

Research Paper

On the Constitutive Equations of Viscoelastic Micropolar Plates and Shells

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Within the framework of the micropolar theory we discuss the constitutive equations of viscoelastic micropolar thin-walled structures, i.e. micropolar plates and shells. Starting from the linear viscoelastic micropolar continuum and using the correspondence principle of the linear viscoelasticity we extend the procedure of reduction of three-dimensional equilibrium equations of elastic shell-like solids to the case of viscoelastic behaviour.

Key Words: Micropolar Continuum; Cosserat Continuum; Viscoelasticity; Through-The-Thickness Integration; Constitutive Equations; Plates and Shells

1. Introduction

The model of the Cosserat or micropolar continuum found recently new applications for modelling of behaviour of materials and structures of complex inner structure, see [12] and references therein. Since the seminal paper by Ericksen and Truesdell [14] the Cosserat model has found numerous applications in construction of various generalized models for beams, plates, and shells, see the review and bibliography in [1]. Within the framework of the direct approach applied in [14], the shell is modelled as a deformable surface at each point of which a set of directors additionally is attached. In the literature are also known theories of plates and shells and theories based on the reduction of three-dimensional micropolar continuum equations, see for example [15, 19, 21, 22, 24], and the review [1], where various averaging procedures in the thickness direction together with the approximation of the displacements and rotations or the force and moment stresses in the thickness direction are applied. In major cases, these considerations are restricted by the elastic behaviour.

For the linear theory of viscoelasticity the application of the correspondence principle gives the possibility to derive the theory of viscoelasticity in the case of thin-walled structures such as plates and shells.

In this paper, we extend the through-the-thickness integration procedure applied in [2] to plates and shells made of viscoelastic micropolar material. The interest to the theory of thin-walled structures made of viscoelastic micropolar continuum is based on prospective applications of this theory to the mechanics of plates and shells made of materials with complex inner structure, for example cellular materials and foams for which the micropolar model is used, see [10, 16] among others. We consider here the variant of plates and shell theory based on six kinematically independent variables, i.e. translations and rotations. This theory can be derived using the direct approach [12, 13, 20] or based on the reduction of the three-dimensional motion equations [9, 18]. The mathematical study of boundary-value problems was performed [4, 5, 11] while various solutions and finite-element calculations are presented [7-9, 13].

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2. Basic Relations of the Viscoelastic Cosserat Continuum

Following [15] we recall the governing equations of the linear micropolar viscoelasticity. Let the micropolar body occupies the domain $\mathcal{V} \in \mathbb{R}^3$. The infinitesimal deformations of the micropolar media are described by two vectorial fields. The first one is the vector of translation $\mathbf{u}$ and the second field is the vector of microrotation $\mathbf{\vartheta}$ given as vector-functions of the position vector $\mathbf{x}$ and time t . From the physical point of view, $\mathbf{u}$ describes a displacement of the particle of a micropolar body while $\mathbf{\vartheta}$ corresponds to the particle rotation. The quasistatic deformations of a micropolar body are described by the equilibrium equations

$$\begin{aligned} \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f} &= \mathbf{0}, \quad \nabla \cdot \boldsymbol{\mu} + \boldsymbol{\sigma}_x + \rho \mathbf{l} = \mathbf{0}, \\ \mathbf{x} &\in \mathcal{V} \end{aligned} \quad (1)$$

where ∇ is the three-dimensional nabla operator, $\mathbf{f}$ and $\mathbf{l}$ are the mass forces and the mass couples vectors, ρ is the density, $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ are the stress tensor and the couple stress tensor, respectively, and $\boldsymbol{\sigma}_x$ denotes the vectorial invariant of the second-order tensor $\boldsymbol{\alpha}$, see [17, 23]. Equation (1)₁ represents the local form of the balance of momentum while Eq. (1)₂ is the balance of moment of momentum.

