GENERALIZED BROWDER-TYPE FIXED POINT THEOREM WITH STRONGLY GEODESIC CONVEXITY ON HADAMARD MANIFOLDS WITH APPLICATIONS

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In this paper, a generalized Browder-type fixed point theorem on Hadamard manifolds is introduced, which can be regarded as a generalization of the Browder-type fixed point theorem for the set-valued mapping on an Euclidean space to a Hadamard manifold. As applications, a maximal element theorem, a section theorem, a Ky Fan-type Minimax Inequality and an existence theorem of Nash equilibrium for non-cooperative games on Hadamard manifolds are established.

Key words: Generalized Browder-type fixed point theorem, Maximal element theorem, Hadamard manifolds, Ky Fan Minimax Inequality, Section theorem, Nash equilibrium.

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1. Introduction

In 1961, [7] established an elementary but very basic geometric lemma for multivalued mappings by the mean of his own generalization of the classical Knaster-Kuratowski-Mazurkievicz theorem [2, 3] used this fact to prove the Fan-Browder fixed-point theorem, which is a more convenient form of Fan’s lemma. Later, by establishing the existence of selection functions for set-valued mappings with open fibers in product spaces, Deguire and Lassonde gave some fixed-point theorems in product spaces for both compact and non-compact domains (see [6]). Readers may consult [1, 5, 8, 9, 11, 13].

On the other hand, in the last few years, several important concepts of nonlinear analysis have been extended from an Euclidean space to a Riemannian manifold setting in order to go further in the study of the convexity theory, the fixed point theory, the variational inequality and related topics. In fact, a manifold is not a linear space. In this setting the linear space is replaced by a Hadamard manifold and the line segment by a geodesic in [14,16,17]. In 2003, [12] introduced and studied the variational inequality on Hadamard manifolds. Some existence theorems of solutions for the variational inequality are proved in [10].

Motivated and inspired by the research works mentioned above, in this paper, we are interested in investigating generalized Browder-type fixed point theorems on Hadamard manifolds. As applications, some existence results of solutions for maximal element theorems, section theorems, a Ky Fan-type Minimax Inequality and non-cooperative games on Hadamard manifolds are obtained.

2. Preliminaries

First we recall some definitions in [4,12]. Let \((M, g)\) be a complete finite-dimensional Riemannian manifold with the Levi-Civita connection \(\nabla\) on \(M\). Let \(x \in M\) and let \(T_xM\) denote the tangent space at \(x\) to \(M\). We denote by \(\langle \cdot, \cdot \rangle_x\) the scalar product on \(T_xM\) with the associated norm \(\| \cdot \|_x\), where the subscript \(x\) is sometimes omitted. For \(x, y \in M\), let \(c : [0, 1] \longrightarrow M\) be a piecewise smooth curve joining \(x\) to \(y\). Then the arc-length of \(c\) is defined by \(l(c) = \int_0^1 \| \dot{c}(t) \| dt\), while the Riemannian
distance from $x$ to $y$ is defined by $d(x, y) := \inf_c l(c)$, where the infimum is taken over all piecewise smooth curves $c : [0, 1] \to M$ joining $x$ to $y$. Recall that a curve $c : [0, 1] \to M$ joining $x$ to $y$ is a geodesic if $c(0) = x$, $c(1) = y$ and $\nabla_c \dot{c} = 0$, $\forall t \in [0, 1]$. A geodesic $c : [0, 1] \to M$ joining $x$ to $y$ is minimal if its arc-length equals its Riemannian distance between $x$ and $y$. By the Hopf-Rinow Theorem (see [4]), if $M$ is additionally connected, then $(M, d)$ is a complete metric space, and there is at least one minimal geodesic joining $x$ to $y$. Moreover, the exponential map at $x$, $\exp_x : T_x M \to M$ is well-defined on $T_x M$. Clearly, a curve $c : [0, 1] \to M$ is a minimal geodesic joining $x$ to $y$ if and only if there exists a vector $v \in T_x M$ such that $\|v\| = d(x, y)$ and $c(t) = \exp_x (t v)$ for each $t \in [0, 1]$. 

