AN EXTENSION OF MEIJER'S G-FUNCTION

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This paper gives a generalization of the Meijer's G-function to two variables. The generalized function yields as special cases the Appell's double hypergeometric functions and the Kampé de Feriét's hypergeometric functions of higher order, besides all those functions which can be defined through the G-symbol.

1. Introduction

Meijer in 1941 defined his G-function by means of a Mellin-Barnes type of integral in the form

$$G^{m,n}_{p,q}\left(x \middle| \begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}\right)$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod\limits_{j=1}^{m} \Gamma[b_j-s] \prod\limits_{j=1}^{n} \Gamma[1-a_j+s]}{\prod\limits_{j=m+1}^{q} \Gamma[1-b_j+s] \prod\limits_{j=n+1}^{p} \Gamma[a_j-s]} x^s \, ds,$$

where an empty product is interpreted as 1, $0 \leqslant m < q$, $0 \leqslant n \leqslant p$, and the parameters are such that no pole of $\Gamma[b_j-s]$, $j = 1, 2, \ldots, m$ coincides with any pole of $\Gamma[1-a_k+s]$, $k = 1, 2, \ldots, n$. The path of integration* L runs from $-i\infty$ to $+i\infty$ so that all poles of $\Gamma[b_j-s]$, $j = 1, 2, \ldots, m$ are to the right, and all the poles of $\Gamma[1-a_k+s]$, $k = 1, 2, \ldots, n$ to the left of L. The integral converges for $p+q < 2(m+n)$ and $|\arg x| < \pi(m+n-\frac{1}{2}p-\frac{1}{2}q)$. The importance of the G-function lies in the great many special functions that can be represented as its particular cases.

The object of this paper is to define a G-function of two variables which not only includes the Meijer's G-function as a particular case but also most of the known functions of two variables, e.g. the Appell's functions F_1, F_2, F_3 and F_4, the Whittaker functions of two variables, etc. Besides including the known functions of two variables as particular cases, it leaves the possibility of defining, through this new G-symbol of two variables, a great many special functions of two variables not hitherto mentioned.

The following notation is used throughout the present paper.

* For all other possible paths of integration, *see* Erdèlyi *et al.* (1953).

Let

$$(a)_m = a(a+1)(a+2)\dots(a+m-1)\,;\ (a)_0 = 1.$$

Also, the symbol (α_p) denotes the sequence of elements $\alpha_1, \alpha_2, \dots, \alpha_p$ and $(\alpha_{m}, _p)$ denotes the sequence $\alpha_m, \alpha_{m+1}, \dots, \alpha_p$.

Thus the double hypergeometric function of higher order in two variables

$$F\left[\begin{array}{c|l|c} p & \epsilon_1, \epsilon_2, \dots, \epsilon_p & \\ t & \gamma_1, \gamma_1', \dots, \gamma_t, \gamma_t' & x \\ s & \delta_1, \delta_2, \dots, \delta_s & y \\ q & \beta_1, \beta_1', \dots, \beta_q, \beta_q' & \end{array}\right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\epsilon_1)_{m+n}\dots(\epsilon_p)_{m+n}(\gamma_1)_m(\gamma_1')_n\dots(\gamma_t)_m(\gamma_t')_n}{(\delta_1)_{m+n}\dots(\delta_s)_{m+n}(\beta_1)_m(\beta_1')_n\dots(\beta_q)_m(\beta_q')_n(1)_m(1)_n} x^m y^n,$$

where $p+t \leqslant s+q+1$ shall be abbreviated to

$$F\left[\begin{array}{c|l|c} p & (\epsilon_p) & \\ t & (\gamma_t)\,;\ (\gamma_t') & x \\ s & (\delta_s) & y \\ q & (\beta_q)\,;\ (\beta_q') & \end{array}\right].$$

The series for the F-functions converges absolutely for all complex values of x and y if $p+t < s+q+1$. In case $p+t = s+q+1$, it converges absolutely for all complex values of x and y such that (Ragab 1963)

$$|x|+|y| < \min(1, 2^{s-p+1}).$$

2. The $G_{p,\,q,\,s,\,t}^{m_1,\,m_2,\,n,\,\nu_1,\,\nu_2}\begin{bmatrix}x\\y\end{bmatrix}$ Function

Consider the double contour integral

$$I = \frac{-1}{4\pi^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \Phi(\xi+\eta)\Psi(\xi,\eta)x^\xi y^\eta d\xi d\eta, \tag{1}$$

where

$$\Psi(\xi,\eta) = \frac{\prod_{j=1}^{m_1}\Gamma[\beta_j-\xi]\prod_{j=1}^{\nu_1}\Gamma[\gamma_j+\xi]\prod_{j=1}^{m_2}\Gamma[\beta_j'-\eta]\prod_{j=1}^{\nu_2}\Gamma[\gamma_j'+\eta]}{\prod_{j=m_1+1}^{q}\Gamma[1-\beta_j+\xi]\prod_{j=\nu_1+1}^{t}\Gamma[1-\gamma_j-\xi]\prod_{j=1+m_2}^{q}\Gamma[1-\beta_j'-\eta]\prod_{j=1+\nu_2}^{t}\Gamma[1-\gamma_j'-\eta]}$$

$$\Phi(\xi+\eta) = \frac{\prod_{j=1}^{n}\Gamma[1-\epsilon_j+\xi+\eta]}{\prod_{j=n+1}^{p}\Gamma[\epsilon_j-\xi-\eta]\prod_{j=1}^{s}\Gamma[\delta_j+\xi+\eta]},$$

and $0 \leqslant m_1 \leqslant q,\ 0 \leqslant m_2 \leqslant q,\ 0 \leqslant \nu_1 \leqslant t,\ 0 \leqslant \nu_2 \leqslant t,\ 0 \leqslant n \leqslant p.$

