ is configuration that differs from x and **F** is gradient of mapping $\varkappa \circ \varkappa_R^{-1}$ i.e.

$$\mathbf{F} = \nabla(\varkappa \circ \varkappa_R^{-1}) = \nabla_{\varkappa_R} \varkappa,$$

then we can determine the first Piola-Kirchhoff stress field by well known formula

$$\mathbf{T}_{\varkappa}^{\varkappa_R} = J_{\mathbf{F}} \mathbf{F}^{-1} \cdot \mathbf{T}_{\varkappa}, \; J_{\mathbf{F}} = |\,\mathrm{d}et\mathbf{F}\,|.$$

Standard arguments yield the balance equation

$$\nabla_{\varkappa_R} \cdot \mathbf{T}_{\varkappa}^{\varkappa_R} + \rho_{\varkappa_R} \mathbf{b} = 0.$$

What will change in the arguments if we replace the gradient **F** by arbitrary smooth tensor field **H**? In order to answer this question, we provide the following calculations. Let

$$\mathbf{T}_{\varkappa}^{\mathbf{H}} = J_{\mathbf{H}} \mathbf{H}^{-1} \cdot \mathbf{T}, \nabla_{\mathbf{H}} = \mathbf{H}^{*} \cdot \nabla_{\varkappa}.$$

The divergence of tensor field $\mathbf{T}_{\varkappa}^{\mathbf{H}}$ may be represented as

$$\nabla_{\mathbf{H}} \cdot \mathbf{T}_{\varkappa}^{\mathbf{H}} = [(\nabla_{\mathbf{H}} J \cdot \mathbf{H}^{-1} + J \nabla_{\mathbf{H}} \cdot (\mathbf{H}^{-1}))] \cdot$$

$$(\mathbf{T}_{\varkappa} + J \cdot (\mathbf{H}^{-1})^{*} \cdot \nabla_{\mathbf{H}} \mathbf{T}_{\varkappa}.$$

The gradient of Jacobian J is

$$\nabla_{\mathbf{H}} J = J(\nabla_{\mathbf{H}} \mathbf{H}) \cdot\cdot \mathbf{H}^{-1} \cdot$$

The divergence of tensor field $\mathbf{H}^{-1}$ may be calculated according to the formula

$$\nabla_{\mathbf{H}} \cdot (\mathbf{H}^{-1}) = -\mathbf{H}^{-1} : (\mathbf{H}^{-1}) \cdot \mathbf{H}^{-1}.$$

This relations gives

$$\nabla_{\mathbf{H}} \cdot \mathbf{T}_{\varkappa}^{\mathbf{H}} = J[(\nabla_{\mathbf{H}} \mathbf{H}) \cdot\cdot \mathbf{H}^{-1} - \mathbf{H}^{-1} : \nabla_{\mathbf{H}} \mathbf{H}] \mathbf{H}^{-1} \cdot$$
$$\mathbf{T}_{\varkappa} + J \nabla_{\varkappa} \cdot \mathbf{T}_{\varkappa}.$$

Recall that

$$J \nabla_{\varkappa} \cdot \mathbf{T}_{\varkappa} = -J \rho_{\varkappa} \mathbf{b} = -\rho_{\mathbf{H}} \mathbf{b}, \; \rho_{\mathbf{H}} = J \rho_{\varkappa}$$

Hence we obtain the following balance equation

$$\nabla_{\mathbf{H}} \cdot \mathbf{T}_{\varkappa}^{\mathbf{H}} + \rho_{\mathbf{H}} \mathbf{b} + \mathbf{s} \cdot \mathbf{T}_{\varkappa}^{\mathbf{H}} = 0,$$

where vector field **s** performs the measure of inhomogeneity and can be expressed in terms of tensor field **H** as follows

$$\mathbf{s} = \mathbf{H}^{-1} : (\nabla_{\mathbf{H}} \mathbf{H}) - (\nabla_{\mathbf{H}} \mathbf{H}) \cdot\cdot \mathbf{H}^{-1},$$

or in more expressive form

$$\mathbf{s} = \mathbf{H}^{-1} : (\nabla_{\mathbf{H}} \mathbf{H}) - (\nabla_{\mathbf{H}} \mathbf{H})^{(321)}),$$

where $\mathbf{A}^{(321)}$ is the isomer of tensor **A** which may be obtained by permutation of elements in dyadic decomposition of **A** according to sequence. Note that equation similar to that obtained above was derived by W Noll [3]. See also the generalization of Piola identity in [10].

The inhomogeneity is the cause of residual stresses which can be determined from the equation

$$\nabla_{\mathbf{H}} \cdot \mathbf{T}_{\varkappa}^{\mathbf{H}} + \mathbf{s} \cdot \mathbf{T}_{\varkappa}^{\mathbf{H}} = 0,$$

and appropriate boundary conditions. If we introduce the covariant derivative ∇ adopted to moving frame generated by eigenvectors of **H** viz.

$$\nabla \cdot \mathbf{u} = \nabla_{\mathbf{H}} \cdot \mathbf{u} + \mathbf{s} \cdot \mathbf{u},$$

that corresponds to Cartan space with connection $\Gamma^{\beta}_{\gamma\alpha}$, then we obtain the balance equation in customary form. This situation has vivid physical interpretation: the observer adopted to the moving frame sees no inhomogeneities in the same way as a *geodesic observer* does not 'feel' any gravitation field in general relativity - see e.g. [5].