**Definition 2.1** — A Hadamard manifold $M$ is a simply-connected complete Riemannian manifold of non-positive sectional curvature.

Throughout the remainder of the paper, we always assume that $M$ is a $m$-dimensional Hadamard manifold. The following result is well-known and will be useful.

**Proposition 2.1** [15] — Let $x \in M$. Then, $\exp_x : T_x M \to M$ is a diffeomorphism, and for any two points $x, y \in M$, there exists a unique normal geodesic joining $x$ to $y$, which is in fact a minimizing geodesic.

A curve $c : [0, 1] \to M$ is a minimal geodesic joining $x$ to $y$, then $c(1) = \exp_x v = y$. By Proposition 2.1, we have $\exp_x^{-1}(y) = v$. Thus $c(t) = \exp_x (t \exp_x^{-1} y)$. The geodesic connecting two points is unique on Hadamard manifold, so just say it geodesic but not necessarily minimal geodesic.

**Remark 2.1** : (i) The exponential mapping and its inverse are continuous on Hadamard manifolds; (ii) For any $p, q \in M$, the geodesic joining $p$ to $q$ is $\exp_p (t \exp_p^{-1} q)$ for $t \in [0, 1]$.

In [12], geodesic convexity is introduced by Nemeth.

**Definition 2.2** [12] — A set $K \subset M$ is said to be geodesic convex if, for any
In this paper, we introduce strongly geodesic convexity on Hadamard manifolds.

**Definition 2.3** — A set \( K \subset M \) is said to be strongly geodesic convex if, for any given \( o \in M \) and for any \( p, q \in K \), \( \exp_o((1-t)\exp_o^{-1}p + t\exp_o^{-1}q) \in K \) for all \( t \in [0,1] \).

**Remark 2.2** : When \( o = p \in K \) and \( q \in K \),
\[
\exp_o((1-t)\exp_o^{-1}p + t\exp_o^{-1}q) = \exp_p(t\exp_p^{-1}q).
\]

Hence we have

\[
\text{strongly geodesic convexity} \Rightarrow \text{geodesic convexity}.
\]

**Lemma 2.1** — \( K \subset M \) is strongly geodesic convex if and only if, for any given \( o \in M \) and for any finite \( \{x_1, \cdots, x_n\} \subset K \), \( \lambda_1, \cdots, \lambda_n \in [0,1] \) with \( \sum_{i=1}^{n} \lambda_i = 1 \), we have \( \exp_o\left(\sum_{i=1}^{n} \lambda_i \exp_o^{-1}x_i\right) \in K \).

**Proof** : It is enough to prove \( \Rightarrow \), because \( \Leftarrow \) is trivial. We use mathematical induction. When \( n = 1,2 \), the statement is true. Suppose that it is true for some \( n \), we only prove that \( \exp_o\left(\sum_{i=1}^{n+1} \lambda_i \exp_o^{-1}x_i\right) \in K \) for any finite \( \{x_1, \cdots, x_{n+1}\} \subset K \), \( \lambda_1, \cdots, \lambda_{n+1} \in [0,1] \) with \( \sum_{i=1}^{n+1} \lambda_i = 1 \). Without loss of generality, we can assume that \( 0 < \lambda_{n+1} < 1 \). Thus,
\[
\sum_{i=1}^{n} \frac{\lambda_i}{1 - \lambda_{n+1}} = 1.
\]