The static boundary conditions have the following form

$$\mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{t}^0, \quad \mathbf{n} \cdot \boldsymbol{\mu} = \mathbf{m}^0 \quad \text{at } S_f. \quad (2)$$

Here $\mathbf{t}^0$ and $\mathbf{m}^0$ are the surface forces and the surface couples acting on the part of the surface S_f of the micropolar body, $S = S_u \cup S_f \equiv \partial \mathcal{V}$. The kinematic boundary conditions consist of the following relations

$$\mathbf{u} = \mathbf{u}^0, \quad \mathbf{\vartheta} = \mathbf{\vartheta}^0 \quad \text{at } S_u, \quad (3)$$

where $\mathbf{u}^0$ and $\mathbf{\vartheta}^0$ are given functions at S_u . The linear strain measures, i.e. the linear stretch tensor $\boldsymbol{\varepsilon}$ and the linear wryness tensor $\boldsymbol{\omega}$, are given by the relations

$$\boldsymbol{\varepsilon} = \nabla \mathbf{u} + \mathbf{\vartheta} \times \mathbf{I}, \quad \boldsymbol{\omega} = \nabla \mathbf{\vartheta}, \quad (4)$$

where $\mathbf{I}$ is the unit three-dimensional tensor and $\times$ the cross product. Let us consider the viscoelastic micropolar isotropic material. For the sake of simplicity

we restrict ourselves by the constitutive equations of the differential type. This means that the constitutive equations are

$$P_0(\partial_t) \boldsymbol{\sigma} = P_1(\partial_t) \boldsymbol{\varepsilon} + P_2(\partial_t) \boldsymbol{\varepsilon}^T + P_3(\partial_t) \mathbf{I} \text{tr} \boldsymbol{\varepsilon}, \quad (5)$$

$$Q_0(\partial_t) \boldsymbol{\mu} = Q_1(\partial_t) \boldsymbol{\omega} + Q_2(\partial_t) \boldsymbol{\omega}^T + Q_3(\partial_t) \mathbf{I} \text{tr} \boldsymbol{\omega}. \quad (6)$$

In Eqs (5) and (6) stands for ∂_t the derivative with respect to time, $P_0, P_1, P_2, P_3, Q_0, Q_1, Q_2, Q_3$, are the polynomials.

Using the Laplace transform of a function $f(t)$

$$\bar{f}(p) = \int_0^\infty f(t) e^{-pt} dt,$$

the constitutive equations (5) and (6) become

$$\begin{aligned} \bar{\boldsymbol{\sigma}} &= \lambda \mathbf{I} \text{tr} \bar{\boldsymbol{\varepsilon}} + \bar{\mu} \bar{\boldsymbol{\varepsilon}}^T + (\bar{\mu} + \bar{\kappa}) \bar{\boldsymbol{\varepsilon}}, \\ \bar{\boldsymbol{\mu}} &= \alpha \mathbf{I} \text{tr} \bar{\boldsymbol{\omega}} + \bar{\beta} \bar{\boldsymbol{\omega}}^T + \bar{\gamma} \bar{\boldsymbol{\omega}}, \end{aligned} \quad (7)$$

where

$$\lambda = \frac{P_3(p)}{P_0(p)}, \quad \bar{\mu} = \frac{P_2(p)}{P_0(p)}, \quad \bar{\kappa} = \frac{P_1(p) - P_2(p)}{P_0(p)}, \quad (8)$$

$$\alpha = \frac{Q_3(p)}{Q_0(p)}, \quad \bar{\beta} = \frac{Q_2(p)}{Q_0(p)}, \quad \bar{\gamma} = \frac{Q_1(p)}{Q_0(p)}, \quad (9)$$

are the Laplace transforms of the relaxation functions of the viscoelastic micropolar material. In fact, (7) with (8), (9) coincide up to notations with the constitutive equations of anisotropic linear micropolar solid, see [15]. Substituting Eqs (7) into Eqs (1) one may derive the equilibrium equations in terms of the kinematical fields $\bar{\mathbf{u}}$ and $\bar{\mathbf{\vartheta}}$. For homogeneous micropolar bodies these equations are

$$\begin{aligned} (\lambda + \mu) \nabla \nabla \cdot \bar{\mathbf{u}} + (\mu + \kappa) \nabla \cdot \nabla \bar{\mathbf{u}} + \kappa \nabla \times \bar{\mathbf{\vartheta}} + \rho \bar{\mathbf{f}} &= \mathbf{0}, \\ (\alpha + \beta) \nabla \nabla \cdot \bar{\mathbf{\vartheta}} + \bar{\gamma} \nabla \cdot \nabla \bar{\mathbf{\vartheta}} + \kappa \nabla \times \bar{\mathbf{u}} - 2\bar{\kappa} \bar{\mathbf{\vartheta}} + \rho \bar{\mathbf{\vartheta}} &= \mathbf{0}. \end{aligned} \quad (10)$$

Remark

In a similar manner other linear viscoelastic constitutive equations (integral equation, complex moduli equations) can be introduced and applied to the representation of the viscoelastic micropolar behaviour.

