ON THE EXISTENCE OF PERIODIC SOLUTIONS TO A \( p \)-LAPLACIAN RAYLEIGH EQUATION

**Bo Du* and Shiping Lu**

*Department of Mathematics, School of Science, Zhejiang Forestry College, Hangzhou 311300, Peoples’ Republic of China

**Department of Mathematics, Anhui Normal University, Wuhu 241000, Peoples’ Republic of China

e-mail: dubo7307@163.com

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By means of the generalized Mawhin’s continuation theorem, we present some sufficient conditions which guarantee the existence of at least one \( T \)-periodic solution for a \( p \)-Laplacian Rayleigh equation.

Key words: Generalized Mawhin’s continuation theorem; periodic solution; \( p \)-Laplacian

1. INTRODUCTION

This paper is devoted to using a new method to study the existence of periodic solutions for a \( p \)-Laplacian Rayleigh equation as follows:

\[
(\varphi_p(x'(t)))' + f(x'(t)) + g(x(t - \tau(t))) = e(t),
\]

where \( \varphi_p(s) = |s|^{p-2}s \), \( p > 1 \), \( \varphi_q = \varphi_{p^{-1}} \), \( \frac{1}{p} + \frac{1}{q} = 1 \); \( f, g, e, \tau \in C(\mathbb{R}, \mathbb{R}) \) with \( e(t) = e(t + T) \) and \( \tau(t) = \tau(t + T); T > 0 \) is a given constant.

In recent years, many authors studied the existence of periodic solutions for \( p \)-Laplacian functional differential equations by using Mawhin’s continuation theorem; for more details, we refer the reader to see [1]-[4] for work on this subject. Now we recall the approach of their studying.
If we denote $L = (\varphi_p(x'(t)))'$, then Mawhin’s continuation theorem can not be used, because the operator $L$ in Mawhin’s continuation theorem is linear, while $(\varphi_p(x'(t)))'$ is no longer linear when $p \neq 2$. In order to use Mawhin’s continuation theorem, it is necessary to change the form of source equation. For example, as for eq. (1), one common method is to transfer the form of eq. (1) to the form as follows:

$$
\begin{align*}
 &x_1'(t) = \varphi_q(x_2(t)), \\
 &x_2'(t) = -f(\varphi_q(x_2(t)) - g(x_1(t - \tau(t))) + e(t).
\end{align*}
$$

Let $L = (x_1', x_2')^T$, then $L$ is a linear operator. Thus Mawhin’s continuation theorem can be used to study the existence of periodic solution to eq. (1). In [1], Cheung and Ren studied the following $p$-Laplacian Liénard equation:

$$
(\varphi_p(x'(t)))' + f(t, x(t))x'(t) + \beta(t)g(x(t - \tau(t))) = e(t),
$$

(3)

by using the above method, transferring (3) to the form:

$$
\begin{align*}
 &x_1'(t) = \varphi_q(x_2(t)), \\
 &x_2'(t) = -f(x_1(t))\varphi_q(x_2(t)) - \beta(t)g(x_1(t - \tau(t))) + e(t).
\end{align*}
$$

After that, in a similar way, in [2], they again studied $p$-Laplacian Rayleigh equation as follows:

$$
(\varphi_p(x'(t)))' + f(x'(t)) + \beta g(x(t - \tau(t))) = e(t),
$$

(4)

transferring (4) to the form:

$$
\begin{align*}
 &x_1'(t) = \varphi_q(x_2(t)), \\
 &x_2'(t) = -f(\varphi_q(x_2(t))) - \beta g(x_1(t - \tau(t))) + e(t).
\end{align*}
$$

In addition, we note that the above method can be applied to studying the existence of periodic solution to $p$-Laplacian neutral equations; in [5], the authors considered the following $p$-Laplacian neutral equation:

$$
(\varphi_p[(x(t) - cx(t - \sigma))'])' + g(t, x(t - \tau(t))) = e(t).
$$

(5)

Transferring (5) to the form:

$$
\begin{align*}
 (Ax_1)'(t) &= \varphi_q(x_2(t)), \\
 x_2'(t) &= -g(t, x_1(t - \tau(t))) + e(t),
\end{align*}
$$

where $Ax(t) = x(t) - cx(t - \sigma)$. 

In a word, most of these papers used this method (changing the form of source equation) for studying $p$-Laplacian equations. Seeing such a fact, we can not but ask “how can we find other ways that we need not to change the form of the source equation?” In 2004, Ge and Ren [6] extended Mawhin’s continuation theorem in order to investigate BVPs with a $p$-Laplacian. This generalized Mawhin’s continuation theorem successfully removes the restriction that $L$ is a linear operator in Mawhin’s continuation theorem. Motivated by the work of Ge and Ren, in this paper we will study the existence of periodic solutions for eq. (1) by using the generalized Mawhin’s continuation theorem. Here, it is worth stating that the form of eq. (1) need not be transferred to the forms of (2) and our results are relevant to deviating arguments which is important in applications.