Since it is true for \( n \), then
\[
y := \exp_o\left(\sum_{i=1}^{n} \frac{\lambda_i}{1 - \lambda_{n+1}} \exp_o^{-1}x_i\right) \in K.
\]
Hence, we have
\[
\exp_o(\lambda_{n+1} \exp_o^{-1}x_{n+1} + (1 - \lambda_{n+1}) \exp_o^{-1}y) \in K.
\]
Since
\[
\exp_o(\lambda_{n+1} \exp^{-1}_o x_{n+1} + (1 - \lambda_{n+1}) \exp^{-1}_o y) = \exp_o \left( \lambda_{n+1} \exp^{-1}_o x_{n+1} + \sum_{i=1}^{n} \exp^{-1}_o x_i \right),
\]
then
\[
\exp_o \left( \sum_{i=1}^{n+1} \lambda_i \exp^{-1}_o x_i \right) \in K.
\]
This completes the proof.

**Definition 2.4** — A set \( S \subset M \), the smallest strongly geodesic convex set containing \( S \) is defined by \( Gco(S) \), called strongly geodesic convex hull of \( S \).

Next, the representation theorem of strongly geodesic convex hull is given.

**Lemma 2.2** — Let \( S \subset M \), and \( o \) be any given point in \( M \).

\[
Gco(S) = \left\{ \exp_o \left( \sum_{i=1}^{n} \lambda_i \exp^{-1}_o x_i \right) : \forall x_1, \cdots, x_n \in S; \lambda_1, \cdots, \lambda_n \in [0, 1], \sum_{i=1}^{n} \lambda_i = 1 \right\}.
\]

**Proof** : Denote

\[
D = \left\{ \exp_o \left( \sum_{i=1}^{n} \lambda_i \exp^{-1}_o x_i \right) : \forall x_1, \cdots, x_n \in S; \lambda_1, \cdots, \lambda_n \in [0, 1], \sum_{i=1}^{n} \lambda_i = 1 \right\}.
\]

(I) We prove \( D \subset Gco(S) \).

For any finite \( \{x_1, \cdots, x_n\} \subset S \subset Gco(S) \), \( \lambda_1, \cdots, \lambda_n \in [0, 1] \) with \( \sum_{i=1}^{n+1} \lambda_i = 1 \), since \( Gco(S) \) is strongly geodesic convex by the definition 2.4, then, by lemma 2.1,

\[
\exp_o \left( \sum_{i=1}^{n} \lambda_i \exp^{-1}_o x_i \right) \in Gco(S).
\]
Thus $D \subset Gco(S)$.

(II) We prove $Gco(S) \subset D$.

For any $x, y \in D$, there exist $m_1, m_2$, finite $\{x_1, \cdots, x_{m_1}\} \subset S$, $\lambda_1, \cdots, \lambda_{m_1} \in [0, 1]$ with $\sum_{i=1}^{m_1} \lambda_i = 1$, and finite $\{y_1, \cdots, y_{m_2}\} \subset S$, $\mu_1, \cdots, \mu_{m_2} \in [0, 1]$ with $\sum_{i=1}^{m_2} \mu_i = 1$ such that

$$x = \exp_o \left( \sum_{i=1}^{m_1} \lambda_i \exp_o^{-1} x_i \right), \quad y = \exp_o \left( \sum_{i=1}^{m_2} \mu_i \exp_o^{-1} y_i \right).$$

Hence, for any $t \in [0, 1]$, we have

$$\exp_o (t \exp_o^{-1} x + (1 - t) \exp_o^{-1} y) = \exp_o \left( \sum_{i=1}^{m_1} t \lambda_i \exp_o^{-1} x_i + \sum_{i=1}^{m_2} (1 - t) \mu_i \exp_o^{-1} y_i \right).$$

Since

$$\sum_{i=1}^{m_1} t \lambda_i + \sum_{i=1}^{m_2} (1 - t) \mu_i = 1,$$

then

$$\exp_o (t \exp_o^{-1} x + (1 - t) \exp_o^{-1} y) \in D.$$

Thus $D$ is strongly geodesic convex. Since $S \subset D$, then $Gco(S) \subset D$ by the definition 2.3.