The sequence of parameters (β_{m_1}), (β'_{m_2}), (γ_{ν_1}), (γ'_{ν_2}) and (ϵ_n) are such that none of the poles of the integrand coincide. The paths of integration are indented, if necessary, in such a manner that all the poles of $\Gamma[\beta_j-\xi]$, $j=1, 2, \ldots, m_1$ and $\Gamma[\beta'_k-\eta]$, $k=1, 2, \ldots, m_2$ lie to the right, and those of $\Gamma[\gamma_j+\xi]$, $j=1, 2, \ldots, \nu_1$ and $\Gamma[\gamma'_k+\eta]$, $k=1, 2, \ldots, \nu_2$ and $\Gamma[1-\epsilon_j+\xi+\eta]$, $j=1, 2, \ldots, n$ lie to the left of the imaginary axis.
The integral (2.1) converges if

$$(2)\qquad \begin{cases} p+q+s+t < 2(m_1+\nu_1+n), \\ p+q+s+t < 2(m_2+\nu_2+n), \\ \text{and} \\ |\arg x| < \pi[m_1+\nu_1+n-\frac{1}{2}(p+q+s+t)], \\ |\arg y| < \pi[m_2+\nu_2+n-\frac{1}{2}(p+q+s+t)]. \end{cases}$$

Evaluating (2.1) by considering the residues at the poles of integrand that lie to the right of imaginary axis, we have

$$(3)\qquad I = \sum_{h=1}^{m_1}\sum_{k=1}^{m_2} x^{\beta_h} y^{\beta'_k} \frac{\prod_{j=1}^{\nu_1}\Gamma[\gamma_j+\beta_h]\prod_{j=1}^{\nu_2}\Gamma[\gamma'_j+\beta'_k]\prod_{j=1}^{m_2}{}'\,\Gamma[\beta'_j-\beta'_k]}{\prod_{j=\nu_1+1}^{t}\Gamma[1-\gamma_j-\beta_h]\prod_{j=1+\nu_2}^{t}\Gamma[1-\gamma'_j-\beta'_k]\prod_{j=1+m_1}^{q}\Gamma[1-\beta_j+\beta_h]}$$

$$\times \frac{\prod_{j=1}^{n}\Gamma[1-\epsilon_j+\beta_h+\beta'_k]\prod_{j=1}^{m_1}{}'\,\Gamma[\beta_j-\beta_h]}{\prod_{j=1+m_2}^{q}\Gamma[1-\beta'_j+\beta'_k]\prod_{j=1+n}^{p}\Gamma[\epsilon_j-\beta_h-\beta'_k]\prod_{j=1}^{s}\Gamma[\delta_j+\beta_h+\beta'_k]}$$

$$\times F\left[\begin{array}{l|l|l} p & (1-\epsilon_p+\beta_h+\beta'_k) & \\ t & (\gamma_t+\beta_h);\ (\gamma'_t+\beta'_k) & (-)^{m_1+p-n+t-\nu_1}x \\ s & (\delta_s+\beta_h+\beta'_k) & (-)^{m_2+p-n+t-\nu_2}y \\ q-1 & (1-\beta_q+\beta_h)^*;\ (1-\beta'_q+\beta'_k)^* & \end{array}\right],$$

where the prime in Π' indicates the omission of the factor of the type $\Gamma[\beta_j-\beta_j]$; the asterisk in the F denotes the omission of the parameter of the type $(1-\beta_h+\beta_h)$. (2.3) converges absolutely for $p+t < s+q$ or $p+t = s+q$ and $|x|+|y| < \min(1, 2^{s-p+1})$.

The right-hand side of (2.3) shall, henceforth, be symbolically denoted by

$$G^{n,\nu_1,\nu_2,m_1,m_2}_{p,t,s,q}\left[\begin{array}{l|l} x & (\epsilon_p) \\ & (\gamma_t);\ (\gamma'_t) \\ y & (\delta_s) \\ & (\beta_q);\ (\beta'_q) \end{array}\right]$$

or $G_{p,t,s,q}^{n,\nu_1,\nu_2,m_1,m_2}\begin{bmatrix}x\\y\end{bmatrix}$ or simply $G\begin{bmatrix}x\\y\end{bmatrix}$, whenever there is no chance of a misunderstanding and is the required extension of Meijer's G-function to two variables.