In what follows we consider the reduction of three-dimensional equilibrium equations to the two-dimensional ones and discuss the corresponding two-dimensional constitutive equations.

3. Micropolar Plate and Shell Equations

Within the framework of the linear theory of micropolar plates and shells which is also named as the Six-Parameter Theory of shells we consider a micropolar shell as a two-dimensional Cosserat continuum, i.e. as a deformable surface $\mathcal{M}$ with 6 degrees of freedom. Each material point of the surface is kinematically similar to an infinitesimal rigid body with three translational and three rotational degrees of freedom. Hence, the deformations of the micropolar plate or shell are described by the translation vector $\mathbf{v}$ and the rotation vector $\boldsymbol{\theta}$ which are defined at the base surface $\mathcal{M}$. Using the direct approach the basics of the linear theory of micropolar shells are summarized in [13, 17], see also Appendix D in [12]. The governing equations of the micropolar shells and plates coincide with the relations of the general 6-parameter nonlinear shell theory presented in [9, 18] in the case of small deformations derived using the through-the-thickness integration procedure.

The balance of momentum and the balance of moment of momentum are formulated as follows

$$\nabla_s \cdot \mathbf{T} + \mathbf{q} = \mathbf{0}, \quad \nabla_s \cdot \mathbf{M} + \mathbf{T}_\times + \mathbf{c} = \mathbf{0}, \quad (11)$$

where $\mathbf{q}$ and $\mathbf{c}$ are the surface loads (forces and moments), $\mathbf{T}$ and $\mathbf{M}$ are the resultant force stress and couple stress tensors, and ∇_s is the surface nabla differential operator. The tensors $\mathbf{T}$ and $\mathbf{M}$ have the properties

$$\mathbf{n} \cdot \mathbf{T} = \mathbf{0}, \quad \mathbf{n} \cdot \mathbf{M} = \mathbf{0}. \quad (12)$$

Hence, $\mathbf{T}$ and $\mathbf{M}$ take the form

$$\begin{aligned} \mathbf{T} &= T_{\alpha\beta} \mathbf{i}^\alpha \otimes \mathbf{i}^\beta + T_{\alpha 3} \mathbf{i}^\alpha \otimes \mathbf{n}, \\ \mathbf{M} &= M_{\alpha\beta} \mathbf{i}^\alpha \otimes \mathbf{i}^\beta + M_{\alpha 3} \mathbf{i}^\alpha \otimes \mathbf{n} \\ (\alpha, \beta &= 1, 2), \end{aligned} \quad (13)$$

where $\mathbf{i}^\alpha$ are the base vectors on the shell surface. Obviously, in this theory of shells the action of the

drilling moment $M_{\alpha 3}$ is taken into account. For example, such possibility may be useful to describe the interaction of the shell and the rigid body or for the description of the deformations of multifolded plates, see [9].

The static and kinematic boundary conditions take the form

$$\begin{aligned} \mathbf{v} \cdot \mathbf{T} &= \mathbf{t}_s^*, \quad \mathbf{v} \cdot \mathbf{M} = \mathbf{m}_s^* \text{ along } \mathcal{C}_f, \\ \mathbf{v} &= \mathbf{v}^0, \quad \boldsymbol{\theta} = \boldsymbol{\theta}^0 \text{ along } \mathcal{C}_u. \end{aligned} \quad (14)$$

Here $\mathbf{v}$ is the vector of the unit normal to, $\mathcal{C}_f$, $\mathbf{v} \cdot \mathbf{n} = 0$, $\mathbf{t}_s^*$ and $\mathbf{m}_s^*$ are external force and couple vectors acting along the boundary $\mathcal{C}_f$, while $\mathbf{u}^0$ and $\boldsymbol{\theta}^0$ are given functions describing the displacements and rotations of the boundary, $\mathcal{C}_u$, respectively, $\mathcal{C} = \mathcal{C}_u \cup \mathcal{C}_f \equiv \partial \mathcal{M}$.

The linear surface strain measures are

$$\boldsymbol{\varepsilon} = \nabla_s \mathbf{v} + \mathbf{A} \times \boldsymbol{\theta}, \quad \boldsymbol{\kappa} = \nabla_s \boldsymbol{\theta}, \quad (15)$$

where $\mathbf{A} \equiv \mathbf{I} - \mathbf{n} \otimes \mathbf{n}$ is the two-dimensional or surface unit tensor.