This completes the proof.

We also need the following result, which are due to Nemeth [12].

**Lemma 2.3** [12] — If $K \subset M$ is nonempty, compact and geodesic convex, then every continuous function $f : K \rightarrow K$ has a fixed point.

By Remark 2.2, we have:
Lemma 2.4 — If $K \subset M$ is nonempty, compact and strongly geodesic convex, then every continuous function $f : K \rightarrow K$ has a fixed point.

Definition 2.5 — Let $X$ and $Y$ be two Hausdorff topological spaces, and $F : X \rightrightarrows Y$ be a set-valued mapping,

(1) If, for any open subset $O$ of $Y$ with $O \supset F(x)$, there exists an open neighborhood $U(x)$ of $x$ such that $O \supset F(x')$ for any $x' \in U(x)$, $F$ is said to be upper semicontinuous at $x \in X$;

(2) If $F$ is upper semicontinuous on each $x \in X$, $F$ is said to be upper semicontinuous on $X$;

(3) If, for any open subset $O$ of $Y$ with $O \cap F(x) \neq \emptyset$, there exists an open neighborhood $U(x)$ of $x$ such that $O \cap F(x') \neq \emptyset$ for any $x' \in U(x)$, $F$ is said to be lower semicontinuous at $x \in X$;

(4) If $F$ is lower semicontinuous on each $x \in X$, $F$ is said to be lower semicontinuous on $X$;

(5) for any $y \in Y$, $F^{-1}(y) := \{x \in X | y \in F(x)\}$.

3. Main Results

Throughout this paper, let $M$ be a Hadamard manifold, $X$ be a nonempty, strongly geodesic convex and compact subset of a Hadamard manifold $M$ and $o$ be any given point in $M$.

Theorem 3.1 — Let $X$ be a nonempty, strongly geodesic convex and compact subset of $M$. Suppose that $F : X \rightrightarrows X$ and $H : X \rightrightarrows X$ are two set-valued mappings with the following conditions:

(1) for any $x \in X$, $GcoH(x) \subset F(x)$;

(2) for any $x \in X$, there exists $y \in X$ such that $x \in intH^{-1}(y)$;

Then there exists $x^* \in X$ such that $x^* \in F(x^*)$. 
PROOF: By condition (2), we have

$$X = \bigcup_{y \in X} \text{int} H^{-1}(y).$$

Since $X$ is nonempty and compact, there exists a finite number of $\text{int} H^{-1}(y_1), \ldots, \text{int} H^{-1}(y_n)$ such that

$$X = \bigcup_{i=1}^{n} \text{int} H^{-1}(y_i).$$

Let $\{\alpha_i|i = 1, \cdots, n\}$ be the partition of unity subordinate to open covering $\{\text{int} H^{-1}(y_i)|i = 1, \cdots, n\}$ of $X$, i.e.,

$$0 \leq \alpha_i(x) \leq 1, \quad \sum_{i=1}^{n} \alpha_i(x) = 1, \forall x \in X, i = 1, \cdots, n;$$

and if $x \not\in \text{int} H^{-1}(y_j)$ for some $j$, then $\alpha_j(x) = 0$.

Now we consider a function $f : X \rightarrow X$, defined by

$$f(x) = \exp_o(\alpha_1(x) \exp_o^{-1} y_1 + \cdots + \alpha_n(x) \exp_o^{-1} y_n), \forall x \in X.$$

Then $f$ is continuous, and by Lemma 2.4, there exists $x^* \in X$ such that $f(x^*) = x^*$.

Let $I = \{i \in \{1, \cdots, n\} | \alpha_i(x^*) > 0\}$, then $\sum_{i \in I} \alpha_i(x^*) = 1$ and $x^* \in \text{int} H^{-1}(y_i) \subset H^{-1}(y_i)$ for all $i \in I$, i.e., $y_i \in H(x^*)$ for each $i \in I$. Thus, we have

$$x^* = f(x^*) = \exp_o \left( \sum_{i \in I} \alpha_i(x^*) \exp_o^{-1} y_i \right) \in \text{Gco} H(x^*) \subset F(x^*).$$

This completes the proof.