3. Certain Particular Cases of $G\begin{bmatrix}x\\y\end{bmatrix}$

(i) $$G_{0,t,0,q}^{0,\nu_1,\nu_2,m_1,m_2}\begin{bmatrix}x\\y\end{bmatrix}=G_{t,q}^{m_1,\nu_1}\left(x\,\middle|\,\begin{matrix}(1-\gamma_t)\\(\beta_q)\end{matrix}\right)G_{t,q}^{m_2,\nu_2}\left(y\,\middle|\,\begin{matrix}(1-\gamma_t')\\(\beta_q')\end{matrix}\right),\ (q\geqslant t).$$

(ii) $$G_{0,t,0,q}^{0,\nu_1,t,m_1,1}\left[\begin{matrix}x\\0\end{matrix}\,\middle|\,\begin{matrix}\cdots\cdots\\(\gamma_t);\ (\gamma_t')\\\cdots\cdots\\(\beta_q);\ 0,\ (\beta_{2,q}')\end{matrix}\right]$$

$$=\frac{\prod\limits_{j=1}^{t}\Gamma[\gamma_j']}{\prod\limits_{j=2}^{q}\Gamma[1-\beta_j']}G_{t,q}^{m_1,\nu_1}\left(x\,\middle|\,\begin{matrix}(1-\gamma_t)\\(\beta_q)\end{matrix}\right),\ (q\geqslant t).$$

(iii) $$G_{n,t,s,q}^{n,t,t,1,1}\begin{bmatrix}x\\y\end{bmatrix}=x^{\beta_1}y^{\beta_1'}\frac{\prod\limits_{j=1}^{t}\left\{\Gamma[\gamma_j+\beta_1]\Gamma[\gamma_j'+\beta_1']\right\}}{\prod\limits_{j=2}^{q}\left\{\Gamma[1-\beta_j+\beta_1]\Gamma[1-\beta_j'+\beta_1']\right\}}$$

$$\times\frac{\prod\limits_{j=1}^{n}\Gamma[1-\epsilon_j+\beta_1+\beta_1']}{\prod\limits_{j=1}^{s}\Gamma[\delta_j+\beta_1+\beta_1']}F\left[\begin{matrix}n\\t\\s\\q-1\end{matrix}\,\middle|\,\begin{matrix}(1-\epsilon_n+\beta_1+\beta_1')\\(\gamma_t+\beta_1);\ (\gamma_t'+\beta_1')\\(\delta_s+\beta_1+\beta_1')\\(1-\beta_q+\beta_1)^*;\ (1-\beta_q'+\beta_1')^*\end{matrix}\,\middle|\,\begin{matrix}-x\\-y\end{matrix}\right],\ (n+t\leqslant s+q).$$

(iv) $$\underset{y\to 0}{Lt}\ G_{p,0,0,q}^{n,0,0,m,1}\left[\begin{matrix}x\\y\end{matrix}\,\middle|\,\begin{matrix}(\epsilon_p)\\\cdots\cdots\\\cdots\cdots\\(\beta_q),\ (0)\end{matrix}\right]=G_{p,q}^{m,n}\left(x\,\middle|\,\begin{matrix}(\epsilon_p)\\(\beta_q)\end{matrix}\right),\ (p\leqslant q).$$

(v) $$G_{1,1,1,1}^{1,1,1,1,1}\begin{bmatrix}x\\y\end{bmatrix}=x^{\beta_1}y^{\beta_1'}\frac{\Gamma[\gamma_1+\beta_1]\,\Gamma[\gamma_1'+\beta_1']}{\Gamma[\delta_1+\beta_1+\beta_1']}$$
$$\times F_1[1-\epsilon_1+\beta_1+\beta_1';\ \gamma_1+\beta_1,\ \gamma_1'+\beta_1';\ \delta_1+\beta_1+\beta_1';\ -x,\ -y].$$

(vi) $$G_{1,1,0,2}^{1,1,1,1,1}\begin{bmatrix}x\\y\end{bmatrix}=x^{\beta_1}y^{\beta_1'}\frac{\Gamma[\gamma_1+\beta_1]\,\Gamma[\gamma_1'+\beta_1']\,\Gamma[1-\epsilon_1+\beta_1+\beta_1']}{\Gamma[1-\beta_2+\beta_1]\,\Gamma[1-\beta_2'+\beta_1']}$$
$$\times F_2[1+\beta_1+\beta_1'-\epsilon_1;\ \gamma_1+\beta_1,\ \gamma_1'+\beta_1';\ 1-\beta_2+\beta_1,\ 1-\beta_2'+\beta_1';\ -x,-y].$$

(vii) $$G^{0,2,2,1,1}_{0,2,1,1}\begin{bmatrix}x\\y\end{bmatrix}=x^{\beta_1}y^{\beta'_1}\frac{\Gamma[\gamma_1+\beta_1]\,\Gamma[\gamma_2+\beta_1]\,\Gamma[\gamma'_1+\beta'_1]}{\Gamma[\delta_1+\beta_1+\beta'_1]}$$
$$\times\Gamma[\gamma'_2+\beta'_1]\,F_3[\gamma_1+\beta_1,\gamma'_1+\beta'_1,\gamma_2+\beta_1,\gamma'_2+\beta'_1;\delta_1+\beta_1+\beta'_1;-x,-y].$$