The boundary value problem for an accreted solid is determined by the equations of equilibrium in $V(\alpha)$ with boundary $\Omega(\alpha)$ parametrically dependent on α i.e.

$$\nabla_{\varkappa_R} \cdot \left[J_{\mathbf{K}}^{-1} \mathbf{K}^* \cdot \frac{\partial W_{\varkappa}^{\varkappa_c}(\mathbf{H}, \mathfrak{X})}{\partial \mathbf{H}^*} \Big|_{\mathbf{H}=\mathbf{K}\cdot\mathbf{F}} \right] + \rho_{\varkappa_R} \mathbf{b} = 0$$

and boundary conditions stated on $\Omega(\alpha)$:

$$\mathbf{n}_{\varkappa_R} \cdot \left. J_{\mathbf{K}}^{-1} \mathbf{K}^* \cdot \frac{\partial W_{\varkappa}^{\varkappa_c}(\mathbf{H}, \mathfrak{X})}{\partial \mathbf{H}^* = 0} \Big|_{\mathbf{H}=\mathbf{K}\cdot\mathbf{F}} \right]_{\Omega(\alpha)} = \mathbf{p}.$$

At the first glance a formal statement of the boundary value problem differs from the classical one only by the fact that the boundary of domain depends parametrically on time. However, there is more profound difference: the elastic potential depends on the tensor field of distortion, determination of which requires additional conditions. Particular form of these conditions depends on the geometrical structure of joined elements, that is in essence of the structure of the bundle of material manifold. If the growth of a body occurs due to continuous influx of prestressed material surfaces to this body then this condition can be written in the form

$$P_{\varkappa_R} \cdot \left[J_{\mathbf{K}}^{-1} \mathbf{K}^* \cdot \frac{\partial W_{\varkappa}^{\varkappa_c}(\mathbf{H}, \mathfrak{X})}{\partial \mathbf{H}^*} \Big|_{\mathbf{H}=\mathbf{K}\cdot\mathbf{F}} \right] \cdot P_{\varkappa_R|\Omega(\alpha)} = J$$

Here $\mathbf{P}_{\varkappa_R} = (\mathbf{E} - \mathbf{n}_{\varkappa_R} \otimes \mathbf{n}_{\varkappa_R})$ is a projector onto the tangent plane to $\Omega(\alpha)$. This equation expresses the fact that the fibers align with the specified tension determined by the surface tensor $\mathcal{J}$, i.e., two-dimensional tensor of second rank defined in the tangent space of the adhering material surface.

5. Determination of Distortion Field

The equation for distortion (implant) tensorial field **K** provided that the growing process is the result of continuous adherence of prestressed surfaces can be obtained from the relations of the theory of material surfaces (theories of solids with a material boundary [6]). Effects of material surface adhering leads to an infinitesimal change of the stress-strain state of an accreted solid but as an elementary act of adhesion occurs during an infinitely small time interval the rate of stress state is finite. This rate can be found from the equations of contact interaction between the spatial body and adhering the material surface. The equation of physical boundary equilibrium (which from the geometrical point of view is a bounding surface of the body in its actual state and from mechanical point of view is a thin film undergoing a membrane state of stress) can be written as

$$\nabla_s \cdot \mathcal{T} + \mathbf{b}_s = \mathbf{n}_{\varkappa} \cdot \mathbf{T}_{\varkappa}|_{\Omega(t)},$$

where ∇_s is the surface nabla operator, $\mathbf{b}_s$ is the surface density of external forces acting on $\Omega(t)$. To complete the boundary value problem formulation the condition on the curvilinear boundary $\partial\Omega(t)$ of the

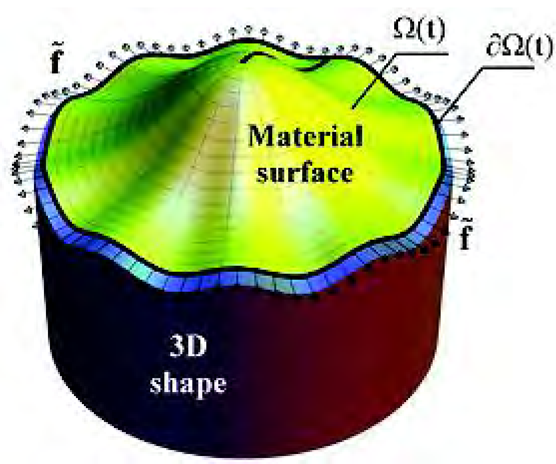

Fig. 6: 3D shape with adhered material surface

surface $\Omega(t)$ must be specified. This conditions may be of the form

$$\tilde{\mathbf{n}} \cdot \mathcal{J}|_{\partial\Omega(t)} = \tilde{\mathbf{f}}.$$

Here $\tilde{\mathbf{n}}$ is an external unit normal to the curve $\partial\Omega(t)$ in the tangent plane and $\tilde{\mathbf{f}}$ is a linear density of forces distributed on the curve $\partial\Omega(t)$.