Remark 3.1: If $H^{-1}(y)$ is open in $X$ for each $y \in X$ and $H(x) \neq \emptyset$ for all $x \in X$, then, for any $x \in X$, there exists $y \in X$ such that $y \in H(x)$, i.e., $x \in H^{-1}(y) = \text{int} H^{-1}(y)$. The condition (2) of theorem 3.1 is obtained. Thus we have the following corollary:
Corollary 3.1 — Let $X$ be a nonempty, strongly geodesic convex and compact subset of $M$. Suppose that $F : X \rightrightarrows X$ and $H : X \rightrightarrows X$ are two set-valued mappings with the following conditions:

1. for any $x \in X$, $GcoH(x) \subset F(x)$;
2. for any $y \in X$, $H^{-1}(y)$ is open in $X$;
3. $H(x) \neq \emptyset$ for all $x \in X$.

Then there exists $x^* \in X$ such that $x^* \in F(x^*)$.

When $H(x) = F(x)$ for all $x \in X$, we have the following corollary, which can be regarded as the Fan-Browder fixed point theorem on Hadamard manifolds.

Corollary 3.2 — Let $X$ be a nonempty, strongly geodesic convex and compact subset of $M$. Suppose that $F : X \rightrightarrows X$ is a set-valued mapping with the following conditions:

1. for any $x \in X$, $F(x)$ is nonempty and strongly geodesic convex in $X$;
2. for any $y \in X$, $F^{-1}(y)$ is open in $X$.

Then there exists $x^* \in X$ such that $x^* \in F(x^*)$.

Next we give a maximal element theorem on Hadamard manifolds by theorem 3.1.

Theorem 3.2 — Let $X$ be a nonempty, strongly geodesic convex and compact subset of $M$. Suppose that $A : X \rightrightarrows X$ and $B : X \rightrightarrows X$ are two set-valued mappings with the following conditions:

1. For each $x \in X$, $x \notin GcoB(x)$.
2. If $A(x) \neq \emptyset$, then there exists $y \in X$ such that $x \in intB^{-1}(y)$.

Then there exists $x^* \in X$ such that $A(x^*) = \emptyset$.

PROOF: Suppose the contrary, then $A(x) \neq \emptyset$ for all $x \in X$. By Theorem 3.1,
there exists $x^* \in X$ such that $x^* \in GcoB(x^*)$, which contradict the fact that $x \not\in GcoB(x)$ for each $x \in X$. This completes the proof.

**Remark 3.2** : If $A(x) \subset B(x)$ for each $x \in X$, and $A(x) \neq \emptyset$, then there exists $y \in X$ such that $y \in A(x) \subset B(x)$. Thus $x \in B^{-1}(y)$. We have the following corollary:

**Corollary 3.3** — Let $X$ be a nonempty, strongly geodesic convex and compact subset of $M$. Suppose that $A : X \rightrightarrows X$ and $B : X \rightrightarrows X$ are two set-valued mappings with the following conditions:

(1) For each $x \in X$, $A(x) \subset B(x)$.

(2) For each $x \in X$, $x \not\in GcoB(x)$.

(3) For each $y \in X$, $B^{-1}(y)$ is open in $X$.

Then there exists $x^* \in X$ such that $A(x^*) = \emptyset$.

In the case when $A = B$, Corollary 3.3 become the following result:

**Corollary 3.4** — Let $X$ be a nonempty, strongly geodesic convex and compact subset of $M$. Suppose that $A : X \rightrightarrows X$ is a set-valued mapping with the following conditions:

(1) For each $x \in X$, $x \not\in GcoA(x)$.