(viii) $$G^{2,0,0,1,1}_{2,0,0,2}\begin{bmatrix}x\\y\end{bmatrix}=x^{\beta_1}y^{\beta'_1}\frac{\Gamma[1-\epsilon_1+\beta_1+\beta'_1]\,\Gamma[1-\epsilon_2+\beta_1+\beta'_1]}{\Gamma[1-\beta_2+\beta_1]\,\Gamma[1-\beta'_2+\beta'_1]}$$
$$\times F_4[1-\epsilon_1+\beta_1+\beta'_1,1-\epsilon_2+\beta_1+\beta'_1;1+\beta_1-\beta_2,1+\beta'_1-\beta'_2;-x,-y].$$

(ix) $$G^{n,0,0,1,1}_{p,0,s,1}\begin{bmatrix}x\\y\end{bmatrix}=x^{\beta_1}y^{\beta'_1}\frac{\prod_{j=1}^{n}\Gamma[1-\epsilon_j+\beta_1+\beta'_1]}{\prod_{j=1}^{s}\Gamma[\delta_j+\beta_1+\beta'_1]\prod_{j=n+1}^{p}\Gamma[\epsilon_j-\beta_1-\beta'_1]}$$
$$\times{}_pF_s\begin{bmatrix}(1-\epsilon_p+\beta_1+\beta'_1);(-)^{n-p}(x+y)\\(\delta_s+\beta_1+\beta'_1)\end{bmatrix},\ (p\leqslant s+1).$$

(x) $$G^{0,1,\nu_2,q,m_2}_{0,t,s,q}\begin{bmatrix}x\\y\end{bmatrix}=\sum_{h=1}^{q}\sum_{k=1}^{m_2}x^{\beta_h}y^{\beta'_k}\Gamma[\gamma_1+\beta_h]$$
$$\times\frac{\prod_{j=1}^{\nu_2}\Gamma[\gamma'_j+\beta'_k]\prod_{j=1}^{q}{}'\,\Gamma[\beta_j-\beta_h]\prod_{j=1}^{m_2}{}'\,\Gamma[\beta'_j-\beta'_k]}{\prod_{j=1}^{s}\Gamma[\delta_j+\beta_h+\beta'_k]\prod_{j=2}^{t}\Gamma[1-\gamma_j-\beta_h]\prod_{j=1+\nu_2}^{t}\Gamma[1-\gamma'_j-\beta'_k]\prod_{j=1+m_2}^{q}\Gamma[1+\beta'_k-\beta'_j]}$$
$$\times F\left[\begin{array}{l|l|l}o&\cdots\cdots&\\t&(\gamma_t+\beta_h);\ (\gamma'_t+\beta'_k)&(-)^{q+t+1}x\\s&(\delta_s+\beta_h+\beta'_k)&(-)^{m_2+t-\nu_2}y\\q-1&(1-\beta_q+\beta_h)^*;\ (1-\beta'_q+\beta'_k)^*&\end{array}\right],\ (t\leqslant s+q).$$

This function may be taken to define an extension of MacRobert's E-function to two variables.

4. Some Simple Properties and Recurrence Relations for $G\begin{bmatrix}x\\y\end{bmatrix}$

In this section certain elementary properties of $G\begin{bmatrix}x\\y\end{bmatrix}$-function are enumerated. The proofs are fairly simple and follow by easy change in variables in the integral (2.1) and hence are omitted.

(A) $$x^{\sigma}y^{\mu}G\begin{bmatrix}x\\y\end{bmatrix}=G\left[\begin{array}{l|l}&(\epsilon_p+\sigma+\mu)\\x&(\gamma_t-\sigma);\ (\gamma'_t-\mu)\\y&(\delta_s-\sigma-\mu)\\&(\beta_q+\sigma);\ (\beta'_q+\mu)\end{array}\right].$$

(B)
$$G^{n,\,\nu_1-1,\,\nu_2-1,\,m_1,\,m_2}_{p,\,t-1,\,s,\,q-1}\left[\begin{array}{c|l} & (\epsilon_p) \\ x & (\gamma_{2,\,t})\,;\;(\gamma'_{2,\,t}) \\ y & (\delta_s) \\ & (\beta_{q-1})\,;\;(\beta'_{q-1}) \end{array}\right]$$

$$= G\left[\begin{array}{c|l} & (\epsilon_p) \\ x & 1-\beta_q,\,(\gamma_{2,\,t})\,;\;1-\beta'_q,\,(\gamma'_{2,\,t}) \\ y & (\delta_s) \\ & (\beta_q)\,;\;(\beta'_q) \end{array}\right],\ (q,\,t,\,\nu_1,\,\nu_2 \geqslant 1).$$

(C)
$$x\frac{\partial}{\partial x}G\begin{bmatrix}x\\y\end{bmatrix} = G\left[\begin{array}{c|l} & (\epsilon_p) \\ x & 1+\gamma_1,\,(\gamma_{2,\,t})\,;\;(\gamma'_t) \\ y & (\delta_s) \\ & (\beta_q)\,;\;(\beta'_q) \end{array}\right] - \gamma_1\,.\,G\begin{bmatrix}x\\y\end{bmatrix}.$$

A similar formula holds for $y\dfrac{\partial G}{\partial y}$.