We restrict ourselves again by the constitutive equations for $\mathbf{T}$ and $\mathbf{M}$ of differential type. We assume the following relations

$$\begin{aligned} A_0(\partial_t) \mathbf{T} &= \mathbf{A} A_1(\partial_t) \text{tr} \boldsymbol{\varepsilon}_\parallel + A_2(\partial_t) \boldsymbol{\varepsilon}_\parallel^T \\ &+ A_3(\partial_t) \boldsymbol{\varepsilon}_\parallel + A_4(\partial_t) \boldsymbol{\varepsilon} \cdot \mathbf{n} \otimes \mathbf{n}, \end{aligned} \quad (16)$$

$$\begin{aligned} B_0(\partial_t) \mathbf{M} &= \mathbf{A} B_1(\partial_t) \text{tr} \boldsymbol{\kappa}_\parallel + B_2(\partial_t) \boldsymbol{\kappa}_\parallel^T \\ &+ B_3(\partial_t) \boldsymbol{\kappa}_\parallel + B_4(\partial_t) \boldsymbol{\kappa} \cdot \mathbf{n} \otimes \mathbf{n}, \end{aligned} \quad (17)$$

Here $\boldsymbol{\varepsilon}_\parallel = \boldsymbol{\varepsilon} \cdot \mathbf{A}$, $\boldsymbol{\kappa}_\parallel = \boldsymbol{\kappa} \cdot \mathbf{A}$ and $A_k(p)$, $B_k(p)$, $k = 0, 1, 2, 3$, are polynomials. We call the constitutive equations for the shell (16) and (17) the Maxwell-type, if $A_0 = 1 + a_0 p$, $B_0 = 1 + b_0 p$, where $1/a_0$ and $1/b_0$ are the relaxation time for the stress resultants and for couple stresses, respectively, and if other polynomials are constants. We call (16) and (17) the Voigt-type constitutive equation, if $A_0 = B_0 = 1$ while other polynomials are linear functions of p .

Using the Laplace transform these constitutive equations become

$$\bar{\mathbf{T}} = \alpha_1 \mathbf{A} \text{tr} \bar{\boldsymbol{\epsilon}}_{||} + \alpha_2 \bar{\boldsymbol{\epsilon}}_{||}^{-T} + \alpha_3 \bar{\boldsymbol{\epsilon}}_{||} + \alpha_4 \bar{\boldsymbol{\epsilon}} \cdot \mathbf{n} \otimes \mathbf{n}, \quad (18)$$

$$\bar{\mathbf{M}} = \beta_1 \mathbf{A} \text{tr} \bar{\boldsymbol{\kappa}}_{||} + \beta_2 \bar{\boldsymbol{\kappa}}_{||}^{-T} + \beta_3 \bar{\boldsymbol{\kappa}}_{||} + \beta_4 \bar{\boldsymbol{\kappa}} \cdot \mathbf{n} \otimes \mathbf{n}, \quad (19)$$

where

$$\alpha_i = \frac{A_i(p)}{A_0(p)}, \quad \beta_i = \frac{B_i(p)}{B_0(p)}, \quad i = 1, 2, 3, 4.$$

The final step in the theory of micropolar viscoelastic shells is the construction of the polynomials A_k and B_k on the base of the dimension reduction of the three-dimensional continuum or by experiments. Further we consider the derivation of the relaxation functions α_i and β_i using the through-the-thickness integration procedure presented in [2, 3].

4. Reduction of Three-Dimensional Constitutive Equations to Two-Dimensional Ones

For simplicity let us consider the undeformed plane geometry, i.e. we restrict ourselves by the theory of plates. The case of shells can be discussed by similar way. Using the correspondence principle [6] from [2] it follows

$$\bar{\mathbf{T}} = \langle \mathbf{A} \cdot \bar{\boldsymbol{\sigma}} \rangle, \quad \bar{\mathbf{M}} = \langle \mathbf{A} \cdot \bar{\boldsymbol{\mu}} \rangle - \langle \mathbf{A} \cdot z \bar{\boldsymbol{\sigma}} \times \mathbf{n} \rangle, \quad (20)$$

where z is the coordinate along the thickness coordinate axis. As a result the two-dimensional relaxation parameters are given by the relations

$$\bar{\alpha}_1 = \Lambda h \equiv \frac{\bar{\lambda}(2\bar{\mu} + \bar{\kappa})}{\bar{\lambda} + 2\bar{\mu} + \bar{\kappa}} h, \quad \bar{\alpha}_2 = \bar{\mu} h, \quad (21)$$