(2) For each $y \in X$, $A^{-1}(y) = \{x \in X | y \in A(x)\}$ is open in $X$.

Then there exists $x^* \in X$ such that $A(x^*) = \emptyset$.

Next we prove a new section theorem on Hadamard manifolds by theorem 3.1.

**Theorem 3.3** — For each $i = 1, \cdots, n$, let $X_i$ be a nonempty, strongly geodesic convex and compact subset of a Hadamard manifold $M_i$. Let

$$X = \prod_{i=1}^{n} X_i, \quad X_{-i} = \prod_{1 \leq j \leq n, j \neq i} X_j.$$
Let \( A_1, \cdots, A_n \) and \( B_1, \cdots, B_n \) be \( 2n \) subsets of \( X \),

\[
A_i(x_i) = \{ x_{-i} \in X_{-i} | (x_i, x_{-i}) \in A_i \}, \quad A_i(x_{-i}) = \{ x_i \in X_i | (x_i, x_{-i}) \in A_i \},
\]

\[
B_i(x_i) = \{ x_{-i} \in X_{-i} | (x_i, x_{-i}) \in B_i \}, \quad B_i(x_{-i}) = \{ x_i \in X_i | (x_i, x_{-i}) \in B_i \}
\]

such that

1. For each \( i = 1, \cdots, n \), and any \( x_{-i} \in X_{-i} \), there exists \( y_i \in X_i \) such that \( x_{-i} \in \text{int} B_i(y_i) \);
2. For each \( i = 1, \cdots, n \), and any \( x_{-i} \in X_{-i} \), \( GcoB_i(x_{-i}) \subset A_i(x_{-i}) \).

Then

\[
\bigcap_{i=1}^{n} A_i \neq \emptyset.
\]

**Proof:** We define set-valued mappings \( F : X \rightrightarrows X \) and \( H : X \rightrightarrows X \) by

\[
F(x) = \prod_{i=1}^{n} F_i(x_{-i}), \quad H(x) = \prod_{i=1}^{n} H_i(x_{-i}),
\]

where

\[
F_i(x_{-i}) = \{ y_i \in X_i | (y_i, x_{-i}) \in A_i \}, \quad H_i(x_{-i}) = \{ y_i \in X_i | (y_i, x_{-i}) \in B_i \}.
\]

By condition (2), \( GcoH(x) \subset F(x) \) for all \( x \in X \). Further, by condition (1), for any \( x \in X \), there exists \( y \in X \) such that \( x \in \text{int} H(y) \). Thus, by theorem 3.1, there exists \( x^* \in X \) such that \( x^* \in F(x^*) \), i.e.,

\[
\bigcap_{i=1}^{n} A_i \neq \emptyset.
\]

Next, we study a Ky Fan-type Minimax Inequality on Hadamard manifolds. First, we define the strongly geodesic concave functions on Hadamard manifolds.

**Definition 3.1** — Let \( K \subset M \) be strongly geodesic convex subset and \( o \) be any given point in \( M \). A function \( f : K \rightarrow R \) is called strongly geodesic concave if,
for any $x_i \in K$, $t_i \geq 0$, $i = 1, \cdots, n$ with $\sum_{i=1}^{n} t_i = 1$, we have

\[ f \left( \exp_o \left( \sum_{i=1}^{n} t_i \exp_o^{-1} x_i \right) \right) \geq \sum_{i=1}^{n} t_i f(x_i); \]

$f$ is called strongly geodesic quasi-concave if, for any $x_i \in K$, $t_i \geq 0$, $i = 1, \cdots, n$ with $\sum_{i=1}^{n} t_i = 1$, we have

\[ f \left( \exp_o \left( \sum_{i=1}^{n} t_i \exp_o^{-1} x_i \right) \right) \geq \min_{i \in \{1, \cdots, n\}} \{ f(x_i) \}. \]

**Theorem 3.4** — Let $X$ be a nonempty, compact and strongly geodesic convex subset of a Hadamard manifold $M$, and let $f : X \times X \rightarrow \mathbb{R}$ be a real-valued function on $X \times X$ such that

1. for each $y \in X$, $x \rightarrow f(x, y)$ is lower semicontinuous on $X$;
2. for each $x \in X$, $y \rightarrow f(x, y)$ is strongly geodesic quasi-concave on $X$;
3. $f(x, x) \leq 0$ for all $x \in X$.