To the best of our knowledge, there is no paper to study the existence of periodic solutions to $p$-Laplacian equations by using generalized Mawhin’s continuation theorem, the main purpose of this paper is to introduce a new method for studying the existence of periodic solutions to $p$-Laplacian functional differential equations. As to other methods for studying the existence of periodic solutions, such as fixed-point theorem, coincidence degree theory, Lyapunov’s second method, Yoshizawa type theorem and Massera type theorem, we refer the reader to see [7]-[10] and the references therein.

2. PRELIMINARIES

In this section, we give some lemmas and definitions which will be used in this paper.

**Definition 2.1** ([6]) Let $X$ and $Z$ be two Banach spaces with norms $\| \cdot \|_X, \| \cdot \|_Z$ respectively. A continuous nonlinear operator

$$M : X \cap \text{dom} M \rightarrow Z$$

is said to be quasi-linear if

(i) $\text{Im} M := M(X \cap \text{dom} M)$ is a closed subset of $Z$;

(ii) $\text{Ker} M := \{ x \in X \cap \text{dom} M : Mx = 0 \}$ is linearly homeomorphic to $\mathbb{R}^n, n < \infty$.

**Definition 2.2** ([6]) Let $\Omega \subset X$ be an open and bounded set with the origin $\theta \in \Omega$. $N_\lambda : \Omega \rightarrow Z, \lambda \in [0, 1]$ is said to be $M$-compact in $\overline{\Omega}$ if there exists subset $Z_1$ of $Z$ satisfying $\dim Z_1 = \dim \text{Ker} M$ and an operator $R : \overline{\Omega} \times [0, 1] \rightarrow X_2$ being continuous and compact such that for $\lambda \in [0, 1]$,

(a) $(I - Q)N_\lambda(\overline{\Omega}) \subset \text{Im} M \subset (I - Q)Z$,

(b) $QN_\lambda x = 0, \lambda \in (0, 1) \Leftrightarrow QN x = 0, \forall x \in \Omega$,

(c) $R(\cdot, 0) \equiv 0$ and $R(\cdot, \lambda)|_{\Sigma_\lambda} = (I - P)|_{\Sigma_\lambda}$.
(d) \( M[P + R(\cdot, \lambda)] = (I - Q)N_{\lambda}, \lambda \in [0, 1], \)

where \( X_2 \) is a the complement space of \( \text{Ker}M \) in \( X \), i.e., \( X = \text{Ker}M \oplus X_2 \), \( P, Q \) are two projectors satisfying \( \text{Im}P = \text{Ker}M, \text{Im}Q = Z_1 \), \( N = N_1, \Sigma_{\lambda} = \{x \in \bar{\Omega} : Mx = N_{\lambda}x\}. \)

Lemma 2.1 ([6]) Let \( X \) and \( Z \) be two Banach spaces with norms \( || \cdot ||_X \cdot || \cdot ||_Z \) respectively and \( \Omega \subset X \) be an open and bounded nonempty set. Suppose

\[
M : X \cap \text{dom}M \to Z
\]

is quasi-linear and \( N_{\lambda} : \bar{\Omega} \to Z, \lambda \in [0, 1] \) is \( M \)-compact in \( \bar{\Omega} \). In addition, if the following conditions hold:

(H1) \( Mx \neq N_{\lambda}x, \forall(x, \lambda) \in \partial \Omega \times (0, 1); \)

(H2) \( QNx \neq 0, \forall x \in \text{Ker}M \cap \partial \Omega; \)

(H3) \( \text{deg}\{JQN, \Omega \cap \text{Ker}M, 0\} \neq 0, J : \text{Im}Q \to \text{Ker}M \) is a homeomorphism.

Then the abstract equation \( Mx = Nx \) has at least one solution in \( \text{dom}(M) \cap \bar{\Omega} \).

Lemma 2.2 ([11]) Let \( p \in (1, +\infty) \) be a constant, \( s \in C(\mathbb{R}, \mathbb{R}) \) such that \( s(t + T) \equiv s(t) \), \( u \in C^1(\mathbb{R}, \mathbb{R}) \) with \( u(t + T) \equiv u(t) \). Then

\[
\int_0^T |u(t) - u(t - s(t))|^p dt \leq 2 \left( \max_{t \in [0, T]} |s(t)| \right)^p \int_0^T |u'(t)|^p dt.
\]

Lemma 2.3 ([12]) Let \( s, \sigma \in C(\mathbb{R}, \mathbb{R}) \) with \( s(t + T) \equiv s(t) \) and \( \sigma(t + T) \equiv \sigma(t) \). Suppose that the function \( t - \sigma(t) \) has a unique inverse \( \mu(t), \forall t \in \mathbb{R} \). Then \( s(\mu(t + T)) \equiv s(\mu(t)) \).