$$\bar{\alpha}_3 = (\bar{\mu} + \bar{\kappa}) h, \quad \bar{\alpha}_4 = (\bar{\mu} + \bar{\kappa}) h,$$

$$\bar{\beta}_1 = \bar{\alpha} h - \bar{\mu} \frac{h^3}{12}, \quad \bar{\beta}_2 = \bar{\beta} h - \Lambda \frac{h^3}{12},$$

$$\bar{\beta}_3 = \bar{\gamma} h + (2\bar{\mu} + \bar{\kappa} + \Lambda) \frac{h^3}{12}, \quad \bar{\beta}_4 = \bar{\gamma} h, \quad (22)$$

where h is the plate thickness. From Eqs (21) and (8) it follows the relations for the tangential relaxation functions

$$\frac{P_3}{P_0} \frac{P_1 + P_2}{P_1 + P_2 + P_3} h = \frac{A_1}{A_0}, \quad \frac{P_2}{P_0} h = \frac{A_2}{A_0},$$

$$\frac{P_1}{P_0} h = \frac{A_3}{A_0} = \frac{A_4}{A_0}. \quad (23)$$

It is clear that determination of the polynomials $A_k(p)$ from the latter equations cannot be performed uniquely. The solution of (23) results in

$$A_0 = P_0(P_1 + P_2 + P_3), \quad A_1 = P_3(P_1 + P_2)h, \quad (24)$$

$$A_2 = P_2(P_1 + P_2 + P_3)h, \quad A_3 = A_4 = P_1(P_1 + P_2 + P_3)h. \quad (25)$$

In a similar way from Eqs (22) and (9) we obtain the equations for the bending relaxation functions

$$\frac{B_1}{B_0} = \frac{Q_3}{Q_0} h - \frac{P_2}{P_0} \frac{h^3}{12}, \quad \frac{B_2}{B_0} = \frac{Q_2}{Q_0}$$

$$h - \frac{P_3(P_1 + P_2)}{P_0(P_1 + P_2 + P_3)} \frac{h^3}{12},$$

$$\frac{B_3}{B_0} = \frac{Q_1}{Q_0} h + \frac{(P_1 + P_2)(P_1 + P_2 + 2P_3)}{P_0(P_1 + P_2 + P_3)} \frac{h^3}{12},$$

$$\frac{B_4}{B_0} = \frac{Q_1}{Q_0} h.$$

From this it follows

$$B_0 = P_0 Q_0 (P_1 + P_2 + P_3),$$

$$B_1 = \left(Q_3 P_0 h - P_2 Q_0 \frac{h^3}{12} \right) (P_1 + P_2 + P_3),$$

$$B_2 = Q_2 P_0 (P_1 + P_2 + P_3) h - P_3 Q_0 (P_1 + P_2) \frac{h^3}{12},$$

$$B_3 = Q_1 P_0 (P_1 + P_2 + P_3) h + Q_0 (P_1 + P_2) (P_1 + P_2 + 2P_3) \frac{h^3}{12},$$

$$B_4 = Q_1 P_0 (P_1 + P_2 + P_3) h.$$

Hence, the differential operators in the constitutive equation for the resultant stress and couple stress tensors $\mathbf{T}$ and $\mathbf{M}$ are more complicated than in the three-dimensional case because the order of derivatives of A_i and B_i is higher than the order of P_k

and Q_k , in general. This means that the viscoelastic properties of the two-dimensional structure are more complicated than their three-dimensional counterparts.

Remark

In the case of shells the stress resultant and couple stress tensors depend on as follows, see [17],

$$\begin{aligned}\bar{T} &= \langle (A - zB)^{-1} \cdot \bar{\sigma} \rangle, \bar{M} = \langle (A - zB)^{-1} \cdot \bar{\mu} \rangle \\ &\quad - \langle (A - zB)^{-1} \cdot z \bar{\sigma} \times n \rangle,\end{aligned}\quad (26)$$

where $B = -\nabla_s n$ is the curvature tensor of the base shell surface. This means that for a curved surface one should obtain more complicated relaxation properties for two-dimensional theories of shell and plate.

5. Examples

Let us consider the simple case of the viscoelastic micropolar constitutive equations and the corresponding two-dimensional equations.