Then there exists $x^* \in X$ such that $f(x^*, y) \leq 0$ for all $y \in X$.

**Proof:** We define the set-valued mapping $A : X \rightrightarrows X$ by

\[ F(x) = \{ y \in X | f(x, y) > 0 \}. \]

Suppose the contrary, then, for any $x \in X$, there exists $y \in X$ such that $y \in F(x)$, i.e., $F(x) \neq \emptyset$ for any $x \in X$. By condition (1), $F^{-1}(y)$ is open in $X$ for all $y \in X$. By condition (2), $F(x)$ is strongly geodesic convex. Hence, by corollary 3.2, there exists $x^* \in X$ such that $x^* \in F(x^*)$, i.e., $f(x^*, x^*) > 0$, which contradict the fact $f(x, x) \leq 0$ for all $x \in X$. This completes the proof.

4. **Application to Games**

Next we shall use the new results to prove some new existence theorems of Nash equilibrium points on Hadamard manifolds.
Consider the \(n\)-person non-cooperative game \(\Gamma\{I, X_i, f_i\}\) on Hadamard manifolds. Assume that

1. \(I = \{1, \cdots, n\}\) is the set of players;
2. for each \(i \in I\), the nonempty set \(X_i\) is the strategy set of \(i\)th player;
3. for each \(i \in I\), \(f_i : \prod_{i \in I} X_i \rightarrow \mathbb{R}\) is the payoff function of \(i\)th player.

We shall note \(X_{-i} = \prod_{j \in I \setminus \{i\}} X_j\), \(x_{-i} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in X_{-i}\), \(x = (x_i, x_{-i}) \in X\). \(x^* = (x^*_i, x^*_{-i}) \in X\) is called a Nash equilibrium point if, for each \(i \in I\),

\[
f_i(x^*_i, x^*_{-i}) = \max_{u_i \in X_i} f_i(u_i, x^*_{-i}).
\]

**Theorem 4.1** — Consider the \(n\)-person non-cooperative game \(\Gamma\{I, X_i, f_i\}\) on Hadamard manifolds satisfying the following conditions:

1. for each \(i \in I\), \(X_i\) is a nonempty, compact and strongly geodesic convex subset of a Hadamard manifold \(M_i\);
2. for each \(i \in I\), \(f_i\) is upper semicontinuous on \(X\);
3. for each \(i \in I\) and each fixed \(u_i \in X_i\), \(f_i(u_i, \cdot)\) is lower semicontinuous on \(X_{-i}\);
4. for each fixed \(u_{-i} \in X_{-i}\), \(f_i(\cdot, u_{-i})\) is strongly geodesic quasi-concave on \(X_i\).

Then there exists at least one Nash equilibrium point \(x^*\).

**Proof:** We define the set-valued mapping \(A(x) = \prod_{i \in I} A_i(x)\), where

\[
A_i(x) = \{y_i \in X_i | f_i(x_i, x_{-i}) < f_i(y_i, x_{-i})\}, \forall x \in X.
\]