Throughout this paper, we assume that \( e(t) \) is not a constant function on \( \mathbb{R} \). Furthermore, we suppose that \( \tau \in C^1(\mathbb{R}, \mathbb{R}) \) with \( \tau'(t) < 1, \forall t \in \mathbb{R} \). It is obvious that the function \( t - \tau(t) \) has a unique inverse denoted by \( \gamma(t) \).

For the sake of convenience, we list the following assumptions which will be used by us to study the existence of \( T \)-periodic solutions to eq. (1).

[A1] There exist constants \( \alpha \geq 0, \beta \geq 0 \) such that

\[
|f(u)| \leq \alpha|u|^{p-1} + \beta, \forall u \in \mathbb{R};
\]

[A2] there exists a constant \( r \geq 0 \) such that

\[
\lim_{|u| \to +\infty} \frac{|g(u)|}{|u|^{p-1}} \leq r \in [0, \infty);
\]
there exists a constant \(D > 0\) such that
\[
g(u) < -|e|_0 - |f(0)| \quad \text{for } u > D
\]
and
\[
g(u) > |e|_0 + |f(0)| \quad \text{for } u < -D.
\]

3. Existence of Periodic Solution for Equation (1)

For convenience of applying Lemma 2.1, we denote \(C_T = \{x|x \in C(\mathbb{R}, \mathbb{R}), x(t + T) = x(t)\}\), \(C_T^1 = \{x|x \in C^1(\mathbb{R}, \mathbb{R}), x(t + T) = x(t)\}\), \(X = C_T^1\) with the norm \(\|x\| := \max\{\|x\|_0, \|x\|_0\}\), where \(\|x\|_0 := \max_{0 \leq t \leq T} |x(t)|\), \(Z = C_T\), then the operators \(M, N_\lambda\) are defined by
\[
M : \text{dom}M \cap X \to Z, \quad (Mx)(t) = (\varphi_p(x'))', \quad t \in \mathbb{R},
\]
\[
N_\lambda : X \to Z, \quad (N_\lambda x)(t) = -\lambda f(x'(t)) - \lambda g(x(t - \tau(t))) + \lambda e(t), \quad t \in \mathbb{R}, \quad \lambda \in [0, 1],
\]
where \(\text{dom}M = \{x \in X : \varphi_p(x') \in C_T^1\}\). For convenience of the proof, let
\[
F(t, x(t), x'(t), x(t - \tau(t))) = -f(x'(t)) - g(x(t - \tau(t))) + e(t),
\]
then \((N_\lambda x)(t) = \lambda F\). By (6) and (7), eq. (1) is equivalent to the operator equation \(Nx = Mx\), where \(N_1 = N\). Then, we have
\[
\text{Ker}M = \{x \in \text{dom}M \cap X : x(t) = a, \quad a \in \mathbb{R}, \quad t \in \mathbb{R}\},
\]
\[
\text{Im}M = \left\{z \in Z : \int_0^T z(s)ds = 0\right\}.
\]
Clearly, \(\text{Ker}M \cong \mathbb{R}, \text{Im}M\) is a closed set in \(Z\). Then we have the following lemma:

Lemma 3.1 — If \(M\) is defined by (6), then \(M\) is a quasi-linear operator.

Let
\[
P : X \to \text{Ker}M, \quad (Px)(t) = x(0), \quad t \in \mathbb{R},
\]
\[
Q : Z \to \mathbb{R}, \quad (Qz)(t) = \frac{1}{T} \int_0^T z(s)ds \quad t \in \mathbb{R}.
\]

Lemma 3.2 — If \(f, g, e, \tau \in C(\mathbb{R}, \mathbb{R})\) with \(e(t) = e(t + T)\) and \(\tau(t) = \tau(t + T)\), then \(N_\lambda\) is \(M\)-compact.

Proof: Let \(Z_1 = \text{Im}Q\). For any bounded set \(\bar{\Omega} \subset X \neq \emptyset\), \(\forall x \in \bar{\Omega}\), from \(x(0) = x(T)\), there exists a point \(\eta \in [0, T]\) such that
\[
x' (\eta) = 0.
\]
Define $R : \bar{\Omega} \times [0, 1] \rightarrow \text{Ker} P$,

$$R(x, \lambda)(t) = \int_0^t \varphi_q \left[ \int_\eta^s \lambda \left( F(r, x(r), x'(r), x(r - \tau(r))) - (QF)(r) \right) dr \right] ds, \quad t \in [0, T],$$

where $F$ is defined by (8), $\eta$ is defined by (9). From $f, g, e, \tau \in C(\mathbb{R}, \mathbb{R})$ with $e(t) = e(t + T)$ and $\tau(t) = \tau(t + T), \forall \lambda \in [0, 1]$, it is easy to see that $R(\cdot, \lambda)$ is relatively compact and continuous. Now, we show that $N_\lambda$ is $M$-compact by four steps, i.e., the conditions of Definition 2.2 are all satisfied.