Defined distortion field allows to specify the geometry on the material manifold, determining the connection on it. The resulting coefficients of affine connection do not possess the symmetry and define a nonzero torsion tensor

$$\mathcal{J}^{\alpha}_{\gamma\beta} = \partial_\gamma (\mathbf{K}^{-1})^{\rho\cdot}_{\cdot\beta} K^{\alpha\cdot}_{\cdot\rho} - \partial_\beta (\mathbf{K}^{-1})^{\rho\cdot}_{\cdot\gamma} K^{\alpha\cdot}_{\cdot\rho}.$$

The non-Euclidean features of reference natural configuration, quantitatively defined by the torsion tensor which characterizes the incompatibility of fibers stress-strain states joined in the process of accretion.

6. Universal Deformations of Growing Solids

In this section some illustrative examples based on the wellknown universal solution of Rivlin-Ericsen type for individual fibers are given.

Example 1. Consider growing incompressible hyperelastic ball (Fig. 7). The material manifold $\mathfrak{M}$ may be represented in $\mathbb{R}^3$ as an open ball with pricked center and radius R that admits fibration into concentric spheres $\mathfrak{M}_{\mathfrak{X}^3}$. Each sphere is a fibre and the open interval $\mathfrak{T} = (0, R)$ is the base of bundle. Last element of material triple $\mathfrak{X}^3 \in \mathfrak{T}$ represents the coordinate on the base and a pair $\mathfrak{X}^1$, $\mathfrak{X}^2$ is the coordinates of the material point on the fibre. We assume the incompressibility of the material that permit the usage of Rivlin-Ericsen universal solutions for individual spherical fibre [6, 7].

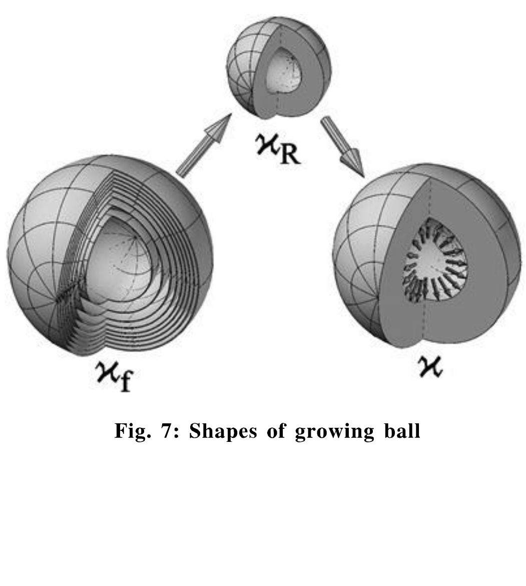

Fig. 7: Shapes of growing ball

Let e_r, e_φ, e_θ be the elements of physical basis of spherical coordinates. Let every fiber hold natural configuration that is the sphere alone immersed in $\mathbb{R}^3$. The union of these fibers is not a configuration because it is not a connected set. But after appropriate individual deformation of each fiber this union can be transformed to the globe. One can obtain the configuration as mentioned, but this configuration is not natural (it is stressed). The corresponding distortion can be defined by the following tensor field

$$\mathbf{K} = \frac{(r^3-\alpha)^{2/3}}{r^2}\mathbf{e}_r \otimes \mathbf{e}_r + \frac{r}{(r^3-\alpha)^{1/3}} (\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi + \mathbf{e}_\theta \otimes \mathbf{e}_\theta). \qquad (6.1)$$

Here $r = r(\mathfrak{X}^3)$ is the spatial (radial) coordinate, $\alpha = \alpha(\mathfrak{X}^3)$ is the distortion parameter. One can treat the field $\mathbf{K}$ as a deformation gradient corresponding to the map $r = \sqrt[3]{r_0^3 + \alpha}$ defined with respect to the individual fiber $\mathfrak{M}_{\mathfrak{X}^3}$, that can be smoothly transformed to natural spherical configuration by $r = r_0$.

Choose one of the possible stressed configurations as a reference configuration and denote it by $\varkappa_R$. The deformation of the reference configuration to the actual one κ is determined by isochoric diffeomorphism $\varkappa_R \to \varkappa$. Under the conditions of central symmetry this map belongs to the assemblage

$$R = \sqrt[3]{r^3 + A},$$

where $\mathbf{A}$ is the parameter of the assemblage. The corresponding deformation gradient is

$$\mathbf{F} = \frac{r^2}{(r^3-A)^{2/3}}\mathbf{e}_r \otimes \mathbf{e}_r + \frac{(r^3-A)^{1/3}}{r} (\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi + \mathbf{e}_\theta \otimes \mathbf{e}_\theta).$$

Full deformation gradient **H** associated with individual fiber may be obtained as a composition of distortion **K** and deformation gradient **F** determined in consistency of three dimensional accreted solid, i.e.

$$\mathbf{H} = \mathbf{K} \cdot \mathbf{F} = \frac{(r^3-\alpha)^{2/3}}{(r^3+A)^{2/3}} \mathbf{e}_r \otimes \mathbf{e}_r + \frac{(r^3+A)^{1/3}}{(r^3-\alpha)^{1/3}} (\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi + \mathbf{e}_\theta \otimes \mathbf{e}_\theta).$$

Corresponding Cauchy-Green strain measure is

$$\mathbf{G} = \mathbf{H} \cdot \mathbf{H}^* = \frac{(r^3-\alpha)^{4/3}}{(r^3+A)^{4/3}} \mathbf{e}_r \otimes \mathbf{e}_r + \frac{(r^3+A)^{1/3}}{(r^3-\alpha)^{1/3}} (\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi + \mathbf{e}_\theta \otimes \mathbf{e}_\theta).$$