By condition (2,3), \(A_i^{-1}(y_i) = \{x \in X | f_i(x_i, x_{-i}) < f_i(y_i, x_{-i})\}\) is open in \(X\) for each \(i \in I\), then \(A^{-1}(y) = \bigcap_{i \in I} A_i^{-1}(y_i)\) is open in \(X\) for all \(y \in X\).
Suppose that there exists \( x \in X \) such that \( x \in GcoA(x) \), then there exist \( o \in M, y^j \in A(x), j = 1, \cdots, m \) and \( t_j \geq 0, j = 1, \cdots, m \) with \( \sum_{j=1}^{m} t_j = 1 \) such that \( x = \exp_o \left( \sum_{j=1}^{m} t_j \exp^{-1} y^j \right) \). Since, for each fixed \( u_{-i} \in X \), \( f_i(\cdot, u_{-i}) \) is strongly geodesic quasi-concave on \( X_i \), then

\[
f_i(x_i, x_{-i}) = f_i \left( \exp_o \left( \sum_{j=1}^{m} t_j \exp^{-1} y^j_i \right), x_{-i} \right) \geq \min_{j \in \{1, \cdots, m\}} f_i(y^j_i, x_{-i}),
\]

which contradict the fact that \( y^j \in A(x) \) for all \( j = 1, \cdots, m \), i.e.,

\[
f_i(x_i, x_{-i}) < f_i(y^j_i, x_{-i}), \ \forall j = 1, \cdots, m.
\]

By corollary 3.4, there exists \( x^* \in X \) such that \( A(x^*) = \emptyset \), i.e., \( f_i(x_i^*, x_{-i}^*) \geq f_i(u_i, x_{-i}^*) \) for all \( u_i \in X_i \) and for each \( i \in I \).

**Theorem 4.2** — Consider the \( n \)-person non-cooperative game \( \Gamma \{ I, X_i, f_i \} \) on Hadamard manifolds satisfying the following conditions:

1. for each \( i \in I \), \( X_i \) is a nonempty, compact and strongly geodesic convex subset of a Hadamard manifold \( M_i \);
2. for each \( i \in I \), \( \sum_{i=1}^{n} f_i \) is upper semicontinuous on \( X \);
3. for each \( y \in X \), \( x \mapsto \sum_{i=1}^{n} f_i(y_i, x_{-i}) \) is lower semicontinuous on \( X \);
4. for each \( x \in X \), \( y \mapsto \sum_{i=1}^{n} f_i(y_i, x_{-i}) \) is strongly geodesic quasi-concave on \( X \).

Then there exists at least one Nash equilibrium point \( x^* \).

**Proof**: We define the function \( \varphi : X \times X \rightarrow \mathbb{R} \), where

\[
\varphi(x, y) = \sum_{i=1}^{n} [f_i(y_i, x_{-i}) - f_i(x_i, x_{-i})], \ \forall y = (y_i, y_{-i}), x = (x_i, x_{-i}) \in X.
\]

By condition (2–4), we can check that

(i) for each \( y \in X \), \( x \mapsto \varphi(x, y) \) is lower semicontinuous on \( X \);
(ii) for each \( x \in X, \) \( y \rightarrow \varphi(x, y) \) is strongly geodesic quasi-concave on \( X; \)

(iii) \( \varphi(x, x) \leq 0 \) for all \( x \in X. \)

By theorem 3.4, there exists \( x^* \in X \) such that \( \varphi(x^*, y) \leq 0 \) for all \( y \in X. \) For each \( i \in I \) and any \( u_i \in X_i, \) let \( y = (u_i, x^*_{-i}) \in X, \) then

\[
\varphi(x^*, y) = f_i(u_i, x^*_{-i}) - f_i(x^*, x^*_{-i}) \leq 0,
\]

\[
f_i(x^*_i, x^*_{-i}) = \max_{u_i \in X_i} f_i(u_i, x^*_{-i}).
\]

5. Conclusion

In this paper, we study the generalized Browder-type fixed point theorem on Hadamard manifolds, which can be regarded as a generalization of the Browder-type fixed point theorem for the set-valued mapping on an Euclidean space to a Hadamard manifold. As applications, maximal element theorems, section theorems, a Ky Fan-type Minimax Inequality and existence theorems of Nash equilibrium for non-cooperative games on Hadamard manifolds are established.

References