**Step 1.** By $Q^2 = Q$, we have $Q(I - Q)N_\lambda(\bar{\Omega}) = 0$, so $(I - Q)N_\lambda(\bar{\Omega}) \subset \text{Ker} Q = \text{Im} M$. On the other hand, $\forall z \in \text{Im} M$, clearly $Qz = 0$, so $z = z - Qz = (I - Q)z$, then $z \in (I - Q)\mathcal{Z}$. So we have

$$(I - Q)N_\lambda(\bar{\Omega}) \subset \text{Im} M \subset (I - Q)\mathcal{Z}.$$

**Step 2.** We show that: $QN_\lambda x = 0$, $\lambda \in (0, 1) \Leftrightarrow QN x = 0$, $\forall x \in \Omega$. Because $QN_\lambda x = \frac{1}{T} \int \lambda F dr = 0$, we get $\frac{1}{T} \int F dr = 0$, i.e., $QN x = 0$. The inverse is true.

**Step 3.** Clearly $R(\cdot, 0) = 0$. $\forall x \in \Sigma_\lambda = \{ x \in \bar{\Omega} : Mx = N_\lambda x \}$, we have $(\varphi_p(x'))' = \lambda F$ and $QF = 0$. Hence

$$R(x, \lambda)(t) = \int_0^t \varphi_q \left[ \int_\eta^s \lambda \left( F(r, x(r), x'(r), x(r - \tau(r))) - (QF)(r) \right) dr \right] ds$$

$$= \int_0^t \varphi_q \left[ \int_\eta^s \lambda F(r, x(r), x'(r), x(r - \tau(r))) dr \right] ds$$

$$= \int_0^t \varphi_q \left[ \int_\eta^s (\varphi_p(x'))' dr \right] ds$$

$$= \int_0^t x'(s) ds$$

$$= x(t) - x(0)$$

$$= [(I - P)x](t).$$

**Step 4.** $\forall x \in \bar{\Omega}$, we have

$$M[Px + R(x, \lambda)](t)$$

$$= \left( \varphi_p \left( \int_0^t \varphi_q \left[ \int_\eta^s \lambda \left( F(r, x(r), x'(r), x(r - \tau(r))) - (QF)(r) \right) dr \right] ds \right) \right)'$$

$$= \left( \int_\eta^t \lambda (F(r, x(r), x'(r), x(r - \tau(r))) - (QF)(r)) dr \right)'$$
\[ = \lambda F - \lambda QF \]
\[ = [(I - Q)N_\lambda x](t). \]

Hence, \( N_\lambda \) is \( M \)-compact in \( \Omega \).

**Theorem 3.1** — Suppose that assumptions \([A1]-[A3]\) hold, then eq. (1) has at least one \( T \)-periodic solution \( x \), if

\[
C_p 2^{\frac{1}{p}} |\tau|_0 \left( \max_{t \in [0,T]} \left| \frac{1}{1 - \tau'(\gamma(t))} \right| \right)^{\frac{1}{q}} \left( \frac{T}{\pi_p} \right)^{\frac{2}{q}} + \alpha T < 1, \]

where

\[
\pi_p = 2 \int_0^{(p-1)\frac{1}{p}} \frac{1}{(1 - \frac{sp}{p-1})^\frac{1}{p}} ds = \frac{2\pi(p - 1)}{p\sin \frac{\pi}{p}}, \quad C_p = \begin{cases} 2^{\frac{2}{q}}, & p > 2; \\ 1, & 1 < p \leq 2. \end{cases} \]

**PROOF:** We complete the proof by three steps.

**Step 1.** Let \( \Omega_1 = \{ x \in \text{dom} M : Mx = N_\lambda x, \lambda \in (0, 1) \} \). We show that \( \Omega_1 \) is a bounded set.

If \( x \in \Omega_1 \), then \( Mx = N_\lambda x \), i.e.,

\[
(\varphi_\rho(x'))' = -\lambda f(x'(t)) - \lambda g(x(t - \tau(t))) + \lambda e(t). \tag{10} \]

Let \( t_0 \) be the global maximum point of \( x(t) \) on \( \mathbb{R} \), then \( x'(t_0) = 0 \) and there exists \( \varepsilon > 0 \) such that \( x'(t) \) is decreasing for \( t \in (\xi - \varepsilon, \xi + \varepsilon) \). So, \( \varphi_\rho(x'(t)) \) is also decreasing for \( t \in (\xi - \varepsilon, \xi + \varepsilon) \) which yields \( (\varphi_\rho(x'(t)))' \leq 0 \), and then from (10),

\[
f(0) + g(x(t_0 - \tau(t_0))) - e(t_0) \leq 0 \]

and

\[
g(x(t_0 - \tau(t_0))) \leq |f(0)| + |e|_0 \]

which together with the assumption \([A3]\), we get \( -D \leq x(t_0 - \tau(t_0)) \leq D \). Because \( x(t) \) is a \( T \)-periodic function, then there exists a constant \( \xi^* \in [0, T] \) satisfying \( t_0 - \tau(t_0) = \xi^* + kT, k \in \mathbb{Z} \), then we have

\[
|x(\xi^*)| \leq D. \tag{11} \]