One can obtain the invariants of above mentioned measure by the formulae

$$I_1(\mathbf{G}) = \frac{(r^3-\alpha)^{4/3}}{(r^3+A)^{4/3}} + 2\frac{(r^3+A)^{2/3}}{(r^3-\alpha)^{2/3}}$$

$$I_2(\mathbf{G}) = \frac{(r^3+A)_\theta^{4/3}}{(r^3-\alpha)^{4/3}} + 2\frac{(r^3-\alpha)^{2/3}}{(r^3+A)^{2/3}}.$$

The inverse measure may be calculated as follows

$$\mathbf{G}^{-1} = \frac{(r^3+A)^{4/3}}{(r^3-\alpha)^{4/3}} \mathbf{e}_r \otimes \mathbf{e}_r + \frac{(r^3-\alpha)^{2/3}}{(r^3+A)^{2/3}} (\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi + \mathbf{e}_\theta \otimes \mathbf{e}_\theta).$$

Consider the equations of equilibrium with the assumption that mass force are absent. Then $\nabla \cdot \mathbf{T} = 0$, where **T** is the Cauchy stress tensor, which may be diadic decomposed as:

$$\mathbf{T} = T_{RR} \mathbf{e}_R \otimes \mathbf{e}_R + T_{\theta\theta} (\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi + \mathbf{e}_\theta \otimes \mathbf{e}_\theta).$$

Suppose that the boundary conditions are formulated in the form:

$$\mathbf{n} \cdot \mathbf{T}|_{\Gamma_1} = \mathbf{B}, \ \mathbf{n} \cdot \mathbf{T}|_{\Gamma_2} = 0. \tag{6.2}$$

Here **B** is the pressure on the inner surface of the globe Γ_1, Γ_2 is the outer surface which is the surface of accretion. Assume that this surface is free of stress. The extra condition determines the tension of accreting fibers *f*, i.e.

$$\mathbf{e}_\theta \cdot \mathbf{T} \cdot \mathbf{e}_\theta |_{\Gamma_2} = T_{\theta\theta} |_{\Gamma_2} = f. \tag{6.3}$$

It is important to note that this extra condition is not the boundary condition for the problem stated for the accreted solid. It is the additional condition that permit to determine the distortion $\mathbf{K}(1)$ as the parameter $\alpha(R) = \alpha(R(r(\mathfrak{X})))$ of the boundary value problem.

The equilibrium equations and boundary conditions under the assumption of central symmetry in spherical coordinates are

$$\frac{\partial T_{RR}}{\partial R} + \frac{1}{2}(T_{RR} - T_{\theta\theta}) = 0, \ T_{RR}|_{r=r_0} = B, T_{RR}|_{r=r_1} = 0,$$

where $r_0 = \sqrt[3]{R_0^3 - A}$, $r_1 = \sqrt[3]{R_1^3 - A}, R_0$ and R_1 are inner and outer radii of the globe in actual configuration $\varkappa$.

The Cauchy stress tensor for the incompressible isotropic hyperelastic material may be introduced by relation [6]:

$$\mathbf{T} = -p\mathbf{E} + 2\frac{\partial W}{\partial I_1}\mathbf{G} - 2\frac{\partial W}{\partial I_2}\mathbf{G}^{-1},$$

where $p = p(R)$ is the pressure that can be determined from the equilibrium equation and W is the hyperelastic potential (elastic energy). If the configuration κ_R is utilized as reference configuration then the hyperelastic potential should be written as [8]:

$$W = J_{\mathbf{K}}^{-1} W_{\varkappa_c}(\mathbf{H}, \mathfrak{X}) = J_{\mathbf{K}}^{-1} W_{\varkappa_c}(\mathbf{H} \cdot \mathbf{F}, \mathfrak{X}).$$

Here $W_{\varkappa_c}$ is a potential gauged locally for every individual fiber with respect to natural configuration of this fiber. By virtue of incompressibility of the considered material there is $J_K = 1$. Therefore

$$W = W_{\varkappa_c}(\mathbf{H} \cdot \mathbf{F}, \mathfrak{X}). \tag{6.4}$$

The substitution of **T** from (6.2_2) into the equilibrium equation gives the equation for pressure p:

$$-\nabla p + 2\left(\begin{array}{l} \mathbf{G}\cdot\nabla\dfrac{\partial W}{\partial l_1} - \mathbf{G}^{-1}\cdot\nabla\dfrac{\partial W}{\partial l_2} + \\ \dfrac{\partial W}{\partial l_1}\nabla\cdot\mathbf{G} - \dfrac{\partial W}{\partial l_2}\nabla\cdot\mathbf{G}^{-1} \end{array}\right) = 0.$$

Taking into account that in central symmetrical case $\nabla p = \mathbf{e}_R dp/dR$, arrive at the equation for $p(R)$:

$$-\frac{dp}{dR} + 2\left\{\frac{(r^3-\alpha)^{4/3}}{(r^3+A)^{4/3}}\frac{d}{dR}\frac{\partial W}{\partial l_1} - \frac{(r^3+A)^{4/3}}{(r^3-\alpha)^{4/3}}\frac{d}{dR}\frac{\partial W}{\partial l_1}\right.$$

$$+\frac{\partial W}{\partial l_1}\left[\frac{d}{dR}\frac{(r^3-\alpha)^{4/3}}{(r^3+A)^{4/3}} + \frac{2}{R}\left(\frac{(r^3-\alpha)^{4/3}}{(r^3+A)^{4/3}} - \frac{(r^3+A)^{4/3}}{(r^3-\alpha)^{4/3}}\right)\right]$$

$$\left.-\frac{\partial W}{\partial l_2}\left[\frac{d}{dR}\frac{(r^3+A)^{4/3}}{(r^3-\alpha)^{4/3}} + \frac{2}{R}\left(\frac{(r^3+A)^{4/3}}{(r^3-\alpha)^{4/3}} - \frac{(r^3-\alpha)^{2/3}}{(r^3+A)^{2/3}}\right)\right]\right\} = 0$$