(D)
$$(\gamma_1+\beta_1)G\begin{bmatrix}x\\y\end{bmatrix} = G\left[\begin{array}{c|l} & (\epsilon_p) \\ x & 1+\gamma_1,\,(\gamma_{2,\,t})\,;\;(\gamma'_t) \\ y & (\delta_s) \\ & (\beta_q)\,;\;(\beta'_q) \end{array}\right]$$

$$+G\left[\begin{array}{c|l} & (\epsilon_p) \\ x & (\gamma_t)\,;\,(\gamma'_t) \\ y & (\delta_s) \\ & 1+\beta_1,\,(\beta_{2,\,q})\,;\,(\beta'_q) \end{array}\right],\ (\nu_1 \geqslant 1).$$

A similar relation holds for $(\gamma'_1+\beta'_1)G$.

(E)
$$(\epsilon_p+\delta_s-2)G\begin{bmatrix}x\\y\end{bmatrix} = G\left[\begin{array}{c|l} & (\epsilon_{p-1}),\,\epsilon_p-1 \\ x & (\gamma_t)\,;\,(\gamma'_t) \\ y & (\delta_s) \\ & (\beta_q)\,;\,(\beta'_q) \end{array}\right]$$

$$+G\left[\begin{array}{c|l} & (\epsilon_p) \\ x & (\gamma_t)\,;\,(\gamma'_t) \\ y & (\delta_{s-1}),\,\delta_s-1 \\ & (\beta_q)\,;\,(\beta'_q) \end{array}\right],\ (p \geqslant 1,\,s \geqslant 1).$$

(F) $$(\epsilon_p+\gamma_1+\gamma_1'-1)G\begin{bmatrix}x\\y\end{bmatrix} = G\left[\begin{array}{c|l} & (\epsilon_{p-1}),\ \epsilon_p-1\\ x & (\gamma_t);\ (\gamma_t')\\ y & (\delta_s)\\ & (\beta_q);\ (\beta_q')\end{array}\right]$$

$$+G\left[\begin{array}{c|l} & (\epsilon_p)\\ x & (\gamma_1+1),\ (\gamma_{2,\,t});\ (\gamma_t')\\ y & (\delta_s)\\ & (\beta_q);\ (\beta_q')\end{array}\right]+G\left[\begin{array}{c|l} & (\epsilon_p)\\ x & (\gamma_t);\ 1+\gamma_1',\ (\gamma_{2,\,t}')\\ y & (\delta_s)\\ & (\beta_q);\ (\beta_q')\end{array}\right],\ (p \geqslant 1,\ \nu_1 \geqslant 1,\ \nu_2 \geqslant 1).$$

(G) $$(\beta_1+\beta_1'+\delta_1-1)G\begin{bmatrix}x\\y\end{bmatrix} = G\left[\begin{array}{c|l} & (\epsilon_p)\\ x & (\gamma_t);\ (\gamma_t')\\ y & (\delta_s)\\ & \beta_1+1,\ (\beta_{2,\,q});\ (\beta_q')\end{array}\right]$$

$$+G\left[\begin{array}{c|l} & (\epsilon_p)\\ x & (\gamma_t);\ (\gamma_t')\\ y & (\delta_s)\\ & (\beta_q);\ \beta_1'+1,\ (\beta_{2,\,q}')\end{array}\right]+G\left[\begin{array}{c|l} & (\epsilon_p)\\ x & (\gamma_t);\ (\gamma_t')\\ y & \delta_1-1,\ (\delta_{2,\,s})\\ & (\beta_q);\ (\beta_q')\end{array}\right],\ (s \geqslant 1).$$

These are only a few representative recurrence relations for $G\begin{bmatrix}x\\y\end{bmatrix}$. However, because of the large number of parameters that occur in it, one can get a variety of other such relations.

5. The $G_{p,\,t,\,s,\,q}^{0,\,\nu_1,\,\nu_2,\,m_1,\,m_2}\begin{bmatrix}x\\y\end{bmatrix}$ Function

This function, although, is a particular case of $G\begin{bmatrix}x\\y\end{bmatrix}$, yet is of special interest. It is easy to see that in the case of $G\begin{bmatrix}x\\y\end{bmatrix}$-function one cannot get the analogue of the important formula,

(1) $$G_{p,\,q}^{m,\,n}\left(x^{-1}\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right) = G_{q,\,p}^{n,\,m}\left(x\,\middle|\,\begin{matrix}(1-b_q)\\(1-a_p)\end{matrix}\right),$$