Hence

\[
|x|_0 = \max_{t \in [0,T]} |x(\xi^*) + \int_{\xi^*}^t x'(s)ds| \leq |x(\xi^*)| + \int_0^T |x'(s)|ds \leq D + \int_0^T |x'(s)|ds. \tag{12} \]

Let \( s - \tau(s) = u \), i.e., \( s = \gamma(u) \), then by using Lemma 2.3, we get

\[
\int_0^T g(x(s - \tau(s)))ds = \int_{-\tau(0)}^{T-\tau(T)} \frac{g(x(u))}{1 - \tau'(\gamma(u))}du = \int_0^T \frac{g(x(u))}{1 - \tau'(\gamma(u))}du. \tag{13} \]
On the other hand, multiplying the two sides of (10) by \(x(t)\) and integrating them over \([0, T]\), then by assumption [A1] and (12), we have

\[
-\int_0^T |x'(t)|^p dt = -\lambda \int_0^T f(x(t))x(t)dt - \lambda \int_0^T g(x(t - \tau(t)))x(t)dt + \lambda \int_0^T e(t)x(t)dt,
\]

\[
\int_0^T |x'(t)|^p dt = \lambda \int_0^T f(x(t))x(t)dt + \lambda \int_0^T g(x(t - \tau(t)))x(t)dt - \lambda \int_0^T e(t)x(t)dt
\]

\[
\leq |x|_0 \int_0^T (\alpha |x'(t)|^{p-1} + \beta)dt + \lambda \int_0^T g(x(t - \tau(t)))x(t)dt + \int_0^T |e(t)x(t)|dt
\]

\[
\leq \lambda \int_0^T g(x(t - \tau(t)))x(t)dt + \beta T|x|_0 + T|e|_0|x|_0 + \alpha|x|_0 \int_0^T |x'(t)|^{p-1} dt
\]

\[
\leq \lambda \int_0^T g(x(t - \tau(t)))x(t)dt + (\beta TD + T|e|_0 D) + (\beta T + T|e|_0) \int_0^T |x'(t)|dt
\]

\[
+ \alpha D \int_0^T |x'(t)|^{p-1} dt + \alpha \int_0^T |x'(t)|dt \int_0^T |x'(t)|^{p-1} dt
\]

\[
\leq \lambda \int_0^T g(x(t - \tau(t)))x(t)dt + (\beta TD + T|e|_0 D) + (\beta + |e|_0 T)^{1+\frac{\alpha}{p}} \left( \int_0^T |x'(t)|^{p} dt \right)^{\frac{1}{p}}
\]

\[
+ \alpha DT^\frac{1}{p} \left( \int_0^T |x'(t)|^{p} dt \right)^{\frac{p-1}{p}} + \alpha T \int_0^T |x'(t)|^{p} dt.
\]  \hspace{1cm} (14)

Now we consider \(\int_0^T g(x(t - \tau(t)))x(t)dt\). Let

\(\Delta_1 = \{t \in [0, T] : |x(t)| > D\}, \Delta_2 = \{t \in [0, T] : |x(t)| \leq D\}, \)

\(E_1 = \{t \in [0, T] : |x(t - \tau(t))| > \rho\}, \ E_2 = \{t \in [0, T] : |x(t - \tau(t))| \leq \rho\}. \)

In view of

\[
C_p 2^\frac{1}{p} |\tau|_0^{\frac{r}{p}} \left( \max_{t \in [0, T]} \left| \frac{1}{1 - \tau'(\gamma(t))} \right| \right)^\frac{1}{q} \left( \frac{T}{\pi p} \right)^\frac{p}{q} + \alpha T < 1,
\]

we have that there exists a constant \(\varepsilon_1 > 0\) such that

\[
C_p 2^\frac{1}{p} |\tau|_0 (r + \varepsilon_1) \left( \max_{t \in [0, T]} \left| \frac{1}{1 - \tau'(\gamma(t))} \right| \right)^\frac{1}{q} \left( \frac{T}{\pi p} \right)^\frac{p}{q} + \alpha T < 1.
\]

For such a constant \(\varepsilon_1\), from assumption [A2], we obtain that there exists a constant \(\rho > 0\) such that

\[
|g(x)| \leq (r + \varepsilon_1)|x|^{p-1}, \text{ whenever } |x| > \rho.
\]  \hspace{1cm} (15)
From (13), (15), Lemma 2.2, Lemma 2.3 and assumption [A3] we get

\[
\int_0^T g(x(t) - \tau(t))x(t)dt = \int_0^T g(x(t - \tau(t)))x(t - \tau(t))dt
\]

\[
+ \int_0^T g(x(t - \tau(t)))(x(t) - x(t - \tau(t)))dt
\]

\[
= \int_0^T \frac{g(x(t))}{1 - \tau'(\gamma(t))} x(t)dt + \int_0^T g(x(t - \tau(t)))(x(t) - x(t - \tau(t)))dt
\]