Integral of this equation is obvious

$$p(R') = p_0 + 2\left[\frac{(r^3-\alpha)^{4/3}}{(r^3+A)^{4/3}}\frac{\partial W}{\partial l_1} - \frac{(r^3+A)^{4/3}}{(r^3-\alpha)^{4/3}}\frac{\partial W}{\partial l_2}\right]$$

$$-4\int_{R_0}^{R'}\left\{\frac{\partial W}{\partial l_2}\left(\frac{(r^3+A)^{4/3}}{(r^3-\alpha)^{4/3}} - \frac{(r^3-\alpha)^{4/3}}{(r^3+A)^{4/3}}\right)\right\}$$

$$-\frac{\partial W}{\partial l_1}\left(\frac{(r^3-\alpha)^{4/3}}{(r^3+A)^{4/3}} - \frac{(r^3+A)^{2/3}}{(r^3-\alpha)^{2/3}}\right)\frac{dR}{R}.$$

Here the upper limit of the integration $R' \in (R_0, R_1)$ is the radial coordinate that determines the position of spherical fibre in actual configuration and p_0 is the integrating constant corresponding to hydrostatic pressure. By changing variables $R \mapsto r$ the resulting expression can be transformed to the form

$$p(r') = 2\left[\frac{(r^3-\alpha)^{4/3}}{(r^3+A)^{4/3}}\frac{\partial W}{\partial l_1} - \frac{(r^3+A)^{4/3}}{(r^3-\alpha)^{4/3}}\frac{\partial W}{\partial l_2}\right]$$

$$-4\int_{r_0}^{r'} p(r') = 2\frac{(r^2(A+\alpha)(2r^3+A-\alpha)}{(r^3-\alpha)^{4/3}(r^3+A)^{5/3}}$$

$$\left(\frac{\partial W}{\partial l_2} + \frac{(r^3-\alpha)^{2/3}}{(r^3+A)^{2/3}}\frac{\partial W}{\partial l_1}\right)dr + p_0.$$

The upper limit of integration $r' \in (r_0, r_1)$ determines the position of spherical fiber in reference configuration $\varkappa_R$. Thereafter

$$T_{RR}(r') = -p_0 + 4\int_{r_0}^{r'}\frac{r^2(A+\alpha)(2r^3+A-\alpha)}{(r^3-\alpha)^{4/3}(r^3+A)^{5/3}}$$

$$\left(\frac{\partial W}{\partial l_2} + \frac{(r^3-\alpha)^{2/3}}{(r^3+A)^{2/3}}\frac{\partial W}{\partial l_1}\right)dr.$$

From the boundary condition $T_{RR|r=r_1} = 0$ it follows that

$$T_{RR}(r') = 4\int_{r_1}^{r'}\frac{r^2(A+\alpha)(2r^3+A-\alpha)}{(r^3-\alpha)^{4/3}(r^3+A)^{5/3}}$$

$$\left(\frac{\partial W}{\partial l_2} + \frac{(r^3-\alpha)^{2/3}}{(r^3+A)^{2/3}}\frac{\partial W}{\partial l_1}\right)dr.$$

Circumferential stresses $T_{\theta\theta}$ may be obtained from the equilibrium equation and expressed in terms of T_{RR} as follows

$$T_{\theta\theta} = T_{RR} + \frac{R}{2}\frac{\partial T_{RR}}{\partial R}.$$

This stresses should be equal to prescribed value of tension f (6.3) in the fibre that is on the current boundary

$$T_{\theta\theta|\Gamma_2} = T_{\theta\theta|r=r_1} = T_{RR|r=r_1} + \frac{\sqrt[3]{r_1^3+A}}{2}\frac{\partial T_{RR}}{\partial R}\Big|_{r=r_1}$$

$$= 2\left[\frac{(A+\alpha)(2r_1^3+A-\alpha)}{(r_1^3-\alpha)^{4/3}(r_1^3+A)^{2/3}}\left(\frac{\partial W}{\partial l_2} + \frac{(r_1^3-\alpha)^{2/3}}{(r_1^3+A)^{2/3}}\frac{\partial W}{\partial l_1}\right)\right]$$

$$\Big|_{r=r_1} = f$$

This permits us to find the following dependence $\alpha = \alpha(r_1, A, f)$. The value of parameter may be calculated from the second boundary condition (6.2). The obtained relation may be formulated in the terms of stress rate $\dot{T}_{RR}$. Indeed, so far as, then

$$\frac{R}{2}\frac{dT_{RR}}{dR}\Big|_{r=r_1} = f.$$

Under the conditions of central symmetry $dR = Vdt$, where V is the rate of motion of accretion surface in material manifold (the velocity of accretion surface). Hence