for the Meijer's G-function. However, we see from (2.3) that

$$(2)\quad \begin{cases} G^{0,\nu_1,\nu_2,m_1,m_2}_{p,t,s,q}\left[\begin{array}{c|l} & (\epsilon_p) \\ x & (\gamma_t);\ (\gamma'_t) \\ y & (\delta_s) \\ & (\beta_q);\ (\beta'_q) \end{array}\right] = \sum_{h=1}^{m_1}\sum_{k=1}^{m_2} x^{\beta_h} y^{\beta'_k} \dfrac{\prod_{j=1}^{\nu_1}\Gamma[\gamma_j+\beta_h]}{\prod_{j=1+\nu_1}^{t}\Gamma[1-\gamma_j-\beta_h]} \\ \times \dfrac{\prod_{j=1}^{\nu_2}\Gamma[\gamma'_j+\beta'_k]\prod'^{m_1}_{j=1}\Gamma[\beta_j-\beta_h]\prod'^{m_2}_{j=1}\Gamma[\beta'_j-\beta'_k]}{\prod_{j=1+\nu_2}^{t}\Gamma[1-\gamma'_j-\beta'_k]\prod_{j=1+m_1}^{q}\Gamma[1-\beta_j+\beta_h]\prod_{j=1+m_2}^{q}\Gamma[1-\beta'_j+\beta'_k]\prod_{j=1}^{p}\Gamma[\epsilon_j-\beta_h-\beta'_k]} \\ \times \dfrac{1}{\prod_{j=1}^{s}\Gamma[\delta_j+\beta_h+\beta'_k]} F\left[\begin{array}{c|l|l} p & (1-\epsilon_p+\beta_h+\beta'_k) & \\ t & (\gamma_t+\beta_h);\ (\gamma'_t+\beta'_k) & (-)^{m_1+t-\nu_1+p}x \\ s & (\delta_s+\beta_h+\beta'_k) & (-)^{m_2+t-\nu_2+p}y \\ q-1 & (1-\beta_q+\beta_h)^*;\ (1-\beta'_q+\beta'_k)^* & \end{array}\right]. \end{cases}$$

valid for $p+t < s+q$ or $p+t = s+q$ and $|x|+|y| < \min(1, 2^{s-p+1})$.

However, if none of the poles of $\Gamma[\gamma_j+\xi], = 1, 2, \ldots \nu_1$ and $\Gamma[\gamma'_j+\eta]$, $\eta = 1, 2, \ldots \nu_2$ coincide, we have from (2.1)

$$(3)\quad \begin{cases} G^{0,\nu_1,\nu_2,m_1,m_2}_{p,t,s,q}\begin{bmatrix} x \\ y \end{bmatrix} = \sum_{h=1}^{\nu_1}\sum_{k=1}^{\nu_2} x^{-\gamma_h} y^{-\gamma'_k} \dfrac{\prod'^{\nu_1}_{j=1}\Gamma[\gamma_j-\gamma_h]\prod'^{\nu_2}_{j=1}\Gamma[\gamma'_j-\gamma'_k]}{\prod_{j=1}^{p}\Gamma[\epsilon_j+\gamma_h+\gamma'_k]\prod_{j=1}^{s}\Gamma[\delta_j-\gamma_h-\gamma'_k]} \\ \times \dfrac{\prod_{j=1}^{m_1}\Gamma[\beta_j+\gamma_h]\prod_{j=1}^{m_2}\Gamma[\beta'_j+\gamma'_k]}{\prod_{j=\nu_1+1}^{t}\Gamma[1+\gamma_h-\gamma_j]\prod_{j=\nu_2+1}^{t}\Gamma[1+\gamma'_k-\gamma'_j]\prod_{j=m_1+1}^{q}\Gamma[1-\gamma_h-\beta_j]\prod_{j=m_2+1}^{q}\Gamma[1-\gamma'_k-\beta'_j]} \\ \times F\left[\begin{array}{c|l|l} s & (1-\delta_s+\gamma_h+\gamma'_k) & \\ q & (\beta_q+\gamma_h);\ (\beta'_q+\gamma'_k) & (-)^{q-m_1-s+\nu_1}x^{-1} \\ p & (\epsilon_p+\gamma_h+\gamma'_k) & (-)^{q-m_2-s+\nu_2}y^{-1} \\ t-1 & (1-\gamma_t+\gamma_h)^*;\ (1-\gamma'_t+\gamma'_k)^* & \end{array}\right], \end{cases}$$

valid for $s+q < t+p$ or $s+q = t+p$ and $\left|\dfrac{1}{x}\right|+\left|\dfrac{1}{y}\right| < \min(1, 2^{p-s+1})$.

Also, by an obvious change in the variables in the integral (2.1) or from (5.2) and (5.3), the important formula,

$$(4)\quad G^{0,\nu_1,\nu_2,m_1,m_2}_{p,t,s,q}\begin{bmatrix} x^{-1} \\ y^{-1} \end{bmatrix} = G^{0,m_1,m_2,\nu_1,\nu_2}_{s,q,p,t}\left[\begin{array}{c|l} & (\delta_s) \\ x & (\beta_q);\ (\beta'_q) \\ y & (\epsilon_p) \\ & (\gamma_t);\ (\gamma'_t) \end{array}\right],$$ is obtained.

This is an extension of the formula (5.1) for the Meijer's G-function and transforms a particular $G\begin{bmatrix}x\\y\end{bmatrix}$-function into one with arguments $1/x$ and $1/y$ and reduces to a relation equivalent to (5.1) for $p=0=s$.