\[
\leq \left( \int_{\Delta_1} + \int_{\Delta_2} \right) \frac{g(x(t))}{1 - \tau'(\gamma(t))} x(t)dt + \int_0^T |g(x(t - \tau(t)))(x(t) - x(t - \tau(t)))dt
\]

\[
\leq Tg_DD \max_{t \in [0,T]} \left| \frac{1}{1 - \tau'(\gamma(t))} \right|
\]

\[
+ \left( \int_0^T |g(x(t - \tau(t)))(x(t) - x(t - \tau(t)))dt \right)^\frac{1}{q} \left( \int_0^T |x(t) - x(t - \tau(t)))dt \right)^\frac{1}{p}
\]

\[
\leq Tg_DD \max_{t \in [0,T]} \left| \frac{1}{1 - \tau'(\gamma(t))} \right|
\]

\[
+ 2^\frac{1}{q} |\gamma|_0 \left( \int_0^T |x'(t)|^p dt \right)^\frac{1}{p} \left[ \left( \int_{E_1} |g(x(t - \tau(t)))(x(t) - x(t - \tau(t)))dt \right)^\frac{1}{q} + \left( \int_{E_2} |g(x(t - \tau(t)))(x(t) - x(t - \tau(t)))dt \right)^\frac{1}{q} \right]
\]

\[
\leq Tg_DD \max_{t \in [0,T]} \left| \frac{1}{1 - \tau'(\gamma(t))} \right|
\]

\[
+ 2^\frac{1}{q} |\gamma|_0 \left( \int_0^T |x'(t)|^p dt \right)^\frac{1}{p} \left( \int_0^T |x(t) - x(t - \tau(t)))dt \right)^\frac{1}{q}
\]

\[
= Tg_DD \max_{t \in [0,T]} \left| \frac{1}{1 - \tau'(\gamma(t))} \right|
\]

\[
+ 2^\frac{1}{q} |\gamma|_0 \left( \int_0^T |x'(t)|^p dt \right)^\frac{1}{p} \left( \int_0^T |x(t) - x(t - \tau(t)))dt \right)^\frac{1}{q}
\]

\[
\leq Tg_DD \max_{t \in [0,T]} \left| \frac{1}{1 - \tau'(\gamma(t))} \right|
\]

\[
+ 2^\frac{1}{q} |\gamma|_0 \left( \int_0^T |x'(t)|^p dt \right)^\frac{1}{p} \left( \int_0^T \frac{|x(t)|^p dt}{1 - \tau'(\gamma(t))} \right)^\frac{1}{q}
\]

\[
+ 2^\frac{1}{q} |\gamma|_0 \left( \int_0^T |x'(t)|^p dt \right)^\frac{1}{p} \left( \max_{t \in [0,T]} \left| \frac{1}{1 - \tau'(\gamma(t))} \right| \right)^\frac{1}{q} \left( \int_0^T |x(t)|^p dt \right)^\frac{1}{q}, \quad (16)
\]