5.1. Maxwell Model

For the Maxwell-type model the polynomials in Eqs (5) and (6) have the form

$$\begin{aligned}P_0 &= 1 + p_0 p, P_1, P_2, P_3 = \text{const.} \\ Q_0 &= 1 + q_0 p.\end{aligned}\quad (27)$$

For the stress resultant tensor T we obtain the constitutive equation of Maxwell-type, i.e. A_0 is the linear polynomial only, while the A_1, A_2, A_3 and A_4 are constants. For the couple stress tensor M we obtain more complicated form of constitutive equation. The simplification is possible if $p_0 = q_0 = 1/\tau$. In this case we use the following polynomials

$$\begin{aligned}B_0 &= P_0(P_1 + P_2 + P_3), \\ B_1 &= \left(Q_3 h - P_2 \frac{h^3}{12} \right) (P_1 + P_2 + P_3), \\ B_2 &= Q_2(P_1 + P_2 + P_3)h - P_3(P_1 + P_2) \frac{h^3}{12}, \\ B_3 &= Q_1(P_1 + P_2 + P_3)h + (P_1 + P_2) \\ &\quad (P_1 + P_2 + 2P_3) \frac{h^3}{12},\end{aligned}$$

$$B_4 = Q_1(P_1 + P_2 + P_3)h,$$

where B_0 is a polynomial while, B_1, B_2, B_3 and B_4 are constants. Thus, we obtain the Maxwell-type two-dimensional constitutive equations in the case when the relaxation times for the stress and couple stress coincide, i.e. when $P_0 \equiv Q_0$, and we have one relaxation time τ . In the general case we obtain more general constitutive equation for M .

5.2. Voigt Model

Let us assume the following form of polynomials in Eqs (5) and (6)

$$\begin{aligned}P_0 &= 1, P_1 = P_1^0(1 + p_1 p), \\ P_2 &= P_2^0(1 + p_2 p), P_3 = P_3^0(1 + p_3 p),\end{aligned}\quad (28)$$

$$\begin{aligned}Q_0 &= 1, Q_1 = Q_1^0(1 + q_1 q), \\ Q_2 &= Q_2^0(1 + q_2 q), Q_3 = Q_3^0(1 + q_3 q),\end{aligned}\quad (29)$$

Here the corresponding polynomials $A_i, i = 1, 2, 3, 4$ are linear functions of p . This means that for T we have the viscoelastic equations which are similar to the standard viscoelastic model [6]. For M we again obtain a more complicated model because in the general case. B_0 is a linear function while B_1, B_2, B_3 and B_4 are quadratic functions of p .

The simplification of the two-dimensional constitutive equation for M is possible after assumption

$$p_1 = p_2 = p_3 = p_4 = q_1 = q_2 = q_3 = q_4 = 1/\tau.$$

In this case we have

$$\begin{aligned}B_0 &= (P_1^0 + P_2^0 + P_3^0), \\ B_1 &= \left(Q_3^0 h - P_2^0 \frac{h^3}{12} \right) (P_1^0 + P_2^0 + P_3^0) \left(1 + \frac{p}{\tau} \right), \\ B_2 &= \left[Q_2^0 (P_1^0 + P_2^0 + P_3^0) h - P_3^0 (P_1^0 + P_2^0) \frac{h^3}{12} \right] \left(1 + \frac{p}{\tau} \right), \\ B_3 &= \left[Q_1^0 (P_1^0 + P_2^0 + P_3^0) h + (P_1^0 + P_2^0) (P_1^0 + P_2^0 + 2P_3^0) \frac{h^3}{12} \right] \\ &\quad \left(1 + \frac{p}{\tau} \right),\end{aligned}$$

$$B_4 = Q_1^0 (P_1^0 + P_2^0 + P_3^0) h \left(1 + \frac{P}{\tau} \right)$$

As for the Maxwell-type model we conclude that the two-dimensional constitutive equations of Voigt-type exist as a special case of the three-dimensional viscoelastic behaviour, i.e. with the same relaxation parameter τ .

Conclusion

We discussed the two-dimensional constitutive equations for resultant force stress and couple stress tensors derived from the constitutive equations of three-dimensional viscoelastic Cosserat (micropolar) continuum. The presented results demonstrate how the viscoelastic properties of three-dimensional

continuum inherit in the constitutive equations for plates and shells. Within the framework of the linear micropolar viscoelasticity with constitutive equations of differential type we show that 2D relaxation functions of shells have more complicated structures than the relaxation function of the bulk material. In particular, even for homogeneous shells the spectrum of relaxation time does not coincide with the spectrum of the bulk material. For inhomogeneous shells the spectrum may depend also on the structure of the shell in the thickness direction and its curvature.

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