6. The Behaviour of $G\begin{bmatrix}x\\y\end{bmatrix}$ for Small Values of x and y

The behaviour of $G\begin{bmatrix}x\\y\end{bmatrix}$ in the neighbourhood of $x=0=y$ follows easily from (2.3). We have

$$G\begin{bmatrix}x\\y\end{bmatrix}=O(|x|^{\beta}|y|^{\beta'}) \text{ as } x \text{ and } y\to 0,$$

where $p+t\leqslant s+q$ and $\beta=\max\ Rl\beta_h$, $h=1,2,\ldots,m_1$ and $\beta'=\max\ Rl\beta'_k$, $k=1,2,\ldots,m_2$.

7. Certain Integrals with $G\begin{bmatrix}x\\y\end{bmatrix}$

In this section we discuss three integrals which involve $G\begin{bmatrix}x\\y\end{bmatrix}$ in the integrand. The first integral gives the behaviour of this function under double Euler-transform. In particular, we evaluate the double integral

$$(1)\qquad \int_0^1\int_0^1 \xi^{-\alpha}(1-\xi)^{\alpha-\beta-1}\eta^{-a}(1-\eta)^{a-b-1}G\begin{bmatrix}x\xi\\y\eta\end{bmatrix}d\xi d\eta.$$

Substituting (2.1) for $G\begin{bmatrix}x\\y\end{bmatrix}$, changing the order of integration and evaluating the inner double integral, we get, on using (2.1) again, that (7.1) equals $\Gamma[\alpha-\beta]\Gamma[a-b]$ times

$$G^{n,\nu_1+1,\nu_2+1,m_1,m_2}_{p,t+1,s,q+1}\left[\begin{array}{c|l} & (\epsilon_p)\\ x & (\gamma_t),\beta;\ (\gamma'_t),b\\ y & (\delta_s)\\ & 1-\alpha,(\beta_q);\ 1-a,(\beta'_q)\end{array}\right],$$

when the set of conditions (2.2) holds along with the restrictions

$$Rl\beta<Rl\alpha<1+Rl\beta_h,\ h=1,2,\ldots,m_1,$$

$$Rlb<Rla<1+Rl\beta'_k,\ k=1,2,\ldots,m_2.$$

Next, let us evaluate the integral

$$(2)\qquad \int_0^{\pi}\int_0^{\pi}(\sin\theta)^{1-2a}\sin(2\lambda+1)\theta\,.\,(\sin\phi)^{1-2b}\sin(2\mu+1)\phi$$

$$\times G\begin{bmatrix}x\sin^{2\sigma}\theta\\ y\sin^{2\sigma}\phi\end{bmatrix}d\theta d\phi,$$

where λ, μ, σ are positive integers, λ and μ may be even zero.

Substituting, once again, the integral (2.1) for $G\begin{bmatrix} x \sin^{2\sigma}\theta \\ y \sin^{2\sigma}\phi \end{bmatrix}$, changing the order of integration and evaluating the inner integral in terms of gamma products with the help of the integral

$$\int_0^\pi \sin(2\lambda+1)\theta\,(\sin\theta)^{1-2\xi}d\theta = \frac{\Gamma[\frac{1}{2}]\Gamma[(3/2)-\xi]\Gamma[\xi+\lambda]}{\Gamma[\xi]\Gamma[2+\lambda-\xi]},\ (Rl\xi < 1),$$

we get that (7.2) equals

$$-\frac{1}{4\pi}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\psi(\xi,\eta)\Phi(\xi+\eta)\frac{\Gamma[(3/2)-a+\sigma\xi]\Gamma[a+\lambda-\sigma\xi]}{\Gamma[a-\sigma\xi]\Gamma[2+\lambda-a+\sigma\xi]}$$

$$\times\frac{\Gamma[(3/2)-b+\sigma\eta]\Gamma[b+\mu-\sigma\eta]x^\xi y^\eta}{\Gamma[b-\sigma\eta]\Gamma[2+\mu-b+\sigma\eta]}d\xi d\eta$$

$$= -\frac{1}{4\pi\sigma}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\frac{\prod_{j=0}^{\sigma-1}\left\{\Gamma\left[\xi+\frac{3-2a+2j}{2\sigma}\right]\Gamma\left[\eta+\frac{3-2b+2j}{2\sigma}\right]\Gamma\left[-\xi+\frac{a+\lambda+j}{\sigma}\right]\right.}{\prod_{j=0}^{\sigma-1}\left\{\Gamma\left[\xi+\frac{2+\lambda-a+j}{\sigma}\right]\Gamma\left[\eta+\frac{2+\mu-b+j}{\sigma}\right]\Gamma\left[-\eta+\frac{b+j}{\sigma}\right]\right.}$$

$$\left.\times\frac{\Gamma[-\eta+\{(b+\mu+j)/\sigma\}]}{\Gamma[-\xi+\{(a+j)/\sigma\}]}\right\}x^\xi y^\eta\psi(\xi,\eta)\Phi(\xi+\eta)d\xi d\eta.$$