$$\frac{dT_{RR}}{Vdt}\Big|_{r=r_1} = \frac{2}{R}f,$$

or introduction of the notion for the stress tensor on the material surface [9] which can be represented here as the following dyadic decomposition

$$J = f(\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi)$$

and the notion **L** for the curvature tensor of mentioned material surface, that is the sphere with radius R, leads to the equation "in terms of velocities":

$$\dot{T}_{RR}\Big|_{r=r_1} = \frac{2V}{R}f = V(\mathcal{T} : \mathbf{L}).$$

Tensor field of distortion (6.1) induces the linear connection on material manifold that becomes flat space with affine connectivity with nontrivial torsion [5]. The non-zero components of corresponding torsion tensor are

$$\Im^{\varphi}_{\cdot r\varphi} = \Im^{\theta}_{\cdot r\theta} = \frac{1}{3r^3 - 3\alpha}\frac{\partial \alpha(r)}{\partial r},$$

$$\Im^{\varphi}_{\cdot \varphi r} = \Im^{\theta}_{\cdot \theta r} = \frac{1}{3r^3 - 3\alpha}\frac{\partial \alpha(r)}{\partial r}. \tag{6.5}$$

It should be noted that the torsion tensor (5) is equal to zero in the case $\alpha = Const$. This situation corresponds to consensual accretion that generates the solid indistinguishable from the instantly created solid. Otherwise if $\alpha = \alpha(\mathfrak{X}^3)$ then the solid obtained as the result of accretion process lacks natural configuration that can be immersed in Euclidian three dimensional space ε. It means that there are no smooth deformations that release the solid from stresses.

Example 2. In this example the deformations are corresponded layerwise to the transformation of a parallelepiped to a hollow circular cylinder (Fig. 8).

The considered problems are related to the universal deformations of an incompressible medium:

$$\mathbf{x}(\mathbf{X}) = \sqrt{\frac{2(\mathbf{i}_1 \cdot \mathbf{X} + a)}{b}}[\mathbf{i}_1 \cos(b\mathbf{i}_2 \cdot \mathbf{X}) + \mathbf{i}_2 \sin(b\mathbf{i}_2 \cdot \mathbf{X})] + \mathbf{i}_3 \otimes \mathbf{i}_3 \cdot \mathbf{X}. \tag{6.6}$$

Here a and b are parameters of the deformation, which will further be assumed to be constant or differentiable functions of the coordinate r, and $\mathbf{i}_1$, ..., $\mathbf{i}_3$ are the elements of cartesian basis. For deformations of the form (6.6), the position gradient **F** and left Cauchy-Green strain tensor **B** can be represented by the following decomposition ($\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$) are the elements of local covariant basis, correspond to the cylindrical co-ordinates):

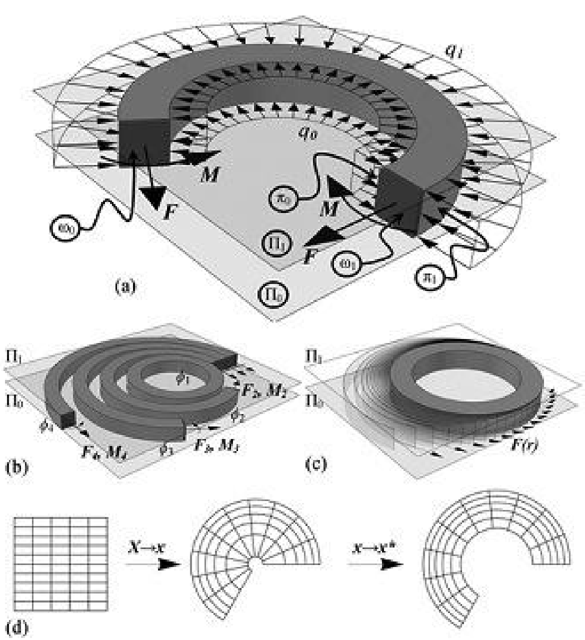

Fig. 8. Shapes of growing cylinder

$$\mathbf{F} = \frac{1}{br}\mathbf{e}_r \otimes \mathbf{i}^1 + b\mathbf{e}_\theta \otimes \mathbf{i}^2 + \mathbf{e}_z \otimes \mathbf{i}^3,$$

$$\mathbf{B} = \frac{1}{b^2 r^2}\mathbf{e}_r \otimes \mathbf{e}_r + b^3 \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_z \otimes \mathbf{e}_z. \qquad (6.7)$$

The Cauchy stress (true stress) **T** can be determined up to the hydrostatic pressure p as follows: $\mathbf{T} = -p\mathbf{I} + J_1\mathbf{B} + J_{-1}\mathbf{B}^{-1}$, where $J_1 = 2\partial W/\partial I_2$ and $J_{-1} = -2\partial W/\partial I_2$ are reaction coefficients of the material. Since the tensor **B** is diagonal for the deformations under study, the stress tensor is also diagonal and its components have the following form:

$$\mathbf{T} = T^{rr}\mathbf{e}_r \otimes + T^{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + T^{zz}\mathbf{e}_z \otimes \mathbf{e}_z,$$

$$T^{rr} = -p + \frac{J_1}{b^2 r^2} + J_{-1}b^2 r^2, T^{\theta\theta} = \frac{p}{r^2}$$

$$+J_1 b^2 + \frac{J_{-1}}{b^2 r^4}, \ T^{zz} = -p + J_1 + J_{-1}.$$

The hydrostatic pressure p can be determined by equation $\nabla \cdot \mathbf{T} = 0$. For the motion (6.6) it can be reduced to the form

$$T^{rr} = \int (J_1 - J_{-1})\left(b^2 r - \frac{1}{b^2 r^3}\right)dr. \qquad (6.8)$$

The above expressions for the stress remain true if the parameter b is assumed to be a function of the variable r. This fact will be used below. But if one assumes that b = constant and takes the explicit expressions for the invariants $I_1 = I_2$ into account, then one can show that $T^{rr} = W + p_0$, where p_0 is the constant of integration.