where \( g_D = \max_{|u| \leq D} |g(u)| \) and \( g_\rho = \max_{|u| \leq \rho} |g(u)| \). Let \( y(t) = x(t + \xi^*) - x(\xi^* \left. \right) \), where \( \xi^* \).
is defined by (11), clearly \( y(0) = y(T) = 0 \). By [13], we have
\[
\int_0^T |y(t)|^p dt \leq \left( \frac{T}{\pi_p} \right)^p \int_0^T |y'(t)|^p dt = \left( \frac{T}{\pi_p} \right)^p \int_0^T |x'(t)|^p dt,
\]
where
\[
\pi_p = 2 \int_0^{(p-1)\frac{1}{p}} \frac{1}{(1 - \frac{sp}{p-1})} ds = \frac{2\pi(p-1)^{\frac{1}{p}}}{p \sin \frac{\pi}{p}}.
\]
Let \( C_p = \begin{cases} \frac{2}{p}, & p > 2; \\ 1, & 1 < p \leq 2. \end{cases} \) and by Minkowski’s inequality, we get
\[
\left( \int_0^T |x(t)|^p dt \right)^{\frac{1}{q}} = \left( \int_0^T |y(t) + x(\xi^*)|^p dt \right)^{\frac{1}{q}}
\leq \left( DT^{\frac{1}{p}} + \left( \frac{T}{\pi_p} \right) \left( \int_0^T |x(t)|^p dt \right)^{\frac{1}{p}} \right)^{\frac{q}{p}}.
\]
Combining (16) and (18), we have
\[
\int_0^T g(x(t) - \tau(t))x(t) dt \leq T g_D D \max_{t \in [0,T]} \left| \frac{1}{1 - \tau'(\gamma(t))} \right| + 2 \frac{1}{p} |\tau|_0 T^{\frac{1}{q}} g_p \left( \int_0^T |x'(t)|^p dt \right)^{\frac{1}{p}}
+ 2 \frac{1}{p} |\tau|_0 (r + \varepsilon_1) \left( \int_0^T |x'(t)|^p dt \right)^{\frac{1}{p}} \left( \max_{t \in [0,T]} \left| \frac{1}{1 - \tau'(\gamma(t))} \right| \right)^{\frac{1}{q}} \left( \int_0^T |x(t)|^p dt \right)^{\frac{1}{q}}
\leq T g_D D \max_{t \in [0,T]} \left| \frac{1}{1 - \tau'(\gamma(t))} \right| + 2 \frac{1}{p} |\tau|_0 T^{\frac{1}{q}} g_p \left( \int_0^T |x'(t)|^p dt \right)^{\frac{1}{p}}
+ C_p 2 \frac{1}{p} |\tau|_0 (r + \varepsilon_1) \left( \max_{t \in [0,T]} \left| \frac{1}{1 - \tau'(\gamma(t))} \right| \right)^{\frac{1}{q}} \left( \frac{T}{\pi_p} \right)^{\frac{p}{q}} \int_0^T |x'(t)|^p dt
+ C_p 2 \frac{1}{p} |\tau|_0 (r + \varepsilon_1) \left( \int_0^T |x'(t)|^p dt \right)^{\frac{1}{p}} \left( \max_{t \in [0,T]} \left| \frac{1}{1 - \tau'(\gamma(t))} \right| \right)^{\frac{1}{q}} \left( \int_0^T |x(t)|^p dt \right)^{\frac{1}{q}} T^{\frac{1}{q}}. \tag{19}
\]
From (19) and (14), we have
\[
\int_0^T |x'(t)|^p dt \leq \lambda \int_0^T g(x(t - \tau(t)))x(t) dt + (\beta T D + T|e_0 D| + (\beta + |e_0|)T^{\frac{1+q}{q}} \left( \int_0^T |x'(t)|^p dt \right)^{\frac{1}{p}}
\]
+ \alpha DT^{\frac{1}{p}} \left( \int_0^T |x'(t)|^p dt \right)^{\frac{p-1}{p}} + \alpha T \int_0^T |x'(t)|^p dt \\
\leq T g_D \max_{t \in [0,T]} \left| \frac{1}{1 - \tau'(\gamma(t))} \right| + 2^{\frac{1}{q}} |\tau|_0 T^{\frac{1}{q}} g_D \left( \int_0^T |x'(t)|^p dt \right)^{\frac{1}{p}} \\
+ C_p 2^{\frac{1}{q}} |\tau|_0 (r + \varepsilon_1) \left( \max_{t \in [0,T]} \left| \frac{1}{1 - \tau'(\gamma(t))} \right| \right)^{\frac{1}{q}} \left( T \frac{T}{\pi_p} \int_0^T |x'(s)|^p ds \right)^{\frac{q}{p}} \\
+ C_p 2^{\frac{1}{q}} |\tau|_0 (r + \varepsilon_1) \left( \int_0^T |x'(t)|^p dt \right)^{\frac{1}{p}} \left( \max_{t \in [0,T]} \left| \frac{1}{1 - \tau'(\gamma(t))} \right| \right)^{\frac{1}{q}} D^{\frac{q}{p}} T^{\frac{q}{p}} \\
+ (\beta T D + T |e|_0 D) + (\beta + |e|_0) T^{\frac{1+q}{q}} \left( \int_0^T |x'(t)|^p dt \right)^{\frac{1}{p}} \\
+ \alpha DT^{\frac{1}{p}} \left( \int_0^T |x'(t)|^p dt \right)^{\frac{p-1}{p}} + \alpha T \int_0^T |x'(t)|^p dt. 

(20)

From

\[ C_p 2^{\frac{1}{q}} |\tau|_0 (r + \varepsilon_1) \left( \max_{t \in [0,T]} \left| \frac{1}{1 - \tau'(\gamma(t))} \right| \right)^{\frac{1}{q}} \left( T \frac{T}{\pi_p} \int_0^T |x'(s)|^p ds \right)^{\frac{q}{p}} + \alpha T < 1, \]

by (20) we have there is a constant \( L_1 > 0 \) such that

\[ \int_0^T |x'(t)|^p dt \leq L_1, \]

combining with (12), we have

\[ |x|_0 \leq D + T^{\frac{1}{q}} \left( \int_0^T |x'(s)|^p ds \right)^{\frac{1}{p}} \leq D + T^{\frac{1}{q}} L_1^{\frac{1}{q}} := L_2. \]