Evaluating the above integral by (2.1), we get that (7.2) equals

$$\frac{\pi}{\sigma}G^{n,\,\nu_1+\sigma,\,\nu_2+\sigma,\,m_1+\sigma,\,m_2+\sigma}_{p,\,t+2\sigma,\,s,\,q+2\sigma}\left[\begin{array}{c|l} & (\epsilon_p) \\ x & (\gamma_{\nu_1}),\,(g_{2\sigma}),\,(\gamma_{\nu_1+1,\,t});\ (\gamma'_{\nu_2}),\,(g'_{2\sigma}),\,(\gamma'_{\nu_2+1,\,t}) \\ y & (\delta_s) \\ & (\beta_{m_1}),\,(b_{2\sigma}),\,(\beta_{m_1+1,\,q});\ (\beta'_{m_2}),\,(b'_{2\sigma}),\,(\beta'_{m_2+1,\,q}) \end{array}\right],$$

where

$$\left.\begin{aligned} g_{j+1} &= \frac{3-2a+2j}{2\sigma}, & g_{j+\sigma+1} &= \frac{1-a+j}{\sigma} \\ g'_{j+1} &= \frac{3-2b+2j}{2\sigma}, & g'_{j+\sigma+1} &= \frac{1-b+j}{\sigma} \end{aligned}\right\} j = 0, 1, 2, \ldots, \sigma-1$$

and

$$\left.\begin{aligned} b_{j+1} &= \frac{a+\lambda+j}{\sigma}, & b_{j+\sigma+1} &= \frac{a-\lambda-1+j}{\sigma} \\ b'_{j+1} &= \frac{b+\mu+j}{\sigma}, & b'_{j+\sigma+1} &= \frac{b-\mu-1+j}{\sigma} \end{aligned}\right\} j = 0, 1, 2, \ldots, \sigma-1.$$

The result being valid under the set of conditions (2.2), $p+t \leqslant s+q$, Rla, $Rlb < 1$.

Similarly, if we use the integral

$$(3)\qquad \int_0^{\pi} (\sin\theta)^{1-2\xi} \cos 2\lambda\theta d\theta = \frac{\Gamma[\frac{1}{2}]\Gamma[1-\xi]\Gamma[\xi+\lambda-\frac{1}{2}]}{\Gamma[(3/2)+\lambda-\xi]\Gamma[\xi-\frac{1}{2}]},\ (Rl\xi < 1)$$

we can obtain the result

$$(4)\qquad \int_0^{\pi}\int_0^{\pi} (\sin\theta)^{1-2a}\cos(2\lambda\theta)(\sin\phi)^{1-2b}\cos(2\mu\phi)\times G\left[\begin{matrix} x\sin^{2\sigma}\theta \\ y\sin^{2\sigma}\phi \end{matrix}\right] d\theta d\phi$$

$$= \frac{\pi}{\sigma} G^{n,\,\nu_1+\sigma,\,\nu_2+\sigma,\,m_1+\sigma,\,m_2+\sigma}_{p,\,t+2\sigma,\,s,\,q+2\sigma}\left[\begin{matrix} & (\epsilon_p) \\ x & (\gamma_{\nu_1}),\,(g_{2\sigma}),\,(\gamma_{\nu_1+1,\,t});\ (\gamma'_{\nu_2}),\,(g'_{2\sigma}),\,(\gamma'_{\nu_2+1,\,t}) \\ y & (\delta_s) \\ & (\beta_{m_1}),\,(b_{2\sigma}),\,(\beta_{m_1+1,\,q});\ (\beta'_{m_2}),\,(b'_{2\sigma}),\,(\beta'_{m_2+1,\,q}) \end{matrix}\right],$$

where

$$\left.\begin{aligned} g_{j+1} &= \frac{1-a+j}{\sigma}, & g_{j+\sigma+1} &= \frac{3-2a+2j}{2\sigma} \\ g'_{j+1} &= \frac{1-b+j}{\sigma}, & g'_{j+\sigma+1} &= \frac{3-2b+2j}{2\sigma} \end{aligned}\right\},\ j = 0, 1, 2, \ldots, (\sigma-1)$$

and

$$\left.\begin{aligned} b_{j+1} &= \frac{2a+2\lambda+2j-1}{2\sigma}, & b_{j+\sigma+1} &= \frac{2a-2\lambda-1+2j}{2\sigma} \\ b'_{j+1} &= \frac{2b+2\mu+2j-1}{2\sigma}, & b'_{j+\sigma+1} &= \frac{2b-2\mu-1+2j}{2\sigma} \end{aligned}\right\},\ j = 0, 1, \ldots, (\sigma-1).$$

The result being valid under the same conditions as for (7.2).

It is proposed to study the differential equation satisfied by $G\left[\begin{matrix} x \\ y \end{matrix}\right]$, its asymptotic behaviour for large values of x and y and the expansion theory in a subsequent communication.

References

Appell, P., and Kampé de Fèriet, M. J. (1926). Fonctions hypergéometriques et hypersphériques, Paris.

Erdèlyi, A., *et al.* (Ed.) (1953). Higher Transcendental Functions, Vol. I. Bateman Project.

Meijer, C. S. (1941). Multiplikations theoreme für die Funktion G^{mn}_{pq} (z). *Proc. Ned. Akad. Wet.*, **44**, 1062–1070.

Ragab, F. M. (1963). Expansions of Kampé de Fèriet's double hypergeometric functions of higher order. *J. reine angew. Math.*, **212**, 113–119.