Deformations of the form (6.6) belong to the class of controllable deformations, i.e., they can be realized by applying specially chosen surface forces to the boundary surfaces. Since the mathematical definition of deformations (6.6) contains the parameters a and b and the stress field is determined up to a constant p_0, it is possible to satisfy the prescribed boundary conditions pointwise on some boundary surfaces and then integrally on the other boundary surfaces, i.e., so that the corresponding net force and the net moment of the surface forces are equal to the prescribed values. The boundary surfaces of the first type are cylindrical surfaces π_0 and π_1 which can be specified analytically by the relations $r = r_0$ and $r = r_1$ with $r_0 = \sqrt{2a/b}$, $r_1 = \sqrt{2(a+\Delta)/b}$. A constant pressure of respective intensity q_0 or q_1 can be prescribed on these surfaces, i.e.,

$$\mathbf{T}\cdot\mathbf{e}^r|_{r=r_0} = q_0\mathbf{e}_r, \ \mathbf{T}\cdot\mathbf{e}^r|_{r=r_1} = q_1\mathbf{e}_r.$$

The expressions obtained above for the radial component of the stress tensor (6.8) permit us to formulate conditions as a system of two nonlinear equations for a and p_0.

$$W|_{r=\sqrt{2a/b}} = q_0 - p_0, \ W|_{r=\sqrt{2(a+\Delta)/b}} = q_1 - p_0. \qquad (6.9)$$

If the reaction coefficients J_1 and J_{-1} are functions of the invariants I_1 and I_2, then system (6.9) must be solved numerically. But if the reaction coefficients are constant, as in the case of the Mooney-Rivlin material $W = C_1(I_1-3) + C_2(I_2-3)$, $C_1 + C_2 = \mu/2$, $C_1 - C_2 = \mu\beta/2$, then it is possible to obtain solutions of system (4) in closed form:

$$a = \frac{\Delta}{2}\left(\sqrt{1 + \frac{1}{\Delta b(\Delta b - \delta)}} - 1\right),$$

$$p_0 = q_0 - \mu\left(ba + \frac{1}{4ba}\right).$$

On the plane boundaries ω_0 and ω_1, the form of the stress distribution is prescribed by the chosen deformation (1), but the net forces on these surface depend on the parameter b and can be chosen so that the conditions on the surfaces ω_0 and ω_1 can be satisfied integrally. The corresponding net force **F** and net moment **M** can be obtained by the formulae

$$\mathbf{F} = \mathbf{e}_{\langle\theta\rangle}h\left\{p_0(r_1 - r_0) + \frac{\mu}{2}\left[b^2\left(r_1^3 - r_0^3\right) + \frac{1}{b^2 r_1} - \frac{1}{b^2 r_0}\right]\right\},$$

$$\mathbf{M} = \mathbf{e}_z\frac{\mu h}{2}\begin{bmatrix} \dfrac{r_1^4 - r_0^4}{4}b^2 + \dfrac{r_1^2 - r_0^2}{2} \\ \left(\dfrac{1}{b^2 r_0 r_1} - b^2 r_0 r_1\right) - \dfrac{1}{b^2}\ln\dfrac{r_1}{r_0} \end{bmatrix}.$$

Discrete Growth

Consider finite sets Ω of bodies that, in the unstressed reference configuration, are parallelepipeds $\iota^k \subset \varepsilon$, $k = 1, 2, ..., n$, deformed by the transformation (6.6) into annual domains $\phi^k \subset \mathrm{E}$, i.e., they are hollow circular cylinders whose bases lie in the planes Π_0 and Π_1. Assume that the sides of the bases Δ^k and h^k of the parallelepipeds ι^k are the same, i.e., $\Delta^k = \Delta$ and $h^k = h$, and their heights H^k are generally distinct. The fact that the results of the deformation are annular domains rather than sectors is expressed by the relation $b^k = 2\Pi/H^k$. Moreover, also assume that the images ϕ^k of the actual configurations of elements of Ω do not intersect pairwise and their union $\bigcup_{k=1}^{n}\phi^k$ is an annular domain in **E**. From the mechanical viewpoint, this condition determines a geometrically ideal contact of elements of Ω along the cylindrical boundary surfaces. The obtained deformed body can be treated as the result of discrete accretion if one assumes that, after such an assembly, there arise constraints on the contacting surfaces, and these constraints prevent elements of Ω from independent deformations. The following types of accretion will be distinguished:

Type 1: Accretion with a Given Actual Geometry

The values of the parameters b^k and a^k are unknown. The radius of the outer boundary surface of the extreme layer is given. Consider a sequence of assemblies of one, two, and more elements of the set Ω, i.e., of the domain $\hat{\phi}^1 = \phi^1$, $\hat{\phi}^2 = \phi^1 \bigcup \phi^1, ...,$ $\hat{\phi}^m = \phi^1 \bigcup \phi^1 \bigcup \cdots \bigcup \phi^m$. Each m-assembly is associated with a system of equations for $m + 1$ unknown variable $\{b^m, \delta^1, \delta^2, ..., \delta^m\}$:

$$\frac{\Delta b^k - \delta^k + \sqrt{(\Delta b^k - \delta^k)^2 + 1 - \delta^k / \Delta b^k}}{(b^k)^2 - b^k\delta^k / \Delta}$$

$$= \frac{\Delta b^{k+1} - \delta^{k+1} + \sqrt{(\Delta b^{k+1} - \delta^{k+1})^2 + 1 - \delta^{k+1} / \Delta b^{k+1}}}{(b^{k+1})^2 - b^{k+1}\delta^{k+1} / \Delta}$$

$$k = 1, ..., m-1, \ \sum\nolimits_{k=1}^{n} \delta^n = \delta,$$

$$= \frac{\Delta b^m - \delta^m + \sqrt{(\Delta b^m - \delta^m)^2 + 1 - \delta^m / \Delta b^m}}{(b^m)^2 - b^m\delta^m / \Delta}.$$

Let us solve the system successively as follows: first, we solve the system of two equations for the 1-assembly $\hat{\phi}^1$ and, thus, determine the elements b^1 and δ^1; then assuming that b^1 is known, we solve the system for the 2-assembly $\hat{\phi}^2$ for the unknowns b^2, δ^1, δ^2, etc. It should be noted that the parameters δ^k do not have to be recalculated anew for each assembly.

Type 2. Accretion with a Given Tension

The values of the parameters b^k and a^k are unknown. The tension F (or the bending moment M) in the extreme layer is given. Consider the sequence of assemblies ϕ^m. Each m-assembly is associated with a system of equations for $m + 1$ unknowns $\{b^m, \delta^1, \delta^2, ..., \delta^m\}$:

$$\frac{\Delta b^k - \delta^k + \sqrt{(\Delta b^k - \delta^k)^2 + 1 - \delta^k / \Delta b^k}}{(b^k)^2 - b^k\delta^k / \Delta}$$

$$= \frac{\Delta b^{k+1} - \delta^{k+1} + \sqrt{(\Delta b^{k+1} - \delta^{k+1})^2 + 1 - \delta^{k+1} / \Delta b^{k+1}}}{(b^{k+1})^2 - b^{k+1}\delta^{k+1} / \Delta}$$

$$k = 1, ..., m-1, \ \sum\nolimits_{k=1}^{n} \delta^n = \delta, \mathbf{e}_{\langle\theta\rangle} F = \mathbf{F}(b^m, \delta^m).$$

Continuous Growth

The parameter b is replaced by a continuous function $b = b(t)$, which will be called the distortion parameter. This allows us to describe continuous accretion realized as a continuous process of adding material surfaces. The tensor field $\mathbf{F} = 1/b(r)r)\,\mathbf{e}_r \otimes \mathbf{i}^1 + b(r)\,\mathbf{e}_r \otimes \mathbf{i}^2 + \mathbf{e}_z \otimes \mathbf{i}^3$ cannot now be the gradient of any deformation.

Type 1. Accretion with a Given Position of the Growth Surface

The volume of the accreted material is given as a function of a time-like parameter $\tau \in (0, t)$, i.e., $V = V(\tau)$. The position of the growth boundary $r_1(\tau)$ is given. Just as in the preceding case, the pressures $q_0(\tau)$ and $q_1(\tau)$ on the boundary surfaces π_0 and π_1 are given. It is required to determine the distortion function and the functions $b(r)$ and the functions $r_0(\tau)$ and $p_0(\tau)$. They can be found by solving the integral equation

$$\mu\int_{\sqrt{r_1(\tau)^2 - V/(\pi h)}}^{r_1(\tau)} \left[b^2(\zeta)\zeta - \frac{1}{b^2(\zeta)\zeta^3} \right] d\zeta + q_0(\tau) = q_1(\tau).$$

Type 2. Accretion with a Given Tension

Just as above, the volume of the accreted material $V = V(\tau)$, the pressures $q_0(\tau)$ and $q_1(\tau)$ on the boundary surfaces π_0 and π_1, and the surface stresses (tension) $g(\tau)$ on the growth boundary are given. It is required to find the distortion function $b(r)$ and $r_0(\tau)$, $r_1(\tau)$, $p_0(\tau)$. They are determined by the equations

$$q_1 + \mu\left(b^2 r^2 - \frac{1}{b^2 r^2} \right) = g(\tau),$$

$$\mu\int_{\sqrt{r_1(\tau)^2 - V/(\pi h)}}^{r_1(\tau)} \left[b^2(\zeta)\zeta - \frac{1}{b^2(\zeta)\zeta^3} \right] d\zeta + q_0(\tau) = q_1(\tau).$$

6. Conclusion

As a result of the surface growth of a solid an "artificial" inhomogeneity is created. In the framework of continuum mechanics it may be treated as a bundle of smooth manifolds with non-Euclidean material connection. The torsion of this connection plays the role of the measure of such inhomogeneity. The torsion depends on initial (implant) distortion that may be found by solving the boundary-value problem for the solid with the material boundary. In some special cases (particularly for incompressible material) it can be done analytically.

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