Let \( \omega(t) = \varphi_p(x'(t)), \) then

\[ \int_0^T f(x'(t))(\varphi_p(x'(t)))' dt = \int_0^T f(\varphi_p(\omega(t))))\omega'(t) dt = 0, \]

hence, multiplying both sides of eq. (10) by \((\varphi_p(x'(t)))'\) and integrating from 0 to \( T, \) we get
\[
\int_0^T |\omega'(t)|^2 dt = -\lambda \int_0^T g(x(t - \tau(t))) \omega'(t) dt + \lambda \int_0^T e(t) \omega'(t) dt
\]
\[
\leq (\max_{|x| \leq L_2} |g(x)| + |e|_0) \int_0^T |\omega'(t)| dt
\]
\[
\leq (\max_{|x| \leq L_2} |g(x)| + |e|_0) T^2 \left( \int_0^T |\omega'(t)|^2 dt \right)^{\frac{1}{2}},
\]
which implies there exists a positive constant \( L_3 \) such that \( \left( \int_0^T |\omega'(t)|^2 dt \right)^{\frac{1}{2}} \leq L_3 \), and then
\[
|\omega|_0 \leq T^2 \left( \int_0^T |\omega'(t)|^2 dt \right)^{\frac{1}{2}} \leq T^2 L_3.
\]
Hence,
\[
|x'|_0 \leq \varphi_q(|\omega|_0) \leq \varphi_q(T^2 L_3) := L_4.
\]
Hence we have
\[
||x|| < \max \{ L_2, L_4 \} + 1 := L.
\]

**Step 2.** Let \( \Omega_2 = \{ x \in \text{Ker} M : QNx = 0 \} \). We shall prove that \( \Omega_2 \) is a bounded set. \( \forall x \in \Omega_2 \), then \( x = a_0, a_0 \in \mathbb{R} \), we have
\[
\int_0^T (g(a_0) + f(0) - e(t)) dt = 0.
\]
From integral mean value theorem, there is a constant \( t_1 \in [0, T] \) such that
\[
g(a_0) + f(0) - e(t_1) = 0,
\]
from assumption \( [A3] \), we have \( |a_0| \leq D \). So \( \Omega_2 \) is a bounded set.

**Step 3.** Let \( \Omega = \{ x \in X : ||x|| < L \} \), then \( \Omega_1 \cup \Omega_2 \subset \Omega \). \( \forall (x, \lambda) \in \partial \Omega \times (0, 1) \), from the above proof, \( Mx \neq N_\lambda x \) is satisfied. Obviously, condition \( (H_2) \) of Lemma 2.1 is also satisfied. Now we prove that condition \( (H_3) \) of Lemma 2.1 is satisfied. Take the homotopy
\[
H(x, \mu) = \mu x + (1 - \mu)JQNx, \ x \in \bar{\Omega} \cap \text{Ker} M, \ \mu \in [0, 1],
\]
where \( J : \text{Im} Q \rightarrow \text{Ker} M \) is a homeomorphism with \( Ja = a, a \in \mathbb{R} \). \( \forall x \in \partial \Omega \cap \text{Ker} M \), we have
\( x = a_1, |a_1| = L > D \), then
\[
H(x, \mu) = a_1 \mu + (1 - \mu) \frac{1}{T} \int_0^T (-g(a_1) - f(0) + e(t)) dt,
\]
\[
a_1 H(x, \mu) = a_1^2 \mu + (1 - \mu) \frac{1}{T} \int_0^T a_1 (-g(a_1) - f(0) + e(t)) dt.
\]
By using assumption [A3], we have \( H(x, \mu) \neq 0 \). And then by the degree theory,

\[
\text{deg}\{JQN, \Omega \cap \text{Ker}M, 0\} = \text{deg}\{H(\cdot, 0), \Omega \cap \text{Ker}M, 0\} = \text{deg}\{H(\cdot, 1), \Omega \cap \text{Ker}M, 0\} = \text{deg}\{I, \Omega \cap \text{Ker}M, 0\} \neq 0.
\]

Applying Lemma 2.1, we reach the conclusion.

**Example 3.1** — As an example we consider the following equation:

\[
(\varphi_3(x'))' + \sin x' + g\left(x\left(t - \frac{1}{2} \sin t\right)\right) = \cos t,
\]

where \( f(x') = \sin x' \), \( \tau(t) = \frac{1}{2} \sin t \), \( e(t) = \cos t \), \( T = 2\pi \),

\[
C_p = \begin{cases} 
-ue^{\sin u} \text{ for } u \geq 0, \\
-\frac{1}{2}u \text{ for } u < 0.
\end{cases}
\]

We have

\[
\lim_{|x| \to +\infty} \frac{|g(x)|}{|x|^{p-1}} = 0 < 2^{-\frac{13}{3}} := r.
\]

From simple calculation, let \( \alpha = \frac{1}{64\pi} \), we have

\[
C_p^{\frac{1}{\frac{p}{2}}} |\tau|_0 r \left( \max_{t \in [0, T]} \left| \frac{1}{1 - \tau'(\gamma(t))} \right| \right)^{\frac{1}{\frac{p}{2}}} \left( \frac{T}{\pi_p} \right)^{\frac{p}{2}} + \alpha T = \frac{7}{8} < 1.
\]

Applying Theorem 3.1, (21) has at least one \( 2\pi \)-periodic solution.

**References**


